SOME KINDS OF NETWORK AND WEAK BASE

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Abstract In this paper, we study some kinds of network, and investigate relations between the kinds of network and the point-countable weak base. It is showed that, if a space has a point-countable kn-network (strong-k-network), then so is its closed compact-covering image.

1. Introduction

Since D. Burke , E. Michael, G. Gruenhage and Y. Tanaka established the fundamental theory on point-countable covers in generalized metric spaces, many topologists have discussed the point-countable covers with various characters. Then, the conceptions of k-network, weak base, cs-network, cs^*-network, wcs^*-network ... were introduced. The study on relations among certain point-countable covers has become one of the most important subjects in general topology. In this paper we shall study some kinds of network, consider relations among certain networks and prove a closed compact-covering mapping theorem on spaces with a point-countable kn-network or strong-k-network.

We adopt the convention that all spaces are T_1, and all mappings are continuous and surjective. We begin with some basic definitions.

1.1. Definition. Let X be a space, $A \subset X$. A collection $\mathcal{F}$ of X is called a *full cover* of A if $\mathcal{F}$ is a finite and each $F \in \mathcal{F}$, there is a closed set $C(F)$ in X with $C(F) \subset F$ such that $A \subset \bigcup\{C(F) : F \in \mathcal{F}\}$.

1.2. Definition. Let X be a space, and $\mathcal{P}$ be a cover of X.

(1) $\mathcal{P}$ is a *k-network* if, whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

(2) $\mathcal{P}$ is a *network* if for every $x \in X$ and U open in X such that $x \in U$, then $x \in \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

(3) $\mathcal{P}$ is a *strong-k-network* if, whenever $K \subset U$ with K compact and U open in X, then there is a full cover $\mathcal{F} \subset \mathcal{P}$ of K such that $\cup\mathcal{F} \subset U$.

(4) $\mathcal{P}$ is a *kn-network* if, whenever $K \subset U$ with K compact and U open in X, then $K \subset (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

(5) $\mathcal{P}$ is a *cs-network* if, whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is an open neighborhood of x, then $\{x\} \cup \{x_m : m \geq k\} \subset P \subset U$ for some $k \in \mathbb{N}$ and some $P \in \mathcal{P}$.

(6) $\mathcal{P}$ is a *cs*-network* if, whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is an open neighborhood of x, then $\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

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(7) $\mathcal{P}$ is a *wcs*-network* if, whenever $\{x_n\}$ is a sequence converging to a point $x \in X$ and U is an open neighborhood of x, then $\{x_{n_i} : i \in I\!N\} \subset P \subset U$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$ and some $P \in \mathcal{P}$.

The following character of kn-network will be used in some next proofs.

1.3. Proposition. *For any space, the following statements are equivalent*

(a) $\mathcal{P}$ is kn-network;

(b) For every $x \in X$ and any open neighborhood U of x, there is a finite subcollection $\mathcal{F}$ of $\mathcal{P}$ such that $x \in (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U$.

Proof. The necessity is trivial.

We only need to prove the sufficiency. Let K be a compact subset of X and U an open set in X such that $K \subset U$. For every $x \in K$, there exists a finite subcollection $\mathcal{F}_x \subset \mathcal{P}$ such that $x \in (\cup\mathcal{F}_x)^o \subset \cup\mathcal{F}_x \subset U$. Then the collection $\{(\cup\mathcal{F}_x)^o : x \in K\}$ covers K. Because K is compact, there are the points $x_1, \dots, x_k$ in K such that the finite subcolection $\{(\cup\mathcal{F}_{x_i})^o : i = 1, \dots, k\}$ covers K. Denote

$$\mathcal{F} = \{F : F \in \mathcal{F}_{x_i},\ i = 1, \dots, k\}.$$

Then, the finite subcollection $\mathcal{F}$ satisfies

$$K \subset \bigcup_{i=1}^{n} (\cup\mathcal{F}_{x_i})^o \subset (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U.$$

1.4. Definition. For a space X and $x \in P \subset X$, P is a *sequential neighborhood* at x in X if, whenever $\{x_n\}$ is sequence converging to x in X, then there is an $m \in I\!N$ such that $\{x_n : n \geq m\} \subset P$.

For a collection of subsets $\mathcal{F}$ of a space X, we write

$$\mathrm{Int}_s(\mathcal{F}) = \{x \in X : \cup\mathcal{F} \text{ is a sequential neighborhood at } x\}.$$

A cover $\mathcal{P}$ of X is called is a *ksn-network* if, whenever $x \in U$ with $x \in X$ and U open in X, then $x \in \mathrm{Int}_s(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$ for some finite $\mathcal{F} \subset \mathcal{P}$.

1.5. Definition. Let X be a space, and $\mathcal{P} = \cup\{\mathcal{P}_x : x \in X\}$ be a family of subsets of X which satisfies that for each $x \in X$,

(1) $x \in P$ for all $P \in \mathcal{P}_x$;

(2) If $U,\ V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is called a *weak base* for X iff a subset G of X is open in X if and only if for each $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

1.6. Definition. Let X be a space, a cover $\mathcal{P}$ of X is called *point-countable* if for every $x \in X$, the set $\{P \in \mathcal{P} : x \in P\}$ is at most countable.

We have the following diagram

$$\begin{array}{ccccc} cs\text{-network} & \Rightarrow & cs^*\text{-network} & \Leftarrow & \text{strong-}k\text{-network} \\ \Uparrow & & \Downarrow & & \Downarrow \\ \text{weak base} & \Rightarrow & wcs^*\text{-network} & \Leftarrow & k\text{-network} \\ & & \Uparrow & & \Uparrow \\ & & ksn\text{-network} & \Leftarrow & kn\text{-network.} \end{array}$$

It is well known from [10] that weak base $\Rightarrow$ cs-network $\Rightarrow$ cs^*-network $\Rightarrow$ wcs^*-network, k-network $\Rightarrow$ wcs^*-network. From the above definitions, it is easily to prove that strong-k-network $\Rightarrow$ k-network, kn-network $\Rightarrow$ k-network, kn-network $\Rightarrow$ ksn-network, and ksn-network $\Rightarrow$ wcs^*-network.

In this paper we shall provide some partial answers to connections between kinds of network and weak base.

2. Main results

The following lemma is due to [5].

2.1. Lemma. *Let $\mathcal{P}$ be a point-countable cs-network for a space X. If $x \in K \cap U$ with U open and K compact, first countable in X, then $x \in Int_K(P \cap K) \subset P \subset U$ for some $P \in \mathcal{P}$.*

First we present some connections between kinds of network

2.2. Proposition. *For any space, if $\mathcal{P}$ is a strong-k-network, then $\mathcal{P}$ is a cs^*-network.*

Proof. Let $\mathcal{P}$ be a strong k-network, a sequence converging $\{x_n\}$ to a point $x \in X$ and an open neighborhood U of x, then there is a full cover $\mathcal{F} \subset \mathcal{P}$ of compact sets $\{x\} \cup \{x_n : n \geq 1\}$ such that $\cup\mathcal{F} \subset U$. From the definition of a full cover, it follows that there exist a $P \in \mathcal{F}$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x\} \cup \{x_{n_i}\} \subset P$ so this shows that $\mathcal{P}$ is a cs^*-network.

2.3. Proposition. *Let X be a locally compact, first countable space. If $\mathcal{P}$ is a point-countable cs-network for X, then $\mathcal{P}$ is a point-countable ksn-network.*

Proof. Let $\mathcal{P}$ be a point-countable cs-network. For every $x \in X$ and any open neighborhood U of x, since X is locally compact, there is a compact neighborhood K of x. By the first countability of X it follows from Lemma 2.1 that there exists $P \in \mathcal{P}$ such that $x \in \mathrm{Int}_K(P \cap K) \subset P \subset U$. Now, let $\{x_n\}$ be an any sequence coverging to x. Because K is a neighborhood of x and $\mathrm{Int}_K(K \cap P)$ is neighborhood of x in K, there is an $m \in I\!N$ such that $\{x\} \cup \{x_n : n \geq m\} \subset \mathrm{Int}_K(K \cap P) \subset P \subset U$. This implies that $x \in \mathrm{Int}(P) \subset P$. Thus, $\mathcal{P}$ is a ksn-network.

2.4. Proposition. *Let X be first countable. If $\mathcal{P}$ is a point-countable cs-network for X, then $\mathcal{P}$ is a k-network.*

Proof. Let $\mathcal{P}$ be a point-countable cs-network. Let K be a compact subset and U an open subset of X such that $K \subset U$. For every $x \in K$, it follows from Lemma 2.1 that

$x \in \mathrm{Int}_K(K \cap P_x) \subset P_x \subset U$ for some $P_x \in \mathcal{P}$. By compactness of K there exist $x_1, \dots, x_m$ in K so that $K \subset \bigcup_{i=1}^{m} \mathrm{Int}_K(K \cap P_{x_i}) \subset \bigcup_{i=1}^{m} P_{x_i} \subset U$. Thus, $\mathcal{P}$ is a point-countable k-network.

Now we shall give some partial answers to the inversion of above implications

2.5. Theorem. *Let X be first countable. Then, $\mathcal{P}$ is a point-countable ksn-network for X if and only if $\mathcal{P}$ is a kn-network.*

Proof. The sufficiency is obvious.

We only need to prove the necessity. Let $\mathcal{P}$ be a ksn-network for X. For every $x \in X$ and any open set U in X such that $x \in U$, there exists a finite subcollection $\mathcal{F} \subset \mathcal{P}$ satisfying $x \in \mathrm{Int}_s(\cup\mathcal{F}) \subset \cup\mathcal{F} \subset U$. By $\{G_n\}$ we denote the countable base of neighborhoods of x such that $G_{n+1} \subset G_n$ for all $n \in \mathbb{N}$. Then there is an $m \in \mathbb{N}$ so that $G_m \subset \cup\mathcal{F}$. Otherwise, for every $n \in \mathbb{N}$ there exists an $x_n \in G_n \setminus (\cup\mathcal{F})$. It is easily seen that the obtained sequence $\{x_n\}$ converges to x but $x_n \notin \cup\mathcal{F}$ for all $n \in \mathbb{N}$. This is contrary to $x \in \mathrm{Int}_s(\cup\mathcal{F})$. Hence, $x \in (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U$. It follows from Proposition 1.3 that $\mathcal{P}$ is a kn-network.

It follows immediately from the proof of Theorem 2.5 that

2.6. Corollary. *Let X be first countable. If $\mathcal{P}$ is a point-countable ksn-network for X, then $\mathcal{P}$ is a k-network.*

2.7. Theorem. *A space X is the first countable if only if X has a point-countable kn-network.*

Proof. Let X be first countable. For every $x \in X$ by $\mathcal{P}_x$ the base of open neighborhoods of x. Let $\mathcal{P} = \cup\mathcal{P}_x$. Then $\mathcal{P}$ is a point-countable weak base.

Conversely, let $\mathcal{P} = \cup\mathcal{P}_x$ be a point-countable kn-network. For every $x \in X$, let $\mathcal{P}_x = \{P \in \mathcal{P} : x \in P\}$ and

$$\mathcal{B}_x = \{(\cup\mathcal{F})^0 : \mathcal{F} \text{ is finite, } \mathcal{F} \supset \mathcal{P}_x$$

2.8. Theorem. *Let X be first countable. Then X has a point-countable wcs^*-network for X if and only if it has a point-countable weak base.*

Proof. The "if" part holds by the above diagram, so we prove the "only if" part. Without loss of generality we may assume that $\mathcal{P}$ is a point-countable wcs^*-network for X which is closed under finite intersections. For every $x \in X$ by $\mathcal{Q}_x = \{Q_n(x) : n \in \mathbb{N}\}$ we denote the countable base of neighborhoods of x such that $Q_{n+1}(x) \subset Q_n(x)$ for all $n \in \mathbb{N}$, and put $\mathcal{P}_x = \{P \in \mathcal{P} : Q_n(x) \subset P$ for some $n \in \mathbb{N}\}$. Then, P is a neighborhood of x for each $P \in \mathcal{P}_x$. Now we show that $\mathcal{B} = \cup\mathcal{P}_x$ is a point-countable weak base.

It is easily seen that for each $x \in X$, $\mathcal{P}_x$ is point-countable, and if $P_1 \in \mathcal{P}_x, P_2 \in \mathcal{P}_x$, then we have $P_1 \cap P_2 \in \mathcal{P}_x$. Now we prove that a subset G of X is open in X if and only if for each $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

In fact, let G be an open subset of X, x any element of G, and $\{P \in \mathcal{P} : x \in P \subset G\} = \{P_m(x) : m \in \mathbb{N}\}$. Assume the contrary that $Q_n(x) \not\subset P_m(x)$ for each $n, m \in \mathbb{N}$.

Then, take $x_{n,m} \in Q_n(x)\backslash P_m(x)$ for every $n, m \in \mathbb{N}$. Now for $n \geq m$ we choose $y_k = x_{n,m}$, where $k = m + \frac{n(n-1)}{2}$. Then the sequence $\{y_k\}$ converges to the point x. Thus, there exist a subsequence $\{y_{k_s}\}$ of $\{y_k\}$, and m, $i \in \mathbb{N}$ such that $\{y_{k_s} : k_s \geq i\} \subset P_m(x) \subset G$. Take $k_s \geq i$ with $y_{k_s} = x_{n,m}$ for some $n \geq m$. Then $x_{n,m} \in P_m(x)$. This is a contradiction.

Conversely, if $G \subset X$ satisfies the following condition: for each $x \in G$ there exists $P \in \mathcal{P}_x$ with $P \subset G$. Then, since P is a neighborhood of x for each $P \in \mathcal{P}_x$, G is a neighborhood of x. Thus, G is open in X. Hence $\mathcal{B} = \cup \mathcal{P}_x$ is a point-countable weak base for X.

Finally, it is well known that spaces with a point-countable cs-network, cs^*-network, or closed k-network are not necessarily preserved by closed maps (even if the domains are locally compact metric). But, spaces with a point-countable k-network are preserved by perfect maps [4]. In the remain part we give some properties of closed compact-covering maps.

The following lemma in [1] shall be used in the proof of Thoerem 2.12

2.9. Lemma. *If $\mathcal{P}$ is a point-countable cover of a set X, then every $A \subset X$ has only countably many minimal finite covers by elements of $\mathcal{P}$.*

2.10. Definition. A mapping $f : X \to Y$ is *compact-covering* if every compact $K \subset Y$ is the image of some compact $C \subset X$.

A mapping $f : X \to Y$ is *perfect* if X is a Hausdorff space, f is a closed mapping and all fibers $f^{-1}(y)$ are compact subsets of X.

2.11. Proposition.([3]) *If $f : X \to Y$ is a perfect mapping, then for every compact subset $Z \subset Y$ the inverse image $f^{-1}(Z)$ is compact.*

2.12. Proposition. *Every a perfect map is compact-covering.*

Proof. It follows directly from their definitions and Proposition 2.10.

2.13. Theorem. *Let $f : X \to Y$ be closed, compact-covering. If X has a point-countable kn-network (strong-k-network), then so does Y respectively.*

Proof. Assume $\mathcal{P}$ is a point-countable kn-network for X. Let Φ be the family of all finite subcollections of $\mathcal{P}$. For $\mathcal{F} \in \Phi$, let

$$M(\mathcal{F}) = \{y \in Y : \mathcal{F} \text{ is a minimal cover of } f^{-1}(y)\}$$

and let $\mathcal{P}' = \{M(\mathcal{F}) : \mathcal{F} \in \Phi\}$. It follows from Lemma 2.8 that $\mathcal{P}'$ is a point-countable collection of subsets of Y. Let us now show that $\mathcal{P}'$ is a kn-network. Let K be compact in Y and U an open subset of Y such that $K \subset U$. As f is compact-covering, there exists a compact set $C \subset X$ such that $f(C) = K$. By continuity of f we obtain an open set $f^{-1}(U)$ in X and $C \subset f^{-1}(U)$. Then, there exists a finite subcollection $\mathcal{F} \subset \mathcal{P}$ such that $C \subset (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset f^{-1}(U)$. Let $\mathcal{F}' = \{M(\mathcal{E}) : \mathcal{E} \subset \mathcal{F}\}$, then $\mathcal{F}'$ is a finite subcollection of $\mathcal{P}'$ and $\cup\mathcal{F}' = \cup\{u \in Y : f^{-1}(u) \subset \cup\mathcal{F}\} \subset U$. If $W = Y \setminus f[X \setminus (\cup\mathcal{F})^o]$, then, because f is closed, it follows that W is open in Y, and $K \subset (\cup\mathcal{F}')^o \subset \cup\mathcal{F}' \subset U$ and therefore the theorem is proved.

The proof of the Theorem in the case X having a strong-k-network is similar.

From Theorem 2.12, Proposition 1.3 and Proposition 2.11, it follows that

2.14. Corollary. *Let $f : X \to Y$ be a perfect map. If X has a point-countable kn-network (strong-k-network), then so does Y respectively.*

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