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Review Article

Maximal Inequalities for Fractional Brownian Motion with Variable Drift

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Abstract: Let BH be a fractional Brownian motion with H∈ (0, 1) and g be a deterministic function. We study the asymptotic behaviour of the tail probability as for fixed x and as for fixed T. Our results partially generalise those obtained by Prakasa Rao in [1].

Keywords: Fractional Brownian motion, Maximal inequalities, Variable drift.

1. Introduction

Let $B^H = (B_t^H)_{t\geq 0}$ be a standard fractional Brownian motion (fBm) with Hurst index , i.e. BH is a centered Gaussian process with covariance function given by

$$R_H(t,\ s) := E[B_t^H B_s^H] = \frac{1}{2}(t^{2\mathbf{H}} + s^{2H} - |t-s|^{2H}), t, s \geq 0.$$

We refer the readers to the monograph [2] for a short survey of properties of fBm. When $H > \frac{1}{2}$, the following limit theorems were proved by Prakasa Rao in [1].

Theorem 1.1. Let $g(t) = a_k t^k + a_{k-1} t^{k-1} + \ ...\ + a_1 t$ be a polynomial of degree k with $a_k > 0$. Then, for any T > 0 and k ≥ 2 we have

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$$\lim_{x\to\infty}\frac{\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x)}{x^2}=-\frac{1}{2T^{2H}}.$$

Theorem 1.2. Let $g(t)=a_k t^k+a_{k-1}t^{k-1}+\ ...\ +a_1 t$ be a polynomial of degree k with $a_k>0$. Then, for any x > 0 and k ≥ 1 we have

$$\lim_{T\to\infty}\sup\frac{\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\leq x)}{T^{2k-2H}}\leq -\frac{a_k^2}{2}.$$

It is known that when $H=\frac{1}{2}$, B_t^H reduces to a standard Brownian motion. In this case, Prakasa Rao's results reduce to those established previously by Jiao [3]. Naturally, one would like to ask the following questions:

Q_1: Are Theorems 1.1 and 1.2 still true when $H<\frac{1}{2}$?

Q_2: Can we remove the polynomial structure of the drift g(t)?

The aim of this paper is to provide an affirmative answer to Q_1 and Q_2. Our method is different from Prakasa Rao's where he mainly uses the classical Slepian's lemma. In the present paper, we employ the techniques of Malliavin calculus which lead us to a shorter proof for more general results.

The rest of the paper is organized as follows. In Section 2, we recall some fundamental concepts of Malliavin calculus. The main results of the paper are stated and proved in Section 3.

2. Preliminaries

It is well known that B_t^H admits the so-called Volterra representation (see, e.g. [4])

$$B_t^H=\int_0^t K(t,s)dB_s\ ,\ t\in[0;T], \tag{2.1}$$

where $(B_t)_{t\geq 0}$ is a standard Brownian motion, K(t, s) = 0 for s ≥ t and

$$K(t,s)=C_H\left[\frac{t^{H-\frac{1}{2}}}{s^{H-\frac{1}{2}}}(t-s)^{H-\frac{1}{2}}-(H-\frac{1}{2})\int_s^t\frac{u^{H-\frac{3}{2}}}{s^{H-\frac{1}{2}}}(u-s)^{H-\frac{1}{2}}du\right],s<t$$

where $C_H=\sqrt{\frac{\pi H(2H-1)}{\Gamma(2-2H)\Gamma(H+\frac{1}{2})^2\sin(\pi(H-\frac{1}{2}))}}$.

Our proofs will be strongly based on techniques of Malliavin calculus. For the reader's convenience, let us recall the definition of Malliavin derivative with respect to Brownian motion *B*, where *B* is used to present B_t^H as in (2.1).

We suppose that $(B_t^H)_{t\in[0,T]}$ is defined on the complete probability space $(\Omega,\mathcal{F},P)$, where

$\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filter generated by the Brownian motion *B*. For $h \in L^2[0,T]$ we denote by *B(h)* the Wiener integral

$$B(h) = \int_0^T h(t)\,dB_t\ .$$

Let S denote the dense subset of $L^2(\Omega, \mathcal{F}, P)$ consisting of smooth random variables of the form

$$F = f(B(h_1),...,B(h_n)), \tag{2.2}$$

where $n \in \mathbb{N}$, $f \in C_b^\infty(\mathbb{R}^n)$, $h_1, ..., h_n \in L^2\,[0, T]$. If F has the form (2.2), we define its Malliavin derivative as the process $DF := \{D_t F; t \in [0;T]\}$ given by

$$D_t F = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}(B(h_1),...,B(h_n))h_k(t).$$

We will denote by $\mathbb{D}^{1,2}$ the space of Malliavin differentiable random variables, it is the closure of S with respect to the norm

$$\|F\|_{1,2}^2 := E|F|^2 + E\left[\int_0^T |D_t F|^2\,dt\right].$$

The next Proposition is a concrete version of Corollary 4.7.4 in [5].

Proposition 2.1. *Let F be in* $\mathbb{D}^{1,2}$. *Assume that*

$$\int_0^T (D_\theta F)^2\,d\theta \le \beta \qquad a.s. \tag{2.3}$$

for some $\beta > 0$. *Then, for all x* > 0, we *have*

$$P(F - E(F) \ge x) \le exp\left(-\frac{x^2}{2\beta}\right). \tag{2.4}$$

Remark 2.1. The random variable -F also satisfies the conditions of Proposition 2.1. We therefore obtain the same bound for the left tail

$$P(F - E(F) \le -x) = P(-F - E(F) \ge x) \le exp\left(-\frac{x^2}{2\beta}\right), x > 0. \tag{2.5}$$

3. The main results

We firstly establish the following technical result which plays a key role in this paper.

Proposition 3.1. Suppose that f is a continuously differentiable function on $\mathbb{R}$ with bounded derivative and g is a continuous function on [0, T]. Let B_t^H be a fBm with $H \in (0,\ 1)$, it holds that

$$P\left(\sup_{t\in[0,\mathrm{T}]}(B_t^H+g(t))\geq x\right)\leq \exp\left(-\frac{(x-c_T)^2}{2\sup_{x\in\mathbb{R}}|f'(x)|^2T^{2H}}\right), x>c_T, \tag{3.1}$$

and

$$P\left(\sup_{t\in[0,\mathrm{T}]}(B_t^H+g(t))\leq x\right)\leq \exp\left(-\frac{(x-c_T)^2}{2\sup_{x\in\mathbb{R}}|f'(x)|^2T^{2H}}\right), 0<x<c_T, \tag{3.2}$$

where $c_T = E\left[\sup_{t\in[0,T]}\left(f(B_t^H)+g(t)\right)\right].$

Proof. If $\sup_{x\in\mathbb{R}}|f'(x)|=0$, then the estimates (3.1) and (3.2) are trivial. Hence, we can and will assume that $\sup_{x\in\mathbb{R}}|f'(x)|>0.$

Consider a countable and dense subset $S_0=\{t_n,\ n\ \geq 1\}$ of [0, T]. Define

$$M_n = sup\{X_{t_1}, X_{t_2}, ..., X_{t_n}\},$$

where $X_t := f(B_t^H)+g(t).$ Because f is continuous differential with bounded derivative, we know from Proposition 1.2.3 in [4] that $X_t\in D^{1,2}$ and

$$D_\theta X_t = f'\left(B_t^H\right)D_\theta B_t^H = f'\left(B_t^H\right)K(t,\theta),\ \theta< t.$$

It is known from Proposition 2.1.10 in [4] that $M_n \in D^{1,2}$ and M_n converges in $L^2(\Omega)$ to $\sup_{t\in[0,T]} X_t$. In order to evaluate the Malliavin derivative of M_n, we introduce the following sets:

$$A_1=\{\mathrm{M}_n=\mathrm{X}_{t_1}\},$$

....

$$A_k=\{\mathrm{M}_n\neq\mathrm{X}_{t_1}, ..., \mathrm{M}_n\neq\mathrm{X}_{t_{k-1}}, \mathrm{M}_n=\mathrm{X}_{t_k}\},\ 2\leq \mathrm{k}\leq \mathrm{n}.$$

By the local property of the operator *D;* on the set A_k the derivatives of the random variables M_n and X_{t_k} coincide. Hence, we can write

$$D_\theta M_n=\sum_{k=1}^n I_{A_k} f'\left(B_{t_k}^H\right)D_\theta X_{t_k} =\sum_{k=1}^n I_{A_k} f'\left(B_{t_k}^H\right)K(t_k,\theta)I_{A_k}.$$

Consequently,

$$(D_\theta M_n)^2=\sum_{k=1}^n\left(f'\left(B_{t_k}^H\right)\right)^2K^2(t_k,\theta)I_{A_k}.$$

And hence,

$$\begin{aligned}\int_0^T(D_\theta M_n)^2d\theta&=\int_0^{t_k}(D_\theta M_n)^2d\theta\\&\leq\sup_{x\in\mathbb{R}}|f'(x)|^2\int_0^{t_n}\sum_{k=1}^n K^2(t_k,\theta)I_{A_k}d\theta \qquad a.s.\end{aligned} \tag{3.3}$$

Denote by *F(t, .)* the antiderivative of K^2 (t, .). Since K(t, θ) = 0 for θ ≥ t we can obtain

$$\begin{aligned}
&\int_0^{t_n}\sum_{k=1}^{n}K^2(t_k,\theta)I_{A_k}d\theta\\
&=\int_0^{t_1}\sum_{k=1}^{n}K^2(t_k,\theta)I_{A_k}d\theta+\int_{t_1}^{t_n}\sum_{k=1}^{n}K^2(t_k,\theta)I_{A_k}d\theta+...+\int_{t_{n-1}}^{t_n}\sum_{k=1}^{n}K^2(t_n,\theta)I_{A_n}d\theta\\
&=\left[F(t_1,t_1)-F(t_1,0)\right]I_{A_1}+...+\left[F(t_n,t_n)-F(t_n,0)\right]I_{A_n}\\
&=E\left|B_{t_1}^H\right|^2I_{A_1}+...+E\left|B_{t_n}^H\right|^2I_{A_n}\\
&=t_1^{2H}I_{A_1}+...+t_n^{2H}I_{A_n}\le T^{2H}.
\end{aligned}\tag{3.4}$$

Combining (3.3) and (3.4) yields

$$\int_0^T(D_\theta M_n)^2d\theta\le\sup_{x\in\mathbb{R}}\left|f'(x)\right|^2T^{2H},a.s.\tag{3.5}$$

The inequalitiy (3.5) shows that the random variable M_n satisfies the condition (2.3) of Proposition 2.1. Consequently, we can get

$$P\left(M_n-E[M_n]\ge x\right)\le\exp\left(-\frac{x^2}{2\sup_{x\in\mathbb{R}}\left|f'(x)\right|^2T^{2H}}\right),x>0.$$

Then, by Fatou's lemma we deduce

$$\begin{aligned}
P\left(\sup_{t\in[0,\,T]}(B_t^H+g(t))-c_T\ge x\right)&\le\liminf_{n\to\infty}P\left(M_n-E[M_n]\ge x\right)\\
&\le\exp\left(-\frac{(x-c_T)^2}{2\sup_{x\in\mathbb{R}}\left|f'(x)\right|^2T^{2H}}\right),x>0,
\end{aligned}$$

which gives us (3.1). Similarly, we can obtain (3.2) by using the estimate (2.5).

So the proof of Proposition is complete.

Remark 3.1. We state Proposition 3.1 in a general form because it can be useful for the other researches. Let us give here an example. Consider the fractional stochastic differential equation

$$x_t=x_0+\int_0^t\sigma(x_s)dB_s^H,t\in[0,T].$$

Under suitable assumptions on σ and H, the Doss-Sussmann representation of x_t is given by (see, e.g. [6, 7])

$$x_t=f\left(B_t^H\right),$$

where f (x) solves the ordinary differential equation:

$$f'(x)=\sigma\left(f(x)\right),f(0)=x_0.$$

Thus f (x) will satisfy the condition of Proposition 3.1 if σ(x) is continuous and bounded on ℝ.

We now are in a position to formulate and prove the first main result which generalises and improves Theorem 1.1.

Theorem 3.1. Let B_t^H be a fBm with H ∈ (0, 1). Fixed T > 0, let g be a continuous function on [0, T]. It holds that

$$\lim_{x\to\infty}\frac{\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x)}{x^2}=-\frac{1}{2T^{2H}}.$$

Proof. Obviously, we have

$$P\left(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x\right)\geq P\big((B_T^H+g(T))\geq x\big)\geq P\big(B_T^H\geq x-g(T)\big).$$

Since B_t^H is a normal random variable with mean zero and variance T^{2H}, we can obtain

$$P\left(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x\right)\geq P\left(Z\geq\frac{x-g(T)}{T^H}\right), \quad (3.7)$$

where Z has a standard normal distribution. Since $\frac{x-g(T)}{T^H}>1$ for sufficiently large x, we can apply Lemma 1 in [2] to get

$$P\left(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x\right)\geq\frac{e^{-\frac{(x-g(T))^2}{2T^{2H}}}}{\frac{6(x-g(T))}{T^H}} \text{ for sufficiently large } x.$$

As a consequence,

$$\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x)\geq-\frac{(x-g(T))^2}{2T^{2H}}-\log(6x-6g(T))+\log T^H,$$

and hence,

$$\liminf_{x\to\infty}\frac{\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x)}{x^2}\geq-\frac{1}{2T^{2H}}. \quad (3.8)$$

On the other hand, we obtain from Proposition 3.1 that

$$P\left(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x\right)\leq\exp\left(-\frac{(x-g(T))^2}{2T^{2H}}\right),$$

which gives us

$$\frac{\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x)}{x^2}\leq-\frac{(x-c_T)^2}{2x^2T^{2H}}.$$

Notice that $c_T=E[\sup_{t\in[0,T]}(B_t^H+g(t))]$ is finite because $c_T\leq E[\sup_{t\in[0,T]}B_t^H]+\sup_{t\in[0,T]}[g(t)]=E[\sup_{t\in[0,1]}B_t^H]T^H+\sup_{t\in[0,T]}[g(t)]$.

Taking the limit x→∞ we get

$$\limsup_{x\to\infty}\frac{\log P(\sup_{t\in[0,\,T]}(B_t^H+g(t))\geq x)}{x^2}\leq-\frac{1}{2T^{2H}}. \quad (3.9)$$

So we can finish the proof by combining (3.8) and (3.9). □

The second main result of this paper is the following theorem.

Theorem 3.2. Let B_t^H be a fBm with H ∈ (0, 1) and g be a continuous function on [0, 1). Assume that there exists a positive constant k > H such that

$$\alpha_k := \lim_{T\to\infty} \frac{g(T)}{T^k} > 0.$$

Then, for any x > 0 we have

$$\limsup_{T\to\infty} \frac{\log P(\sup_{t\in[0,\,T]} (B_t^H + g(t)) \le x)}{T^{2k-2H}} \le -\frac{\alpha_k^2}{2}. \tag{3.10}$$

and

$$\liminf_{T\to\infty} \frac{\log P(\sup_{t\in[0,\,T]} (B_t^H + g(t)) \ge x)}{T^{2k-2H}} \ge -\frac{\alpha_k^2}{2}. \tag{3.11}$$

Proof. It is clear that $c_T = E(\sup_{t\in[0,T]} B_t^H + g(t)) \ge E(B_T^H + g(T)) = g(T) \to \infty.$ as $T\to\infty$.

Hence x < c_T for sufficiently large T. Once again, we apply Proposition 3.1 to get

$$\frac{\log P(\sup_{t\in[0,\,T]} (B_t^H + g(t)) \le x)}{T^{2k-2H}} \le -\frac{(x-c_T)^2}{2T^{2k}} \text{ for sufficiently large T,}$$

which leads us to the following

$$\limsup_{T\to\infty} \frac{\log P(\sup_{t\in[0,\,T]} (B_t^H + g(t)) \le x)}{T^{2k-2H}} \le -\frac{1}{2}\left(\lim_{T\to\infty} \frac{c_T}{T^k}\right). \tag{3.12}$$

Since $\lim_{T\to\infty} \frac{c_T}{T^k} \ge \lim_{T\to\infty} \frac{g(T)}{T^k} = \alpha_k > 0.$ This, together with (3.12), yields

$$\limsup_{T\to\infty} \frac{\log P(\sup_{t\in[0,\,T]} (B_t^H + g(t)) \le x)}{T^{2k-2H}} \le -\frac{\alpha_k^2}{2}.$$

Thus the estimate (3.10) was proved.

The remaining of the proof is to show (3.11). Because $\alpha_k > 0$ and $k > H$, we have $\lim_{T\to\infty} \frac{x-g(T)}{T^H} = -\infty$ for any x>0. Hence,

$$P\left(Z \ge \frac{x-g(T)}{T^H}\right) \ge \frac{1}{2} \ge P\left(Z \ge \frac{g(T)-x}{T^H}\right)$$

for sufficiently large T. Recalling (3.7) and using Lemma 1 in [3], we have

$$P\left(\sup_{t\in[0,\,T]} (B_t^H + g(t)) \ge x\right) \ge \frac{e^{-\frac{(g(T)-x)^2}{2T^{2H}}}}{\frac{6(g(T)-x)}{T^H}} \text{ for sufficiently large T.}$$

We therefore obtain

$$\log P(\sup_{t\in[0,\,\mathrm{T}]}(B_t^H + g(t)) \ge x) \ge -\frac{(g(T)-x)^2}{2T^{2H}} - \log(6g(T)-6x) + \log T^H$$

and

$$\liminf_{T\to\infty} \frac{\log P(\sup_{t\in[0,\,\mathrm{T}]}(B_t^H + g(t)) \ge x)}{T^{2k-2H}} \ge -\frac{1}{2}\left(\lim_{T\to\infty}\frac{g(T)}{T^k}\right)^2 = -\frac{\alpha_k^2}{2}.$$

The proof of Theorem is complete.

We end up this paper with a remark.

Remark 3.2. The method used in the paper can be applied to a larger class of Gaussian processes of form

$$Y_t = \int_0^t k(t,s)dB_s, t \in [0,T],$$

where the Volterra kernel k(t, s) is continuous and satisfies the function $t \mapsto E|Y_t|^2 = \int_0^t k^2(t,\ s)ds$ is non-decreasing. Here we note that the non-decreasing property of $E|Y_t|^2$ is used to prove the inequality (3.5).

For example, when $Y_t = \int_0^T e^{-\lambda(t-s)} dB_s$ is an Ornstein-Uhlenbeck process we have

$$\limsup_{x\to\infty} \frac{\log P(\sup_{t\in[0,\,\mathrm{T}]}(Y_t + g(t)) \ge x)}{x^2} = -\frac{1}{2\int_0^T k^2(T,s)ds} = -\frac{\lambda}{1-e^{-2\lambda T}}.$$

3. Conclusion

Thus, we have generalized Rao's studies of fractional Brownian motion with continuous drift, H ∈ (0, 1). And we got the answers to question 1 one and question 2 who are the two issues raised in the introduction. In these proofs we also use images of the Malliavin’s calculus, which are quite different from Rao's.

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