

Secondary Mathematics Knowledge in Econometrics

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Abstract: Econometrics (economic measure) can be defined as a social science in which economic and mathematical knowledge are co-present in the analysis of economic phenomena. Knowledge already taught in secondary mathematics education must be transformed into tools for modeling phenomena of economic in reality. In this paper, we are going to explain the difficulties of students when they have to mobilize the two objects of knowledge: the slope of the straight line and the logarithms.

Keywords: Mathematics knowledge in secondary education, slope of the straight line, logarithm, econometrics.

1. Secondary mathematics knowledge in econometrics

In this paper, we just mention two objects of knowledge:

- Linear function $y = ax + b$
- Concept of logarithm

These two objects of knowledge are researched from observing students' difficulties when we teach econometrics in the bachelor of economics training program.

- Note 1: Let $y = 24,25 + 0,78x$, where x is the income and y is the expenditure.

When the lecturer raised this question:

If the income is increased a unit of currency, how will the expenditure change?

Almost the students from classes we observed could not answer this question.

- Note 2 : Let $y = ax^b$ (Model 1)

When the lecturer asked this question:

How do we change Model 1 – a nonlinear model to a linear model as the following form $y^* = ax^* + b$?

None of the students had any idea about using logarithm for this case.

The following presentation might explain for the difficulties of students when they mobilize the two mentioned objects of knowledge. On the other hand, we will clarify some roles of each one.

2. Role of the straight line and its slope

2.1. In econometrics

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As mentioned in the introduction, econometrics applies economic and mathematical knowledge in measuring economic relations in reality. For example, to forecast the average consumption by income, we can base on the fundamental psychological law of Keynes (1936): “The fundamental psychological law ... is that men [women] are disposed, as a rule and on average, to increase their consumption as their income increases, but not as much as the increase in their income.”

The econometrician starts describing this law in mathematical language:

“In short, Keynes postulated that the **marginal propensity to consume (MPC)**¹, the rate of change of consumption for a unit (say, a dollar) change in income, is greater than zero but less than 1.” ([10], p. 4)

The point is finding a function which expresses the relationship between expenditure and income, where expenditure is the dependent variable and income is the independent variable. Thus, the econometrician has to set up a mathematical model for this law.

“Although Keynes postulated a positive relationship between consumption and income, he did not specify the precise form of the functional relationship between the two.” ([10], p. 4).

Choosing which type of function needs statistical researches, we can start from a linear function because of its simplicity in mathematical technique and as we can always approximate a nonlinear function by a linear one in the vicinity of the independent variable.

¹ If we use a differentiable function $C(I)$ to express the relationship between expenditure C by the income I then the Marginal Propensity To Consume (MPC) is the derivative $C'(I)$.

“For simplicity, a mathematical economist might suggest the following form of the Keynesian consumption function:

$$Y = \beta_1 + \beta_2 X \quad (I.3.1)$$

where Y = consumption expenditure and X = income, and where β_1 and β_2 known as the **parameters** of the model, are, respectively, the **intercept** and **slope** coefficients.

The slope coefficient β_2 measures the MPC. Geometrically, Eq. (I.3.1) is shown in Figure I.1.

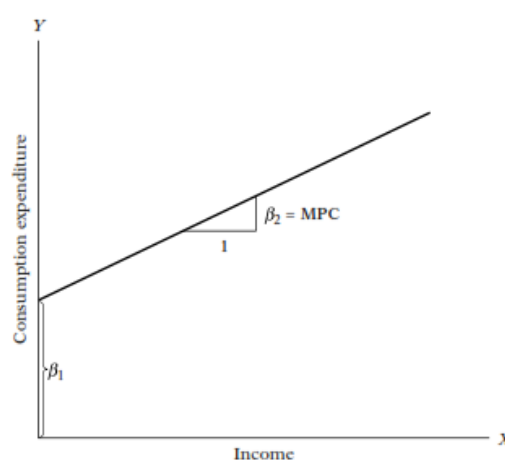


Figure I.1. Keynesian consumption function”. ([10], p. 4)

Therefore, the slope of a straight line is the derivative of the linear function. It measures the slope of that straight line and shows the rate of change of the dependent variable y while the independent variable increases (or decreases) a unit.

2.2. In mathematics teaching in high school

In mathematics teaching at high school in Vietnam, straight line objects appear in all main subjects: Geometry, Algebra and Calculus.

Analysing some current secondary textbooks

If we just consider the straight line when having its function, this object firstly appears in

the Algebra program of grade 7th with the equation $y = ax$ (the straight line that passes through the origin).

The more general equation is presented in the Algebra program of grade 9th ($y = ax+b$). At this time, the meaning of the slope of a straight line is mentioned.

- The first meaning of the slope is that: the sign of the slope defines the direction of variation of a linear function.

“A linear function $y = ax + b$ ($a \neq 0$) is defined for every real value x and has these properties:

- a) increasing in R when $a > 0$.
- b) decreasing in R when $a < 0$.” ([2], p. 47)

This meaning is given to students by these types of tasks (in the exercises): define the variation (decreasing or increasing) of a linear function, find the parameter m that makes a linear function decreasing (or increasing).

It is important to notice that: when the first meaning is mentioned, the term “slope” has not appeared yet.

- The meaning that “the slope is the tangent of the angle formed by the straight line and Ox ”

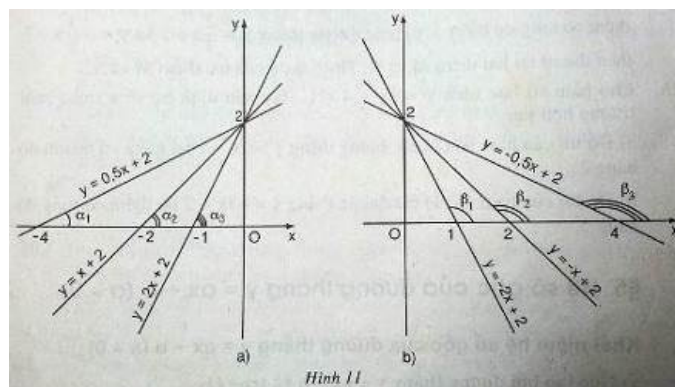
is just informally constructed at high schools. The explanation in Maths teacher’s book has shown the reason is that the trigonometric values of obtuse angles have not been defined.

“[...] At high school, students have not known how to find the angle α when $\tan\alpha$ is negative. Thus, when the slope of the straight line $y = ax + b$ is negative, students have to find an indirect way to calculate the angle formed by this line and Ox .

[...] Finally, through two already known examples, teachers finalize the problem about a direct way to calculate angle α formed by the straight line $y = ax + b$ and Ox in case $a > 0$, and an indirect way to calculate angle α in case $a < 0$ ($\alpha = 180^\circ - \alpha'$ với $\alpha' < 90^\circ$ và $\tan\alpha' = -a$.)” ([3], p. 70-71)

The above explanation relates to this type of task: calculate the angle formed by the straight line $y = ax + b$ and Ox . Textbook presents the technique to solve this type of task by plotting graph and then calculate the tangent of the acute angle.

In the theory part of the textbook, the term “slope” appears after an activity that already has the solution and is illustrated by this graph:



? Figure 11a) presents the graph of this function (with $a > 0$):

$$y = 0,5x + 2; y = x + 2; y = 2x + 2.$$

Figure 11b) presents the graph of this function (with $a < 0$):

$$y = -2x + 2; y = -x + 2; y = -0,5x + 2$$

a) Compare these three angles: $\alpha_1, \alpha_2, \alpha_3$, and compare the respective values of the coefficients in each functions (if $a > 0$), and then draw a conclusion.

b) Repeat with a) when $a < 0$.

By considering graphs of these above mentioned functions, we may say:

- When the coefficient is positive ($a > 0$), the angle formed by the straight line $y = ax + b$ and Ox is acute. The bigger the coefficient is, the greater the angle is, but this angle is still smaller than 90° .

- When the coefficient is negative ($a < 0$), the angle formed by the straight line $y = ax + b$ and Ox is obtuse. The bigger the coefficient is, the greater the angle is, but this angle is still smaller than 180° .

Since there is a relationship between the coefficient and the angle formed by the straight line $y = ax + b$ and Ox , a is called the slope of this line." ([2], p. 56-57)

The relationship between the slope and the oriented angle is mentioned; however, the link with the gradient or the grow rate of a function depends on variable has not been clarified.

Analysing some current highschool textbooks

- The meaning "signs of slopes define the direction of variation of the straight line function" is repeated in Algebra of grade 10th program. Besides, the case that a slope equals 0 is mentioned as well.

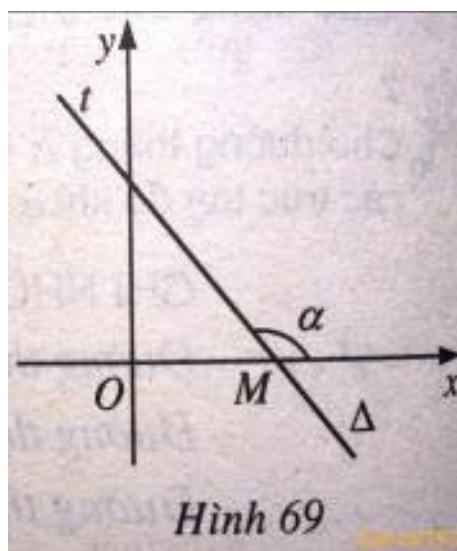
- The meaning "the slope is the tangent of the angle formed by the straight line and Ox " is mentioned in Geometry of grade 10th program. At this time, the straight line function is considered more generally, including the case that a straight line does not have the slope.

"Notice

Now consider the straight line Δ with the general equation $ax + by + c = 0$.

If $b \neq 0$, this equation has been transformed into $y = kx + m$ (3).

With $k = -\frac{a}{b}, m = -\frac{c}{b}$, where k is the slope of the straight line Δ and (3) is called the Slope Intercept Form of Δ .



Geometrical meaning of slopes (Fig.69)

Now consider the straight line $\Delta: y = kx + m$.

When $k \neq 0$, let M is the intersection point of Δ and Ox , and M_t is a ray of Δ that lies on Ox . Then, if α is the angle formed by the two rays M_t and M_x , the slope of Δ is the tangent of angle α , that is $k = \tan \alpha$.

When $k = 0$, Δ is the straight line that is parallel or coincide with Ox ." ([7], p. 77-78)

However, in the exercise, there is no type of task which recalls this meaning.

- Another meaning of slope may appear informally in the textbook: the slope of a straight line is the ratio between the ordinate and the abscissa of a directional vector of the equation of that straight line (if it has a slope).

- When researching the derivative in Calculus in grade 11th and 12th, the knowledge "the slope of the tangent is the derivative at the contact of a curve" is emphasized throughout this type of task: write the tangential equation of a curve at a contact.

In calculating derivative technique, every students have to learn by heart the rule $(ax + b)' = a$. However, this does not guarantee the meaning "the slope of the tangent is the derivative of the straight line function" is formed in students' mind.

Besides, the research of Le Thi Hoai Chau (2014) has indicated that the meaning "the rate of change" of the derivative has not appeared in the recent teaching Maths at high school of Vietnam.

Thus, analyzing some recent highschool text books (especially in the exercise part for students) has indicated that these following meanings about the slope and the relationship between them has not been clarified.

- The slope is the derivative of the straight line function.

- The slope calculates the gradient of a straight line and shows the change rate of y when x changes a unit.

This fact explains the difficulties of students as we mentioned in the first note when teaching econometric at economics universities.

In the next part, we are going to present some analytical results about the instrumental role of logarithms related to the second note.

3. Instrumental role of logarithms

3.1. The significant feature of logarithms

Some historical researches have shown that John Napier (1550-1617) is one of the first people who used logarithms (although he did not define this definition officially). His logarithmic tables were established in 1614. The aim of this research is doing the addition, subtraction, division into two or three in these tables would replaced the multiplication, division, taking square root and cube root of positive real numbers alternatively. Nowadays, we have already known that these tables are the logarithm of positive real numbers with the *nap* base, which might be expressed by e base like this: $\log_{nap} x = 10^7 \cdot \log_{\frac{1}{e}} \left(\frac{x}{10^7} \right)$.

Hieu Nguyen Viet (2013) re-presented some examples about using Napier's logarithmic table. Here is an example:

"Example 1: Let $a = 10.000.000$ and $b = 5.000.000$. Find the square root of the product $a \cdot b$.

Napier found $c = \sqrt{a \cdot b}$ in this way:

+ Take the Napier logarithm of both a and b , he got $\log_{nap} a = 0$; $\log_{nap} b = 6931470$.

+ Find $\log_{nap} c$ by this formula $\log_{nap} c = \frac{\log_{nap} a + \log_{nap} b}{2} = 3465735$.

+ Look up the logarithmic table, found the square root of the product $a \cdot b$ was approximately 7071068." ([5], tr. 9)

Nowadays, we know many ways to define the logarithms, for example:

- The logarithmic function is the reverse of the exponential function: $y = a^x$ (where $a > 0$ and $a \neq 1$).

- The logarithmic function is defined by the formula $y = \log_a x = \frac{\ln x}{\ln a}$, while $\ln a$ is the flat area limited by a hyperbol, which has equation $y = \frac{1}{x}$, the horizontal axis and the two straight lines $x = 1, x = a$ (where $a > 0$ and $a \neq 1$).

- The logarithmic function is defined by the formula $y = \log_a x = \frac{\ln x}{\ln a}$ (where $a > 0$ and $a \neq 1$), while $y = f(x) = \ln x$ is the solution unic of the functional equation $f(x.t) = f(x) + f(t)$.

No matter how we define the logarithms, its feature is still $f(x.t) = f(x) + f(t)$. An easier way to state is that *the logarithm converts multiplication to addition, and hence, it converts powers to multiplication.*

$$\ln A = \tan x \cdot \ln(1 + (\sin x - 1)) = \frac{\ln(1 + (\sin x - 1))}{\sin x - 1} \cdot \frac{\sin x - 1}{\cot x}$$

And $\frac{\sin x - 1}{\cot x} = \sin x \cdot \frac{\sin x - 1}{\cos x}$. Thus,

$$\lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x - 1}{\cot x} = 0$$

Finally: $\lim_{x \rightarrow \frac{\pi}{2}} \ln A = 0$, i.e:

$$\lim_{x \rightarrow \frac{\pi}{2}} A = \lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x} = e^0 = 1 \text{'' ([9], p. 46)}$$

- Converting multiplication to the addition property of logarithms allows us to find the derivative of some functions more easily. For example:

3.2. Some applications of logarithms

With the characteristic mentioned above, the logarithm has many applications. We introduce some applications in teaching Maths at the beginning years of University - College and in some other sciences, such as Physics, Chemistry. Especially, we are going to present some instrumental roles of logarithms in econometrics (and statistics, in general).

- Converting powers to multiplication property of logarithms allows us to solve the exponential equation like $a^{f(x)} = b^{g(x)}$, to calculate the derivative of some functions like $y = f(x)^{g(x)}$ or to find the limit of undefined types: $1^\infty, 0^0, \infty^0$. For example:

“8. Find these limits: [...]”
 5) $\lim_{x \rightarrow \frac{\pi}{2}} (\sin x)^{\tan x}$ ”([9], p. 36)

Here is the solution in the text book:

“5) Let $A = (\sin x)^{\tan x} = [1 + (\sin x - 1)]^{\tan x}$.

“3. Find the derivative of these functions: 5)

$$y = \sqrt[3]{\frac{1+x^3}{1-x^3}} \text{ ([9], p. 54)}$$

Solution: $y = \sqrt[3]{\frac{1+x^3}{1-x^3}} = \left(\frac{1+x^3}{1-x^3}\right)^{\frac{1}{3}}$.

Therefore, $\ln y = \frac{1}{3} \cdot [\ln(1+x^3) - \ln(1-x^3)]$

$$\frac{y'}{y} = \frac{1}{3} \left(\frac{3x^2}{1+x^3} + \frac{3x^2}{1-x^3} \right) = x^2 \cdot \left[\frac{1}{1+x^3} + \frac{1}{1-x^3} \right].$$

$$\text{So, } y' = \frac{2x^2}{1-x^6} \sqrt[3]{\frac{1+x^3}{1-x^3}}, |x| \neq 1. \text{'' ([9], p. 60)}$$

- These indicated applications in mathematics have helped in solving some problems in some other natural science subjects. Furthermore, some other roles of logarithms allow us to convert some quantities, which have too big or too small values into an easier control bound. For example:

+ The $pH = -\log C_{H^+}$ (decimal logarithm)

with C_{H^+} is the molar concentration of ion H^+ in the mixture and has a very small value, about from 10^{-14} to 1. Thanks to logarithms, the pH rate of a mixture ranges from 0 to 14.

+ The strength of an earthquake $M = \log \frac{I}{I_0}$ (Richter), where I_0 is the standard amplitude and I is the amplitude of an oscillation of an earthquake. The ratio might be very big. It ranges from 1 to 10^{10} . Through logarithms, the strength of an earthquake is expressed by 10 unit in Richter scale.

In econometrics (and statistics, in general)

As mentioned above, in mathematical technique aspect, a linear model is easier for researching than a nonlinear one. This is not an exception when applying mathematics in economic research. By this point, the logarithm has made its special benefit thanks to the significant feature: converts multiplication to addition and powers to multiplication.

“Consider the following model, known as the **exponential regression model**:

$$Y_i = \beta_1 X_i^{\beta_2} e^{u_i} \quad (6.5.1)$$

which may be expressed alternatively as

$$\ln Y_i = \ln \beta_1 + \beta_2 \ln X_i + u_i \quad (6.5.2)$$

where $\ln =$ natural log (i.e, log to base e and where $e = 2,718$).

If we write (6.5.2) as

$$\ln Y_i = \alpha + \beta_2 \ln X_i + u_i \quad (6.5.3)$$

where $\alpha = \ln \beta_1$, this model is linear in the parameters α and β_2 , linear in the logarithms of the variables Y and X , and can be estimated by OLS regression. Because of this linearity, such models are called **log-log**, **double-log**, or **log-linear** models.

[...]

$$Y_i^* = \alpha + \beta_2 \ln X_i^* + u_i \quad (6.5.4)$$

where $Y_i^* = \ln Y_i$ and $X_i^* = \ln X_i$.” ([10], p. 175 - 176)

Estimating and researching model (6.5.4) is done more easily than model (6.5.1). Gujarati (2004) explained this benefit accompanied with an example in economics:

“One attractive feature of log-log model, which has made it popular in applied work, is that the slope coefficient β_2 measures **the elasticity**² of Y with respect to X , that is, the percentage change in Y for a given (small) percentage change in X . Thus, if Y represents the quantity of a commodity demanded and X its unit price, β_2 measures the price elasticity of demand, a parameter of considerable economic interest. If the relationship between quantity demanded and price is as shown in Figure 6.3a, the double-log

² In a simpler way, in model (6.5.3), if X increases (or decreases) 1% then Y will change (it will increase or decrease depends on the sign of β_2) $|\beta_2|$ %.

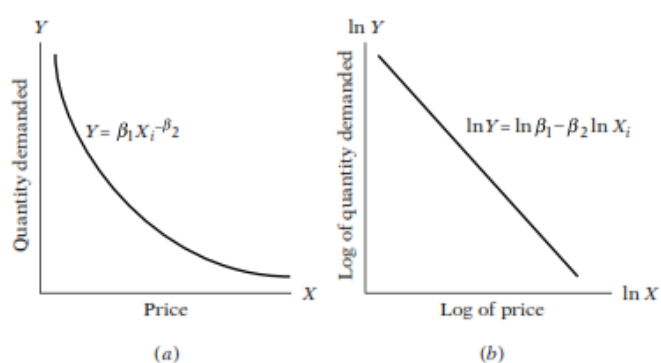


FIGURE 6.3 Constant-elasticity model.

Transformation as shown in Figure 6.3b will then given the estimate of the price elasticity ($-\beta_2$)." ([10], p. 176 - 177)

Beside the instrumental role of logarithm in the above excerpt, we also know an example about the reason for the appearance of a non-linear kind of function in the reality. The graph of this kind of function $y = \beta_1 x^{-\beta_2}$ (through its graph, the author implied that β_1, β_2 are positive and $\beta_2 \neq 1$) shows that if the price is increased then in general, the quantity demanded will be decreased and going to reach 0. This is a rule in economics and it is easier to understand for highschool students. This nonlinear kind of function has a reason for its appearance and is worthy for researching (instead of giving a function at first and then research it as the traditional way of teaching Maths).

The economic knowledge has enriched the real problems, beside pure problems of natural science - especially in Physics (because this science has an historic distribution in the rise of many mathematic knowledge), and hence, has helped serving teaching by modeling.

Beside the mentioned kind of function, logarithm has allowed converting some kind of nonlinear functions to linear functions. For example, $Y_t = Y_0(1+r)^t$, where Y_t : the total

amount of the principal and the interest after t terms with compound interest when saving money, Y_0 is the principal, r is the interest (this formula is presented in the current Algebra - Calculus text books of grade 11th). After using the logarithms, we will get a linear model $Y^* = \alpha + \beta t$ with $Y^* = \ln Y_t$; $\alpha = \ln Y_0$ and $\beta = \ln(1+r)$.

In conclusion, thanks to its feature, the logarithm is an instrument to convert a nonlinear function to a linear function.

Moreover, when researching statistical data, the demanding of transforming very big data into a more controllable bound is asked. Thus, the statistical analysing software always constructs the logarithm (with natural base or decimal base) and allows the choosing logarithmic scale when plotting graph.

3.3. Logarithm in Maths teaching at high school in Vietnam

Vietnamese textbooks define the definition of logarithm with base a ($a > 0$ and $a \neq 1$) of a positive number b first, and then define the logarithmic function:

"Let a and b are positive numbers and $a \neq 1$. The number α which satisfies the equality is called the logarithm with base a of b and noted as $\log_a b$. $\alpha = \log_a b \Leftrightarrow a^\alpha = b$ " ([4], p. 62).

The research of Hieu Nguyen Viet (2012) has shown the almost unique instrumental role of logarithm in the current highschool textbooks is solving exponential and logarithmic equations. The more specifically:

+ More than 80% number of tasks in exercise of the two Algebra - Calculus textbooks of grade 11th are the solving exponential and logarithmic equations type.

+ About 17% tasks related to simplifying or calculating logarithmic expressions.

+ Just about 3% questions (1 task in each text book) related to simplifying operations of logarithm. For example:

“Example 6. In order to calculate $2,1^{3,2}$, they follow these steps:

- Calculate $\log 2,1^{3,2}$:
 $\log 2,1^{3,2} = 3,2 \log 2,1 \approx 1,0311$

- Then ... $2,1^{3,2} \approx 10^{1,0311} \approx 10,7424$ ” ([8], p. 88).

+ The simplifying operation role of derivation has just appeared in the proofs in the theory part in textbooks. As it does not appear in exercise so this role is predicted not to be taught to most of the students.

In conclusion, the required teaching contents, which are presented in the textbooks about the logarithm object, has not been enough to be taught to students about the characteristic property “transforming multiplication into addition” of this knowledge.

4. Conclusion

The researches that we have presented are examples for a way to determine the element for answering the questions: What do we teach Maths for? Which contents do we teach?

Considering the instrumental role of objects of Maths knowledge in highschool in some other natural science subjects (instead of just considering in Maths) has helped clarifying the reason why an object of knowledge is chosen in teaching and which meaning of them we must teach.

Moreover, the results getting through researching method presented in our paper is totally appropriate with the mentioned trend of teaching in Vietnam for the hope of totally renovating highschool education - teaching by modeling, intergrated teaching, and intersubjects teaching.

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