Local polynomial convexity of union of two graphs with CR isolated singularities

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Abstract. We give sufficient conditions so that the union of two graphs with CR isolated singularities in $\mathbb{C}^2$ is locally polynomially convex at a singular point. Using this result and some ideas in previous work, we obtain a new result about local approximation continuous function.

1. Introduction

We recall that for a given compact $K$ in $\mathbb{C}^n$, by $\hat{K}$ we denote the polynomial convex hull of $K$ i.e.,

$$\hat{K} = \{z \in \mathbb{C}^n : |p(z)| \leq \|p\|_K \text{ for every polynomial } p \text{ in } \mathbb{C}^n\}.$$

We say that $K$ is polynomially convex if $\hat{K} = K$. A compact $K$ is called locally polynomially convex at $a \in K$ if there exists the closed ball $B(a)$ centered at $a$ such that $B(a) \cap K$ is polynomially convex.

A smooth real manifold $S \subset \mathbb{C}^n$ is said to be totally real at $a \in S$ if the tangent plane $T_S(a)$ of $S$ at $a$ contains no complex line. A point $a \in S$ is not totally real that will be called a CR singularity. By the result of Wermer, if $K$ is contained in totally real smooth submanifolds of $\mathbb{C}^2$ then $K$ is locally polynomially convex at all point $a \in K$ (see [1], chapter 17). Note that union of two polynomially convex sets which can be not polynomially convex set. Let $D$ be a small closed disk in the complex plane, centered at the origin and

$$M_1 = \{(z, \bar{z}) : z \in D\}; M_2 = \{(z, \bar{z} + \varphi(z)) : z \in D\},$$

where $\varphi$ is a $C^1$ function in neighborhood of 0, $\varphi(z) = o(|z|)$. Then $M_1, M_2$ are totally real(locally contained in a totally real manifold), so that $M_1, M_2$ are locally polynomially convex at 0. The local polynomially convex hull of $M_1 \cup M_2$ is essentially studied by Nguyen Quang Dieu (see [2,3]).

Let

$$X_1 = \{(z, \bar{z}^n) : z \in D\}, X_2 = \{(z, \bar{z}^n + \varphi(z)) : z \in D\},$$

where $n \geq 1$ is integer and $\varphi$ is a $C^1$ function in neighborhood of 0, $\varphi(z) = o(|z|^n)$. If $n > 1$ then $X_1$ and $X_2$ is not totally real at 0, so we can not deduce that $X_1$ and $X_2$ are locally polynomially at 0 by the Wermer’s work. However, using the results about local approximation of De Paepe (see [4,5]) or the work of Bharali (see [5]), we can conclude that $X_1$ and $X_2$ are locally polynomially convex at 0. In this paper, we will investigate the local polynomially hull of $X_1 \cup X_2$ at 0. The ideas of proof takes

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from [2] and [3]. An appropriate tool in this context is Kallin's lemma (see [6,7]): Suppose $X_1$ and $X_2$ are polynomially convex subsets of $\mathbb{C}^n$, suppose there is polynomial $p$ mapping $X_1$ and $X_2$ into two polynomially convex subsets $Y_1$ and $Y_2$ of the complex plane such that $0$ is a boundary point of both $Y_1$ and $Y_2$ and with $Y_1 \cap Y_2 = \{0\}$. If $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex, then $X_1 \cup X_2$ is polynomially convex. Several instances of such a situation, motivated by questions of local approximation, were studied by O’Farell, De Paepe and Nguyen Quang Dieu (see [8-10],...).

Let $f$ be a continuous function on $D$. We denote that $[z^2, f^2; D]$ is the function algebra which consisting of uniform limit on $D$ of all polynomials in $z^2$ and $f^2$. Using polynomial convexity theory, it can be shown that $[z^2, f^2; D] = C(D)$ for some choices a $C^1$ function $f$, which behaves like $\overline{z}$ near the origin (see [9-11],...). By the known result about approximation of O’Farrell, Preskenis and Walsh [12] :if $X$ is polynomially convex subset of the real manifold $M$, $K$ is a compact subset of $X$ such that $X \setminus K$ is totally real. Then, if $f$ is continuous function on $X$ and $f$ can be uniform approximated by polynomials on $K$ then $f$ can be uniform approximated by polynomials on $X$, and the techniques developed in [13], we give a class function $f$ which behaves like $z^n$ such that $[z^2, f^2; D] = C(D)$.

2. The main results

We always take the graphs $X_1$ and $X_2$ of the form (*). For each $r > 0$ we put

$$X_i^r = X_i \cap \{(z, w) : |z| \leq r\}, \quad i = 1, 2.$$  

Now we come to the main results of this paper.

**Theorem 2.1.** Let $m, n$ be positive integers with $m > n$. Let $\varphi$ be a $C^1$ function which is defined near 0 of the form

$$\varphi(z) = \begin{cases} \sum_{k=\infty}^{+\infty} a_k z^k z^{m-k} + f(z) & z \neq 0 \\ 0 & z = 0, \end{cases}$$

where $f(z)$ is a $C^1$ function and $f(z) = o(|z|^m)$. Suppose that there exists $l \leq \frac{m}{2}$ such that

$$|a_l| > \sum_{k \neq l} |a_k|$$  \hspace{1cm} (1)

and $\frac{m-2l}{n}$ is integer. Then $X_1 \cup X_2$ is locally polynomially convex at 0.

**Proof.** Consider the polynomial $p(z, w) = \overline{\alpha} z^{m-2l+n} + \alpha w^{\frac{m-2l}{n} + 1}$ with $\alpha$ choose later. Thus $p(X_1) = \overline{\alpha} z^{m-2l+n} + \alpha z^{m-2l+n}$ belongs to real axis and

$$p(X_2) = \overline{\alpha} z^{m-2l+n} + \alpha \left( z^n + \sum_{k=-\infty}^{+\infty} a_k z^k z^{m-k} + f(z) \right)^{\frac{m-2l}{n} + 1} =$$

$$= \overline{\alpha} z^{m-2l+n} + \alpha \left( z^n + \sum_{k=-\infty}^{+\infty} a_k z^k z^{m-k} + o(|z|^m) \right)^{\frac{m-2l}{n} + 1} + o(|z|^m).$$

From $p(X_1) = \overline{\alpha} z^{m-2l+n} + \alpha z^{m-2l+n} \in \mathbb{R}$, we obtain

$$\text{Im } p(X_2) = \text{Im}(\alpha \left( \frac{m-2l}{n} + 1 \right) z^{m-2l} + \sum_{k=-\infty}^{+\infty} a_k z^k z^{m-k} + o(|z|^m)).$$
Choose \( \alpha = i \bar{\alpha} \). It follows that

\[
\text{Imp}(X_2) \geq |z|^{2m-2l} \left( \frac{m-2l}{n} + 1 \right) \left( |a_l| - \sum_{k \neq l} |a_k| \right) > 0
\]  

(2)

for any \( z \neq 0 \) in a small neighborhood of 0, by (1). It implies that \( p(X_2) \cap \mathbb{R} = \{0\} \). On the other hand, from the inequality (2) we see that

\[
p^{-1}(0) \cap X_2^* = \emptyset.
\]

It is elementary to check that

\[
p^{-1}(0) \cap X_1^* = \left\{ (\rho \exp(i\theta), \rho^n \exp(-n i \theta)) : 0 \leq \rho \leq r \right\},
\]

with a constant \( \theta \). Obviously,

\[
p^{-1}(0) \cap X_1^*
\]

is polynomially convex for \( r \) small enough. Thus \( p^{-1}(0) \cap (X_1^* \cup X_2^*) \) is polynomially convex for \( r \) small enough. By Kallin’s lemma (mentioned in introduction) we conclude that \( X_1^* \cup X_2^* \) is polynomially convex for \( r \) small enough. The proof is completed.

**Remark.** 1) In the Theorem 1 we can replace \( X_1 \) by \( X_1' = \{ (z, \bar{z}^n - \varphi(z)) : z \in D \} \). Then, as \( p \) in Theorem 1 we obtain the estimate

\[
\text{Imp}(X_1') < 0,
\]

for any \( z \neq 0 \) in a small neighborhood of 0. On the other hand \( p^{-1}(0) \cap (X_1'^* \cup X_2^*) = \emptyset \) for \( r \) small enough. By Kallin’s lemma we may conclude that \( X_1' \cup X_2 \) is locally polynomially convex.

2) This result includes the more restricted case \( n = 1 \) that is studied by Nguyen Quang Dieu (see [2]).

The following Proposition shows that if we replace \( l > \frac{m}{2} \) we may get nontrivial hull of \( X_1^* \cup X_2^* \).

**Proposition 2.2.** Let \( n, p \) be positive integers and

\[
X_1 = \{(z, \bar{z}^n) : z \in D \}; \quad X_2 = \{(z, \bar{z}^n + z^p \bar{z}^{n+p}) : z \in D \}.
\]

Then \( X_1 \cup X_2 \) is not locally polynomially convex at 0.

**Proof.** For each \( t > 0 \), let \( W_t = \{(z, w) : z^n w = t\} \). Consider the sets

\[
P_t := W_t \cap X_1 = \{(z, \bar{z}^n) : |z| = t^{\frac{1}{2n}} \};
\]

\[
Q_t := W_t \cap X_2 = \{(z, \bar{z}^n + z^p \bar{z}^{n+p}) : |z| = s \},
\]

where \( s \) is unique positive solution of the equation \( s^{2n} + s^{2p+2n} = t \). By the maximum modulus principle we see that the hull of \( X_1^* \cup X_2^* \) will contain an open subset of \( W_t \) bounded by two closed curves \( P_t \) and \( Q_t \) for any \( t > 0 \) small enough and hence \( X_1 \cup X_2 \) is not locally polynomially convex at 0.
Theorem 2.3. Let \( m \) be a positive even integer and let \( n \) be an odd integer such that \( m > n \). Let \( g \) be a \( C^1 \) function which is defined near \( 0 \) of the form
\[
g(z) = \begin{cases} \bar{z}^n + \sum_{k=-\infty}^{+\infty} a_k \bar{z}^k z^{m-k} + f(z) & z \neq 0 \\ 0 & z = 0, \end{cases}
\]
where \( f \) is a \( C^1 \) function and \( f(z) = o(|z|^m) \). Suppose that there exists \( \ell \) such that \( \frac{m-2\ell}{n} \) is a positive integer and
\[
|a_\ell| > \sum_{k \neq \ell} |a_k|.
\] (3)
Then the functions \( z^2 \) and \( g^2(z) \) separate points near \( 0 \). Moreover, \(|z^2, g^2; D| = C(D)\) for \( D \) small enough.

We need the next lemma (see [7,8]) for the proof of Theorem 2.1.

Lemma 2.4. Let \( X \) be a compact subset of \( C^2 \), and let \( \pi: C^2 \to C^2 \) be defined by \( \pi(z, w) = (z^m, w^n) \). Let \( \pi^{-1}(X) = X_{11} \cup \ldots \cup X_{kl} \cup \ldots \cup X_{mn} \) with \( X_{mn} \) compact, and \( X_{kl} = \{(p^kz, \tau^l w) : (z, w) \in X_{mn}\} \) for \( 1 \leq k \leq m, 1 \leq l \leq n \), where \( \rho = \exp\left(\frac{2\pi i}{m}\right) \) and \( \tau = \exp\left(\frac{2\pi i}{n}\right) \). If \( P(\pi^{-1}(X)) = C(\pi^{-1}(X)) \), then \( P(X) = C(X) \).

Proof of Theorem 2.3. First we show that the functions \( z^2 \) and \( g^2(z) \) separate points near \( 0 \). Clearly points \( a \) and \( b \) with \( a \neq -b \) are separated by \( z^2 \). Now assume that \( g^2(z) \) takes the same value at \( a \) and \( -a \) for some \( a \neq 0 \). Set
\[
h(z) = \begin{cases} \sum_{k=-\infty}^{+\infty} a_k \bar{z}^k z^{m-k} + f(z) & z \neq 0 \\ 0 & z = 0, \end{cases}
\]
it follows that \( h(a) = -h(-a) \). As \( m \) is even, we have
\[
\sum_{k=-\infty}^{+\infty} a_k \bar{a}^k a^{m-k} = \frac{-f(a) - f(-a)}{2}.
\]
Dividing both sides by \( \bar{a}^{m-l} a^l \) we obtain
\[
a_l + \sum_{k \neq \ell} a_k \bar{a}^{l-k} = \frac{-f(a) - f(-a)}{2a^{m-l} \bar{a}^l}.
\]
By the inequality (3) and the fact that \( f(z) = o(|z|^m) \), we arrive at a contradiction if we choose the disk \( D \) sufficiently small.

Next we consider for a small closed disk \( D \) the set \( \tilde{X} \) which is the inverse of the compact \( X = \{(z, g^2(z)) : z \in D \} \) under the map \( (z, w) \mapsto (z^2, w^2) \). We have \( \tilde{X} = X_1 \cup X_2 \cup X_3 \cup X_4 \) where
\[
X_1 = \{(z, \bar{z}^n + h(z)) : z \in D\};
\]
\[
X_2 = \{(-z, -\bar{z}^n - h(z)) : z \in D\} = \{(z, \bar{z}^n - h(-z)) : z \in D\};
\]
\[
X_3 = \{-z, \bar{z}^n + h(z)\};
\]
\[
X_4 = \{(z, -\bar{z}^n - h(z)) : z \in D\} = \{(-z, \bar{z}^n - h(-z)) : z \in D\};
\]
By Remark 1), $X_1 \cup X_2$ is polynomially convex for $D$ small enough. We have $X_3 \cup X_4$ is the image of $X_1 \cup X_2$ under the biholomorphic map $(z, w) \mapsto (-z, w)$. So $X_3 \cup X_4$ is also polynomially convex with $D$ sufficiently small.

Now we consider the polynomial $q(z, w) = z^n w$. Then $q$ maps $X_1 \cup X_2$ to an angular sector situated near the positive real axis, while $p$ maps $X_3 \cup X_4$ to such sector situated near the negative real axis. The sectors only meet at the origin. Applying Kallin’s lemma we get $\tilde{X} = X_1 \cup X_2 \cup X_3 \cup X_4$ is polynomially convex with $D$ small enough. Furthermore, notice that $\tilde{X} \setminus \{0\}$ is totally real (locally contained in a totally real manifold), by an approximation theorem of O’Farrell, Preskenis and Walsh (mentioned in introduction), we get that every continuous function on $\tilde{X}$ can be uniformly approximated by polynomials. By the Lemma 2.4, we see that the same is true for $X$, which is equivalent to the fact that our algebra equals $C(D)$.

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References