On stability of Lyapunov exponents

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Abstract. In this paper we consider the upper (lower) - stability of Lyapunov exponents of linear differential equations in $\mathbb{R}^n$. Sufficient conditions for the upper - stability of maximal exponent of linear systems under linear perturbations are given. The obtained results are extended to the system with nonlinear perturbations.

Keywords: Lyapunov exponents, upper (lower) - stability, maximal exponent.

1. Introduction

Let us consider a linear system of differential equations

\[ \dot{x} = A(t)x; \quad t \geq t_0 \geq 0. \]  \hfill (1)

where $A(t)$ is a real $n \times n$ - matrix function, continuous and bounded on $[t_0; +\infty)$. It is well known that the above assumption guarantees the boundedness of the Lyapunov exponents of system (1). Denote by

$\lambda_1; \lambda_2; \ldots; \lambda_n \ (\lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n)$

the Lyapunov exponents of system (1).

Definition 1. The maximal exponent $\lambda_n$ of system (1) is said to be upper - stable if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that for any continuous on $[t_0; +\infty)$ $n \times n$ - matrix $B(t)$, satisfying $\|B(t)\| < \delta$, the maximal exponent $\mu_n$ of perturbed system

\[ \dot{x} = (A(t) + B(t))x, \]  \hfill (2)

satisfies the inequality

\[ \mu_n < \lambda_n + \epsilon. \]  \hfill (3)

If $\|B(t)\| < \delta$ implies $\mu_1 > \lambda_1 - \epsilon$, we say that the minimal exponent $\lambda_1$ of system (1) is lower - stable.

In general, the maximal (minimal) exponent of system (1) is not always upper (lower) - stable [1]. However, if system (1) is reducible (in the Lyapunov sense) then its maximal (minimal) exponent is upper (lower) - stable. In particular, if system (1) is periodic then it has this property [2,3]. A problem arises: In what conditions the maximal (minimal) exponent of nonreducible systems is upper (lower) - stable? The aim of this paper is to show a class of nonreducible systems, having this property.

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2. Preliminary lemmas

Lemma 1. Let system (1) be regular in the Lyapunov sense. The maximal exponent $\lambda_n$ is upper-stable if only if the minimal exponent of the adjoint system to (1) is lower-stable.

Proof. We denote by

$$\alpha_1; \alpha_2; ..., \alpha_n \ (\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n)$$

the Lyapunov exponents of the adjoint system to (1):

$$\dot{y} = -A^*(t)y.$$ (4)

According to the Perron theorem, we have

$$\lambda_1 + \alpha_1 = 0, \ \lambda_n + \alpha_n = 0.$$ (5)

If the maximal exponent $\lambda_n$ of system (1) is upper-stable then the minimal exponent $\alpha_n$ of system (4) is lower-stable. In fact, denoting by

$$\beta_1; \beta_2; ..., \beta_n \ (\beta_1 \geq \beta_2 \geq ... \geq \beta_n)$$

the Lyapunov exponents of adjoint system to (2), we have

$$\beta_1 + \mu_1 = 0, \ \beta_n + \mu_n = 0.$$ (6)

Hence

$$\beta_n = -\mu_n > -\lambda_n - \epsilon = \alpha_n - \epsilon \ \text{if} \ |B^*(t)|| < \delta.$$ (7)

Conversely, suppose that the minimal exponent $\alpha_n$ is lower-stable, then if (7) is satisfied we have

$$\beta_n \geq \alpha_n - \epsilon.$$ (7)

Then

$$\mu_n = -\beta_n < -\alpha_n + \epsilon = \lambda_n + \epsilon.$$ (7)

Which proves the lemma.

Consider now a nonlinear system of the form

$$\dot{x} = A(t)x + f(t, x).$$ (8)

Lemma 2. (Principle of linear inclusion) [1] Let $x(t)$ be an any nontrivial solution of system (8). There exists a matrix $F(t)$ such that $x(t)$ is a solution of the linear system

$$\dot{y} = [A(t) + F(t)]y.$$ (8)

Moreover, if $f(t, x)$ satisfies the condition

$$|f(t, x)| \leq g(t)|x|; \forall t \geq t_0; \forall x \in \mathbb{R}^n,$$ (8)

then matrix $F(t)$ satisfies the inequality

$$\|F(t)\| \leq g(t); \forall t \geq t_0.$$ (8)

The proof of Lemma 2 is given in [1].
3. Main results

3.1. Stability of system with the linear perturbations

In this section we consider systems of two linear differential equations in $R^2$:

\[ \dot{x} = A(t)x \quad (9) \]
\[ \dot{x} = A(t)x + B(t)x. \quad (10) \]

We denote by $\mu_1; \mu_2$ and $\lambda_1; \lambda_2$ ($\mu_1 \leq \mu_2; \lambda_1 \leq \lambda_2$) the exponents of systems (9) and (10) respectively. Let:

\[ A(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}; \quad B(t) = \begin{pmatrix} b_{11}(t) & b_{12}(t) \\ b_{21}(t) & b_{22}(t) \end{pmatrix} \]

We suppose that $A(t), B(t)$ are real matrix functions, continuous on $[t_0; +\infty)$ and $\sup_{t \geq t_0} \|A(t)\| = M < +\infty.$

**Theorem 1.** Let system (9) be regular and there exists a constant $C > 0$ such that

\[ \int_{t_0}^{\infty} \sqrt{|a_{22}(t) - a_{11}(t)|^2 + |a_{21}(t) + a_{12}(t)|^2} \, dt \leq C < +\infty, \]

then the maximal exponent $\lambda_2$ of system (9) is upper - stable.

**Proof.** Let

\[ W(t) = \sqrt{|a_{22}(t) - a_{11}(t)|^2 + |a_{21}(t) + a_{12}(t)|^2}. \]

According to the Perron theorem [1,4] there exists an orthogonal matrix function $U(t)$ (i.e. $U^*(t) = U^{-1}(t), \forall t \geq t_0$) such that by the following transformation

\[ x = U(t)y \quad (11) \]

the system $\dot{x} = A(t)x$ is reduced to

\[ \dot{y} = P(t)y \quad (12) \]

where $P(t)$ is a matrix of the triangle form:

\[ P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ 0 & p_{22}(t) \end{pmatrix} \]

The matrix $P(t)$ is defined as $P(t) = U^{-1}(t)A(t)U(t) - U^{-1}(t)\dot{U}(t)$.

Now we show that if matrix $A(t)$ is bounded on $[t_0; +\infty)$, then matrix $P(t)$ is also bounded on this interval, i.e. exists a constant $M_1 > 0$ such that $\|P(t)\| \leq M_1, \forall t \geq t_0$. Indeed, let:

\[ \tilde{A}(t) = (\tilde{a}_{ij}(t)) = U^{-1}(t)A(t)U(t); \quad V(t) = (v_{ij}(t)) = U^{-1}(t)\dot{U}(t). \]

It is easy to show that $V^*(t) = -V(t)$. This implies $v_{ii}(t) = 0, \forall i = 1, 2$. Thus, we get

\[ v_{ij}(t) = \begin{cases} -\tilde{a}_{ji}(t) & \text{if } i < j \\ 0 & \text{if } i = j \\ \tilde{a}_{ij}(t) & \text{if } i > j. \end{cases} \]

Since $A(t), U(t), U^{-1}(t)$ are bounded, matrix $P(t)$ is also bounded on $[t_0; +\infty)$. Let $\|P(t)\| \leq M_1, \forall t \geq t_0$. Taking the same Perron transformation to system (10), we obtain

\[ \dot{x} = \dot{U}(t)y + U(t)\dot{y} = A(t)x + B(t)x \]
\[ \begin{align*}
&\Leftrightarrow U(t)\dot{y} = A(t)x + B(t)x - \dot{U}(t)y \\
&\Leftrightarrow U(t)\dot{y} = A(t)U(t)y + B(t)U(t)y - \dot{U}(t)y \\
&\Leftrightarrow \dot{y} = [U^{-1}(t)A(t)U(t) - U^{-1}(t)\dot{U}(t)]y + U^{-1}(t)B(t)U(t)y.
\end{align*} \]

Denoting \( Q(t) = U^{-1}(t)B(t)U(t) \), the last equation is in the form

\[ \dot{y} = P(t)y + Q(t)y. \] (13)

Writing triangle matrix \( P(t) \) as follows:

\[ P(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ 0 & p_{22}(t) \end{pmatrix} = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix} + \begin{pmatrix} 0 & p_{12}(t) \\ 0 & 0 \end{pmatrix} \]

and putting \( \tilde{P}(t) = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix} \); \( \tilde{Q}(t) = Q(t) + \begin{pmatrix} 0 & p_{12}(t) \\ 0 & 0 \end{pmatrix} \),

we have

\[ \dot{y} = \tilde{P}(t)y + \tilde{Q}(t)y. \] (14)

Taking the linear transformation \( y = Sz \) with

\[ S = \begin{pmatrix} \frac{M_1}{\delta} & 0 \\ 0 & \sqrt{\frac{M_1}{\delta}} \end{pmatrix}, \]

from (14) we get the following equivalent equation

\[ \dot{z} = S^{-1}\tilde{P}(t)Sz + S^{-1}\tilde{Q}(t)Sz = \tilde{P}(t)z + S^{-1}\tilde{Q}(t)Sz. \] (15)

Denoting by \( \tilde{Q}(\tau) \) the similar matrix of matrix \( \tilde{Q}(\tau) \), we have

\[ \tilde{Q}(\tau) = S^{-1}\tilde{Q}(\tau)S = S^{-1}Q(\tau)S + S^{-1} \begin{pmatrix} 0 & p_{12}(\tau) \\ 0 & 0 \end{pmatrix} S, \]

which gives

\[ \|\tilde{Q}(\tau)\| \leq \|S^{-1}Q(\tau)S\| + \|S^{-1} \begin{pmatrix} 0 & p_{12}(\tau) \\ 0 & 0 \end{pmatrix} S\|. \] (16)

The solutions of the homogeneous system \( \dot{z} = \tilde{P}(t)z \) is defined as follows

\[ \dot{z} = \tilde{P}(t)z \Leftrightarrow \begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} p_{11}(t) & 0 \\ 0 & p_{22}(t) \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Leftrightarrow \begin{cases}
z_1(t) = C_1 e^{\int_{t_0}^{t} p_{11}(\tau) d\tau} \\
z_2(t) = C_2 e^{\int_{t_0}^{t} p_{22}(\tau) d\tau}.
\end{cases} \]

Therefore

\[ \Phi(t, \tau) = \begin{pmatrix} e^{\int_{t_0}^{t} p_{11}(s) ds - \int_{t_0}^{\tau} p_{11}(s) ds} & 0 \\ 0 & e^{\int_{t_0}^{t} p_{22}(s) ds - \int_{t_0}^{\tau} p_{22}(s) ds} \end{pmatrix} \]

is the Cauchy matrix of this system.

The solution satisfied the initial condition \( z(t_0) = z_0 \) of nonhomogeneous system (15) is given by [5]

\[ z(t) = \Phi(t, t_0)z_0 + \int_{t_0}^{t} \Phi(t, \tau)S^{-1}\tilde{Q}(\tau)Sz(\tau) d\tau, \]

which is the same as \( \Phi^{-1}(t, t_0)z(t) = z_0 + \int_{t_0}^{t} \Phi^{-1}(t, t_0)\Phi(t, \tau)S^{-1}\tilde{Q}(\tau)Sz(\tau) d\tau \)

or \( \Phi^{-1}(t, t_0)z(t) = z_0 + \int_{t_0}^{t} \Phi(t_0, \tau)S^{-1}\tilde{Q}(\tau)S\Phi(\tau, t_0)\Phi^{-1}(\tau, t_0)z(\tau) d\tau. \)
Then
\[ \| \Phi^{-1}(t, t_0) z(t) \| \leq \| z_0 \| + \int_{t_0}^t \| \Phi(t_0, \tau) S^{-1} \tilde{Q}(\tau) S \Phi(\tau, t_0) \| \| \Phi^{-1}(\tau, t_0) z(\tau) \| \, d\tau \]  
(17)

\((t \geq \tau, \tau \geq t_0)\)

Denoting by \( \tilde{q}_{ij}(t) \) the elements of matrix \( \tilde{Q}(t) \) and let
\[ D = \Phi(t_0, \tau) S^{-1} \tilde{Q}(\tau) S \Phi(\tau, t_0), \]
we have
\[
D = \begin{pmatrix}
0 & e^{-\int_{t_0}^t p_{11}(s) \, ds} \\
0 & e^{-\int_{t_0}^t p_{22}(s) \, ds}
\end{pmatrix} S^{-1} \begin{pmatrix}
\tilde{q}_{11}(\tau) & \tilde{q}_{12}(\tau) \\
\tilde{q}_{21}(\tau) & \tilde{q}_{22}(\tau)
\end{pmatrix} S \begin{pmatrix}
0 & e^{\int_{t_0}^t p_{11}(s) \, ds} \\
0 & e^{\int_{t_0}^t p_{22}(s) \, ds}
\end{pmatrix},
\]

We can verify that
\[
\left\| S^{-1} \begin{pmatrix}
0 & p_{12}(\tau) \\
0 & 0
\end{pmatrix} S \right\| = \left\| \begin{pmatrix}
0 & p_{12}(\tau) \sqrt{\frac{\delta}{M_1}} \\
0 & \sqrt{\frac{\delta}{M_1}}
\end{pmatrix} \right\| \leq \sqrt{\delta} \sqrt{M_1}.
\]

Since
\[ \| Q(\tau) \| = \| U^{-1}(\tau) B(\tau) U(\tau) \| \leq \| U^{-1}(\tau) \| \| B(\tau) \| \| U(\tau) \| \leq 1.\delta.1 = \delta, \]
denoting \( \max\{1 + \sqrt{\frac{1}{M_1}}, 1 + \sqrt{\frac{1}{M_1}}\} = M_2 \) and choosing \( \delta \) small enough such that \( 0 < \delta < 1 \), we have
\[ \left\| S^{-1} Q(\tau) S \right\| = \left\| \begin{pmatrix}
q_{11}(\tau) & q_{12}(\tau) \sqrt{\frac{\delta}{M_1}} \\
q_{21}(\tau) \sqrt{\frac{1}{M_1}} & q_{22}(\tau)
\end{pmatrix} \right\| \leq \max\{\delta(1 + \sqrt{\frac{\delta}{M_1}}); \delta(1 + \sqrt{\frac{1}{M_1}})\} \]
\[ = \max\{\sqrt{\delta}(\sqrt{\delta} + \delta \sqrt{\frac{1}{M_1}}); \sqrt{\delta}(1 + \sqrt{\frac{1}{M_1}}) \leq \sqrt{\delta} \max\{1 + \sqrt{\frac{1}{M_1}}, 1 + \sqrt{\frac{1}{M_1}}\} := \sqrt{\delta} M_2. \]

Consequently, applying the above inequalities to (16), we have \( \| \tilde{Q}(\tau) \| \leq 2M_2 \sqrt{\delta}. \)

Now, we establish the norm of matrix \( D \) as follows:

It is known that in \( R^2 \) orthogonal matrix \( U(t) \) has just one of two the following forms:

a) \( U(t) = \begin{pmatrix}
\cos \phi(t) & \sin \phi(t) \\
\sin \phi(t) & -\cos \phi(t)
\end{pmatrix} \); b) \( U(t) = \begin{pmatrix}
\cos \phi(t) & -\sin \phi(t) \\
\sin \phi(t) & \cos \phi(t)
\end{pmatrix} \).

Without loss of the generality we suppose that matrix \( U(t) \) has the form a). In this case, we have
\[ U^{-1}(t) = \begin{pmatrix}
\cos \phi(t) & \sin \phi(t) \\
\sin \phi(t) & -\cos \phi(t)
\end{pmatrix}. \]

Since in Perron transformation \( x = U(t)y \), where \( U(t) \) is a orthogonal matrix, the diagonal elements of matrix \( P(t) \) and matrix \( U^{-1}(t) A(t) U(t) \) are the same \( p_{11}(t) \) and \( p_{22}(t) \). Therefore we obtain that
\[ p_{22}(t) - p_{11}(t) = [a_{22}(t)] - [a_{11}(t)] \cos 2\phi(t) - [a_{21}(t) + a_{12}(t)] \sin 2\phi(t). \]

It is easy to see that, there is a function \( \psi(t) \) such that
\[ p_{22}(t) - p_{11}(t) = \sqrt{[a_{22}(t)] - [a_{11}(t)]^2 + [a_{21}(t) + a_{12}(t)]^2 \cos[2\phi(t) + \psi(t)]} \]
\[ = W(t) \cos[2\phi(t) + \psi(t)]. \]
Since \[|\mathbf{q}_{ij}(t)| \leq |\tilde{q}(t)| \leq 2M_2\sqrt{\delta}, \]
we have

\[
\|D\| = \left\| \begin{pmatrix}
\tilde{q}_{11}(\tau) & \tilde{q}_{12}(\tau) e^{\int_{0}^{\tau} [p_{22}(s) - p_{11}(s)] ds} \\
\tilde{q}_{21}(\tau) e^{\int_{0}^{\tau} [p_{11}(s) - p_{22}(s)] ds} & \tilde{q}_{22}(\tau)
\end{pmatrix} \right\| 
\leq 2M_2\sqrt{\delta} [2 + e^{\int_{0}^{\tau} [p_{22}(s) - p_{11}(s)] ds} + e^{\int_{0}^{\tau} [p_{11}(s) - p_{22}(s)] ds}]
\]
\[
= 2M_2\sqrt{\delta} [2 + e^{\int_{0}^{\tau} W(s) \cos[2\phi(s) + \psi(s)] ds} + e^{\int_{0}^{\tau} W(s) \cos[2\phi(s) + \psi(s) - \pi] ds}].
\]

From the assumptions \[\int_{t_0}^{+\infty} W(t) dt \leq C < +\infty, \]
we have

\[
\|D\| \leq 2M_2\sqrt{\delta} (2 + 2e^C) = M_3\sqrt{\delta} \text{ where } M_3 := 2M_2(2 + 2e^C).
\]

Applying the last inequality to (17), we get

\[
\|\Phi^{-1}(t, t_0)z(t)\| \leq \|z_0\| + \int_{t_0}^{t} M_3\sqrt{\delta} \|\Phi^{-1}(\tau, t_0)z(\tau)\| d\tau. \tag{18}
\]

\[(t \geq \tau, s \geq t_0)\]

According to the Gronwall - Belman inequality \([1, 4, 5], \)
we have

\[
\|\Phi^{-1}(t, t_0)z(t)\| \leq \|z_0\| e^{M_3\sqrt{\delta} \int_{t_0}^{t} d\tau} = \|z_0\| e^{M_3\sqrt{\delta}(t-t_0)}
\]

\[
\Rightarrow \left\{ \begin{aligned}
e^{-\int_{t_0}^{t} p_{11}(\tau) d\tau} & z_1(t) \leq \|z_0\| e^{M_3\sqrt{\delta}(t-t_0)} \\
e^{-\int_{t_0}^{t} p_{22}(\tau) d\tau} & z_2(t) \leq \|z_0\| e^{M_3\sqrt{\delta}(t-t_0)}
\end{aligned} \right. \Rightarrow \left\{ \begin{aligned}z_1(t) & \leq \|z_0\| e^{M_3\sqrt{\delta}(t-t_0)} e^{\int_{t_0}^{t} p_{11}(\tau) d\tau} \\
 & = \|z_0\| e^{M_3\sqrt{\delta}(t-t_0)} e^{\frac{1}{M_2} \int_{t_0}^{t} p_{11}(\tau) d\tau} \\
z_2(t) & \leq \|z_0\| e^{M_3\sqrt{\delta}(t-t_0)} e^{\int_{t_0}^{t} p_{22}(\tau) d\tau}.
\end{aligned} \right.
\]

Using properties of Lyapunov exponents, we get

\[
\begin{cases} \chi[z_1] = \chi[\|z_0\| e^{M_3\sqrt{\delta}(t-t_0)}] + \chi[\int_{t_0}^{t} p_{11}(\tau) d\tau] = M_3\sqrt{\delta} + \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} p_{11}(\tau) d\tau \\
\chi[z_2] = \chi[\|z_0\| e^{M_3\sqrt{\delta}(t-t_0)}] + \chi[\int_{t_0}^{t} p_{22}(\tau) d\tau] = M_3\sqrt{\delta} + \lim_{t \to +\infty} \frac{1}{t} \int_{t_0}^{t} p_{22}(\tau) d\tau.
\end{cases}
\]

It is clear that in Perron transformations the Lyapunov exponents are unchanged \([1,4]. \)
Thus, for any small enough given \(\epsilon > 0, \)
choosing \(0 < \delta < (\frac{\epsilon}{M_3})^2, \)
we obtain that

\[
\begin{cases} \chi[x_1] = \chi[z_1] \leq \lambda_1 + \epsilon & \text{or} & \mu_1 \leq \lambda_1 + \epsilon \\
\chi[x_2] = \chi[z_2] \leq \lambda_2 + \epsilon & \mu_2 \leq \lambda_2 + \epsilon.
\end{cases}
\]

The same result is proved for the case, when matrix \(U(t)\) has form b.

The proof of theorem is completed.

**Corollary 1.** Suppose that all assumptions of Theorem 1 hold. Then the minimal exponent of system (9) is lower - stable.

**Proof.** From Lemma 1 it follows that minimal exponent of system (9) is lower - stable if the maximal exponent of adjoint system \(\dot{x} = -A^*(t)x\) to this system is upper - stable. According to Theorem 1, the last requirement will be satisfied if the following inequality holds

\[
\int_{t_0}^{\infty} \sqrt{[-a_{22}(t) + a_{11}(t)]^2 + [-a_{21}(t) - a_{12}(t)]^2} \, dt \leq C < +\infty
\]

\[
\Leftrightarrow \int_{t_0}^{\infty} \sqrt{[a_{22}(t) - a_{11}(t)]^2 + [a_{21}(t) + a_{12}(t)]^2} \, dt \leq C < +\infty.
\]
This proves the corollary.

3.2. Stability of systems with nonlinear perturbations

We consider the following linear system with nonlinear perturbation in $\mathbb{R}^n$:

$$\dot{x} = A(t)x + f(t, x).$$

(19)

Since the system (19) is nonlinear, it is difficult to study its spectrum [5]. However under the suitable conditions we can obtain some results on it, for example, to study supremum of its all exponents. Let us denote this supremum by $\mu_{\text{sup}}$.

**Definition 2.** The maximal exponent $\lambda_n$ of homogeneous system $\dot{x} = A(t)x$ is said to be upper-stable under the nonlinear perturbation $f(t, x)$ if for any given $\epsilon > 0$ there exists $\delta = \delta(\epsilon) > 0$ such that if following inequality holds $\|f(t, x)\| \leq \delta \|x\|$, then

$$\mu_{\text{sup}} < \lambda_n + \epsilon.$$ 

(20)

We consider now the system (9) and (19) in $\mathbb{R}^2$. For this space the following result is obtained:

**Theorem 2.** Suppose that:

i) System (9) is regular and there exists a constant $C > 0$ such that

$$\int_{t_0}^{\infty} \sqrt{|a_{22}(t) - a_{11}(t)|^2 + |a_{21}(t) + a_{12}(t)|^2} \, dt \leq C < +\infty.$$

ii) Function $f(t, x)$ is continuous on $[t_0; +\infty)$ and there exists a function $g(t) > 0$, $\forall t > t_0$, satisfying the condition:

$$\|f(t, x)\| \leq g(t)\|x\|, \quad \forall t \geq t_0$$

Then maximal exponent $\lambda_2$ of system (9) under perturbation $f(t, x)$ is upper-stable.

**Proof.** We denote by $x_0(t) = x(t_0, x_0, t)$ the solution of system (19), which satisfies initial condition $x_0(t_0) = x_0$. Denote by $F_{x_0}(t)$ the function matrix corresponding to this solution in the sense of Lemma 2, i.e. for this solution there exists a function matrix $F_{x_0}(t)$ such that $x_0(t)$ is a solution of the following linear system

$$\dot{x} = A(t)x + F_{x_0}(t)x, \quad (x_0 \in \mathbb{R}^2),$$

(21)

where $\|F_{x_0}(t)\| \leq g(t), \forall t \geq t_0$. We denote by $\mu_{x_0}^2 \leq \mu_{x_0}^2$ the elements of spectrum of nonlinear system (19). According to Theorem 1, for every given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\|F_{x_0}(t)\| \leq \delta \quad \text{implies} \quad \mu_{x_0}^2 < \lambda_2 + \frac{\epsilon}{2}, \forall x_0 \in \mathbb{R}^2.$$

From $\|F_{x_0}(t)\| \leq g(t) \leq \delta$, we have

$$\mu_{x_0}^2 \leq \lambda_2 + \frac{\epsilon}{2}, \forall x_0 \in \mathbb{R}^2.$$ 

Therefore, we obtain that

$$\mu_{\text{sup}} = \sup_{x_0 \in \mathbb{R}^2} \mu_{x_0}^2 \leq \lambda_2 + \frac{\epsilon}{2} < \lambda_2 + \epsilon.$$ 

The proof is therefore completed.
Corollary 2. Suppose that conditions i) and ii) of Theorem 2 hold and the function $g(t)$ in condition ii) satisfies the condition
\[ \lim_{t \to +\infty} g(t) = 0. \]
Then maximal exponent $\lambda_2$ of system (9) under perturbation $f(t, x)$ is upper-stable.

Proof. For every given $\varepsilon > 0$ there exists $\delta > 0$ such that
\[ \|F_{x_0}(t)\| \leq \delta \quad \text{implies} \quad \mu_2^{x_0} < \lambda_2 + \frac{\varepsilon}{2}, \quad \forall x_0 \in R^2. \]
Since $\lim_{t \to +\infty} g(t) = 0$, for $\delta > 0$ there exists $T = T(\delta) \geq t_0$ such that $0 < g(t) < \delta$, $\forall t \geq T$. Thus, if $t \geq T$ then $\|F_{x_0}(t)\| \leq g(t) \leq \delta$. Taking to limit as $t \to +\infty$, we have
\[ \mu_2^{x_0} \leq \lambda_2 + \frac{\varepsilon}{2}, \quad \forall x_0 \in R^2. \]
Taking to supremum over all $x_0 \in R^2$, we have
\[ \mu_{\text{sup}} = \sup_{x_0 \in R^2} \mu_2^{x_0} \leq \lambda_2 + \frac{\varepsilon}{2} < \lambda_2 + \varepsilon. \]

The proof is therefore completed.

Example. Consider the system
\[
\begin{align*}
\dot{x}_1 &= (1 + \frac{1}{t^2})x_1 \\
\dot{x}_2 &= \sqrt{3} \cdot \frac{t^2}{t^2} x_1 + (1 + \frac{2}{t^2})x_2
\end{align*}
\]
(22)

It is easy to see that this system is nonreducible and nonperiodic. We can show that for this system:
\[ \lambda_1 = \lambda_2 = 1 \quad \text{and} \quad \lim_{t \to +\infty} \frac{1}{t} \int_1^t \text{Sp}A(s)ds = 2. \]
Therefore, system (22) is regular. We can see also for this system:
\[ W(t) = \sqrt{[(1 + \frac{2}{t^2}) - (1 + \frac{1}{t^2})]^2 + (\sqrt{3} \cdot \frac{t^2}{t^2})^2} = \frac{2}{t^2}. \]
Therefore, we get
\[ \int_1^t W(s)ds = 2 - \frac{2}{t} \leq 2, \quad \forall t \geq 1. \]
Thus, system (22) satisfies all conditions of Theorem 1. Its maximal exponent is upper-stable.

References