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Original Article

On Stability for Hybrid System under Stochastic Perturbations

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Abstract: The aim of this paper is to find out suitable conditions for almost surely exponential stability of communication protocols, considered for nonlinear hybrid system under stochastic perturbations. By using the Lyapunov-type function, we proved that the almost surely exponential stability remain be guaranteed as long as a bound on the maximum allowable transfer interval (MATI) is satisfied.

Keywords: Networked Control System, almost surely exponential stability, maximum allowable transfer interval, Lyapunov function.

1. Introduction

In recent years, Networked Control Systems (NCS) were addressed strongly in the control community because of its extensive applications in wireless as well as wireline. The pioneering papers were proposed by Walsh, Beldiman and Bushnell [10, 11, 12]. They introduced about stability of control systems with deterministic protocol. More recently, quite many articles and literatures referred to study stability of hybrid systems by specifically showing the Lyapunov-type function and bounds on the maximum allowable transfer interval (MATI), see [1, 2, 3, 4, 8, 6, 9, 13] for more details. This paper is divided into two sections. Beside Introduction, we state Preliminary and main problem in the second section.

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In [5], the authors solved entirely for researching the stable types of solution of hybrid systems, modelled as follows:

$$\dot{x}(t) = f(x(t), e(t)), t \in (t_k, t_{k+1}), \quad (1a)$$
$$\dot{e}(t) = g(x(t), e(t)), t \in (t_k, t_{k+1}), \quad (1b)$$
$$\dot{\tau}(t) = 1, t \in (t_k, t_{k+1}), \quad (1c)$$
$$\tau(t_k^+) = 0, \quad (1d)$$
$$x(t_k^+) = x(t_k), \quad (1e)$$
$$e(t_k^+) = b_k h(k, e(t_k)) + (1 - b_k) e(t_k), k = 0,1,2,... \quad (1f)$$

Remind here that the variable b_k belongs to the set $\{0,1\}$. If $b_k = 1$ then transmission is successful, and the protocol h determines the updated error. While if $b_k = 0$ then the error remains unchanged at the t_k. We get a sequence $(b_k)_{k\in\mathbb{N}}$. Let $S := \{0,1\}$ and the probability space $(S^{\mathbb{N}}, F_b, P)$ with the sequence space

$$S^{\mathbb{N}} := \{(b_k)_{k\in\mathbb{N}} : b_k \in S, \forall k \in \mathbb{N}\}$$

where the σ-algebra $F_b := 2^S \times 2^S \times ...$ and the probability P satisfying

$$P(b \in S^{\mathbb{N}} : b_k = 1) = p, \forall k \in \mathbb{N}.$$

We also assume that the random variables b_k are independently and identically distributed.

Motivated from this paper, we concern to hybrid system in which exogenously stochastic perturbation is a Wiener process. This is, up to now, one of proposed problems remain have not been solved yet. To solve the problem, we make use of tools as introduced in [5] by defining τ_{MATI} or choosing the Lyapunov function W for protocol. We also, of course, use other tools for stochastic stability from [7] in order to support our proof.

2. Preliminary and main result

Let us now consider the perturbed hybrid system that is of form

$$dx(t) = f_1(x(t), e(t))dt + f_2(x(t), e(t))d\mathrm{w}(t), t \in (t_k, t_{k+1}), \quad (2a)$$
$$de(t) = g_1(x(t), e(t))dt + g_2(x(t), e(t))d\mathrm{w}(t), t \in (t_k, t_{k+1}), \quad (2b)$$
$$\dot{\tau}(t) = 1, t \in (t_k, t_{k+1}), \quad (2c)$$
$$\tau(t_k^+) = 0, \quad (2d)$$
$$x(t_k^+) = x(t_k), \quad (2e)$$
$$e(t_k^+) = b_k h(k, e(t_k)) + (1 - b_k) e(t_k), k = 0,1,2,... \quad (2f)$$

where $x \in \mathbb{R}^n$ is the state of the system, $e \in \mathbb{R}^n$ is the error at the controller, h is the update function that models the particular protocol, τ is a timer to constrain both the transmission interval and the transmission delay, and $w(t)$ is a Wiener process. In this paper, suppose that f_1, f_2, g_1 and g_2 satisfy Lipschitz and linear growth conditions which guarantee the existence and uniqueness of the solution of (2). Assume furthermore that $f_1(0,0) = f_2(0,0) = g_1(0,0) = g_2(0,0)$ and $h(k,0) = 0$ for all $k \in \mathbb{N}$. So system (2) has the solution $\xi(t) := (x(t); e(t)) = (0,0)$ corresponding to the initial value $\xi^* := (x^*, e^*) = (0,0)$.

Now, we introduce the concept of almost surely exponential stability, which can be found in Mao [7].

Definition 1 *Consider the system* (2). *The solution* $\xi^* = (x^*, e^*) = (0,0)$ *of* (1) *is called almost surely exponentially stable, if for all* ξ_0

$$\limsup_{t\to\infty} \frac{1}{t} \log \|\xi(t,0,\xi_0,b)\| < 0, \text{ almost surely.}$$

We need the following assumptions for the stability of network and system.

Assumption (A1) *The probability* $p \in (0,1)$ *of successful transmission of the k-th sampling time is identical for all* $k \in \mathbb{N}$ *and independent of* $k \in \mathbb{N}$.

Assumption (A2) *The stochastic perturbations b and w are mutually independent. Put* F_b *is the σ-algebra generated by* $(b_k)_{k\in\mathbb{N}}$, *and* F_w *is the σ-algebra generated by* $\{w(t)\}_{t\ge 0}$. *The system (2) defined on a probability space (Ω, F, P) where* $F = \sigma\{F_b \cup F_w\}$. *Hereafter, we use notation* $E_b(.)$ *instead of* $E_b(.\,|\,F_b)$ *and* $E_w(.)$ *instead of* $E_w(.\,|\,F_w)$.

Assumption (A3) Lyapunov functions for the protocol and the perturbed system.

(i) There exist constants $0 < a_1, a_2$, $0 < \lambda < 1$ *such that for all* $e \in \mathbb{R}^n$:

$$a_1\|e\|^2 \le W(e) \le a_2\|e\|^2 \tag{3}$$

$$W(h(k,e)) \le \lambda W(e). \tag{4}$$

(ii) The evolution of Lyapunov function W is bounded in the sense that there exist a constant $\alpha \ge 0, \beta \in \mathbb{R}$ *and a continuous function* $H : \mathbb{R}^n \to \mathbb{R}_+$ *such that for all* $x, e \in \mathbb{R}^n$:

$$\frac{\partial W}{\partial e}.g_1(x,e) = \left\langle \frac{\partial W}{\partial e}, g_1^T(x,e) \right\rangle \le 2\alpha W(e) + \beta H(x) \tag{5}$$

(iii) There exist a C^2 *Lyapunov function V and constants* $b_1, b_2, b_3 > 0$ *such that for all* $x, e \in \mathbb{R}^n$

$$b_1\|x\|^2 \le V(x) \le b_2\|x\|^2 \tag{6}$$

$$LV(x) := \frac{\partial V}{\partial x}.f_1(x,e) + \frac{1}{2} f_2^T(x,e).\frac{\partial^2 V}{\partial x^2}.f_2(x,e) \le -b_3 V(x), \tag{7}$$

where

$$f_i^T(x,e)=\left[f_i^{(1)}\cdots f_i^{(n)} \right] \text{ is the transpose of } f_i(x,e)\in \mathbb{R}^n, i=1,2$$

$$g_i^T(x,e)=\left[g_i^{(1)}\cdots g_i^{(n)} \right] \text{ is the transpose of } g_i(x,e)\in \mathbb{R}^n, i=1,2$$

$$\frac{\partial^2 V}{\partial x^2}=V_{xx}:=\begin{bmatrix} \frac{\partial^2 V}{\partial x_1\partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_n\partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n\partial x_n} \end{bmatrix}_{n\times n}, \quad \frac{\partial^2 V}{\partial x\partial e}=\frac{\partial^2 V}{\partial e\partial x}:=\begin{bmatrix} \frac{\partial^2 V}{\partial x_1\partial e_1} & \cdots & \frac{\partial^2 V}{\partial x_1\partial e_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_n\partial e_1} & \cdots & \frac{\partial^2 V}{\partial x_n\partial e_n} \end{bmatrix}_{n\times n}$$

$$\frac{\partial V}{\partial x}=\left[\frac{\partial V}{\partial x_1}\cdots\frac{\partial V}{\partial x_n}\right],\ \frac{\partial V}{\partial x}.f_1(x,e)=\left\langle \frac{\partial V}{\partial x}, f_1^T(x,e)\right\rangle,\ \frac{\partial V}{\partial e}.g_1(x,e)=\left\langle \frac{\partial V}{\partial e}, g_1^T(x,e)\right\rangle.$$

Here, τ_{MATI} follows from the equation

$$\dot{\phi}=-2\alpha\phi-\gamma(\phi^2+1),\phi(0)=\eta^{-1}. \tag{8}$$

We choose $\tau(\eta)$ such that for all $\tau\in\left[0,\tau(\eta)\right]$ we have

$$\phi_\eta(\tau)\in\left[\eta,\eta^{-1}\right], \tag{9}$$

see [5] for more details.

Theorem 2 *Consider the system* (2). *Assume that (A1), (A2) and (A3) hold. If there exist* $\eta\in(0,1)$ *and* $\gamma>0$ *as defined in* (8) *satisfying*

$$g_2^T(x,e).\frac{\partial^2 W}{\partial e^2}.g_2(x,e)\le 2\left[(2\gamma\eta-b_3)W(e)-\beta H(x)\right] \text{ for almost all } x,e\in\mathbb{R}^n \tag{10}$$

then the solution $\xi^*=(0,0)$ *of system* (2) *is almost surely exponentially stable.*

Proof: We first assume that system (2a), (2b) is almost surely exponentially stable. Consider Lyapunov-type function

$$U(\xi,\tau)=U(x,e,\tau):=V(x)+\gamma\phi(\tau)W(e). \tag{11}$$

It follows that

$$b_1\|x\|^2\le V(x)\le b_2\|x\|^2, a_1\|e\|^2\le \mathrm{W(e)}\le \mathrm{a}_2\|e\|^2, \eta\le\phi(\tau)\le\frac{1}{\eta}.$$

We yield

$$b_1\|x\|^2+\gamma\eta a_1\|e\|^2\le U(x,e,\tau)=V(x)+\gamma\phi(\tau)\mathrm{W(e)}\le b_2\|x\|^2+\gamma\eta^{-1}a_2\|e\|^2$$

and

$$m\|\xi\|^2=m\|(x,e)\|^2\le U(x,e,\tau)\le M\|(x,e)\|^2=M\|\xi\|^2 \tag{12}$$

where $m=\min\{b_1,\gamma\eta a_1\}, M=\max\{b_2,\gamma\eta^{-1}a_2\}$.

By Ito's formula and Assumption (A3), we can derive that

$$
\begin{aligned}
dU(x,e,\tau) &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial e}de + \frac{\partial U}{\partial \tau}d\tau + \frac{1}{2}\left[f_2^T(x,e).\frac{\partial^2 U}{\partial x^2}.f_2(x,e) + g_2^T(x,e).\frac{\partial^2 U}{\partial e^2}.g_2(x,e)\right]dt \\
&= \frac{\partial V}{\partial x}\left[f_1(x,e)dt + f_2(x,e)dw\right] + \gamma\phi(\tau)\frac{\partial W}{\partial e}\left[g_1(x,e)dt + g_2(x,e)dw\right] \\
&\quad + \gamma\dot{\phi}(\tau)W(e)\dot{\tau}dt + \frac{1}{2}\left[f_2^T(x,e).\frac{\partial^2 V}{\partial x^2}.f_2(x,e) + \gamma\phi(\tau)g_2^T(x,e).\frac{\partial^2 W}{\partial e^2}.g_2(x,e)\right]dt \\
&= LV(x)dt + \left[\gamma\phi(\tau)\frac{\partial W}{\partial e}.g_1(x,e) + \gamma\dot{\phi}(\tau)W(e) + \frac{1}{2}\gamma\phi(\tau)g_2^T(x,e).\frac{\partial^2 W}{\partial e^2}.g_2(x,e)\right]dt \\
&\quad + \left[\frac{\partial V}{\partial x}.f_2(x,e) + \gamma\phi(\tau)\frac{\partial W}{\partial e}.g_2(x,e)\right]dw
\end{aligned} \tag{13}
$$

and

$$
\begin{aligned}
&\gamma\phi(\tau)\frac{\partial W}{\partial e}.g_1(x,e) + \gamma\dot{\phi}(\tau)W(e) + \frac{1}{2}\gamma\phi(\tau)g_2^T(x,e).\frac{\partial^2 W}{\partial e^2}.g_2(x,e) \\
&= \gamma\phi(\tau)\frac{\partial W}{\partial e}.g_1(x,e) + \gamma\left[-2\alpha\phi(\tau) - \gamma(\phi^2(\tau)+1)\right]W(e) + \frac{1}{2}\gamma\phi(\tau)g_2^T(x,e).\frac{\partial^2 W}{\partial e^2}.g_2(x,e) \\
&\overset{(5),(8),(10)}{\leq} \gamma\phi(\tau)\left[2\alpha W(e) + \beta H(x)\right] - 2\gamma\alpha\phi(\tau)W(e) - \gamma^2\left[\phi^2(\tau)+1\right]W(e) \\
&\quad + \gamma\phi(\tau)\left[(2\gamma\eta - b_3)W(e) - \beta H(x)\right] \\
&= 2\gamma\alpha\phi(\tau)W(e) + \gamma\beta\phi(\tau)H(x) - 2\gamma\alpha\phi(\tau)W(e) - \gamma^2\phi^2(\tau)W(e) - \gamma^2 W(e) \\
&\quad + \gamma\phi(\tau)\left[(\gamma\eta + \gamma\eta - b_3)W(e) - \beta H(x)\right] \\
&\overset{(9)}{\leq} -\gamma\phi(\tau)b_3W(e).
\end{aligned} \tag{14}
$$

Therefore

$$
\begin{aligned}
dU(x,e,\tau) &\overset{(7),(14)}{\leq} -b_3V(x)dt - \gamma\phi(\tau)b_3W(e)dt + \left[\frac{\partial V}{\partial x}.f_2(x,e) + \gamma\phi(\tau)\frac{\partial W}{\partial e}.g_2(x,e)\right]dw \\
&= -b_3U(x,e,\tau)dt + \left[\frac{\partial V}{\partial x}.f_2(x,e) + \gamma\phi(\tau)\frac{\partial W}{\partial e}.g_2(x,e)\right]dw.
\end{aligned} \tag{15}
$$

This implies

$$
dE_w\left[U(x,e,\tau)\right] \leq -b_3E_w\left[U(x,e,\tau)\right]dt\,. \tag{16}
$$

For each $k = 1,2,\ldots$, integrating both sides of (16) from t_{k-1}^+ to t_k, we get

$$
\begin{aligned}
E_w\left[U(x(t_k,b),e(t_k,b),\tau(t_k))\right] &\leq E_w\left[U(x(t_{k-1}^+,b),e(t_{k-1}^+,b),\tau(t_{k-1}^+))\right] + \int_{t_{k-1}^+}^{t_k} E_w\left[-b_3U(x,e,\tau)\right]dt \\
&\leq E_w\left[U(x(t_{k-1}^+,b),e(t_{k-1}^+,b),\tau(t_{k-1}^+))\right].
\end{aligned} \tag{17}
$$

If at time t_k transmission is successful, i.e. if $b_k = 1$, then

$$
U(x(t_k^+,b),e(t_k^+,b),\tau(t_k^+)) \leq V(x(t_k,b)) + \eta^{-2}\lambda\gamma\phi(\tau(t_k))W(e(t_k,b)).
$$

On the other hand if transmission fails, i.e. if $b_k = 0$ then

$$U(x(t_k^+,b),e(t_k^+,b),\tau(t_k^+)) \le V(x(t_k,b)) + \eta^{-2}\gamma\phi(\tau(t_k))W(e(t_k,b)).$$

These give

$$\begin{aligned} & E_b\left\{E_w\left[U(x(t_k^+,b),e(t_k^+,b),\tau(t_k^+))\middle|(x(t_k^+,b),e(t_k^+,b))\right]\right\} \\ & \le p\left\{E_w\left[V(x(t_k,b))\right] + \eta^{-2}\lambda\gamma\phi(\tau(t_k))E_w\left[W(e(t_k,b))\right]\right\} \\ & \quad + (1-p)\left\{E_w\left[V(x(t_k,b))\right] + \eta^{-2}\gamma\phi(\tau(t_k))E_w\left[W(e(t_k,b))\right]\right\} \\ & < E_w\left[U(x(t_k,b),e(t_k,b),\tau(t_k))\right] - \kappa\gamma\eta E_w\left[W(e(t_k,b))\right] \end{aligned} \tag{18}$$

where $\kappa := 1-(1-p+p\lambda)\eta^{-2}$.

From (16) it follows that

$$E_w\left[U(\xi(t,b),\tau(t))\right] = E_w\left[U(x(t,b),e(t,b),\tau(t))\right] < e^{-b_3(t-t_k)}E_w\left[U(\xi(t_k^+,b),\tau(t_k^+))\right].$$

Taking expectation in *b*, we obtain

$$e^{b_3 t}E_b\left\{E_w\left[U(\xi(t,b),\tau(t))\right]\right\} < e^{b_3 t_k}E_b\left\{E_w\left[U(\xi(t_k^+,b),\tau(t_k^+))\right]\right\} \tag{19}$$

and

$$\begin{aligned} 0 & \le e^{b_3 t_k}E_b\left\{E_w\left[U(\xi(t_k^+,b),\tau(t_k^+))\right]\right\} = e^{b_3 t_k}E_b\left\{E_w\left[U(\xi(t_k^+,b),\tau(t_k^+))\middle|\xi(t_k,b)\right]\right\} \\ & \overset{(17),(18)}{\le} ME_w\left\|\xi_0\right\|^2 - \kappa\gamma\eta\sum_{i=0}^{k} e^{b_3 t_i}E_b\left\{E_w\left[W(e(t_i,b))\right]\right\}. \end{aligned} \tag{20}$$

From (12), (19) and (20), it follows that

$$\begin{aligned} me^{b_3 t}E_b\left\{E_w\left\|\xi(t,b)\right\|\right\} & \le e^{b_3 t}E_b\left\{E_w\left[U(\xi(t,b),\tau(t))\right]\right\} \\ & < e^{b_3 t_k}E_b\left\{E_w\left[U(\xi(t_k^+,b),\tau(t_k^+))\right]\right\} \le ME_w\left\|\xi_0\right\|^2. \end{aligned}$$

Hence

$$E_b\left\{E_w\left\|\xi(t,b)\right\|\right\} \le \frac{M}{m}E_w\left\|\xi_0\right\|^2 e^{-b_3 t}, \forall t \ge 0. \tag{21}$$

From the system (2), we have

$$x(t) = x(t_k) + \int_{t_k}^{t} f_1 ds + \int_{t_k}^{t} f_2 dw(s)$$

and

$$e(t) = e(t_k) + \int_{t_k}^{t} g_1 ds + \int_{t_k}^{t} g_2 dw(s) .$$

In addition, the conditions $f_1(0,0) = f_2(0,0) = g_1(0,0) = g_2(0,0)$ lead to exist a positive constant *K* such that

$$\begin{cases} \left\| f_1(x,e)\right\|^2 \vee \left\| f_2(x,e)\right\|^2 \le K\left\|(x,e)\right\|^2 \\ \left\|g_1(x,e)\right\|^2 \vee \left\|g_2(x,e)\right\|^2 \le K\left\|(x,e)\right\|^2 \end{cases} \tag{22}$$

Therefore, we obtain

$$\begin{aligned} E_w\|x(t)\|^2 &\le 3\left[E_w x^2(t_k)+E_w\left(\int_{t_k}^t f_1 ds\right)^2+E_w\left(\int_{t_k}^t f_2 dw(s)\right)^2\right] \\ &\le 3\left[E_w x^2(t_k)+(t-t_k)\int_{t_k}^t E_w f_1^2 ds+\int_{t_k}^t E_w f_2^2 ds\right] \\ &\overset{(22)}{\le} 3\left[E_w x^2(t_k)+\bar{\tau}K\int_{t_k}^t E_w\|\xi(s)\|^2 ds+K\int_{t_k}^t E_w\|\xi(s)\|^2 ds\right] \\ &\le 3\left[E_w x^2(t_k)+(\bar{\tau}+1)K\int_{t_k}^t E_w\|\xi(s)\|^2 ds\right] \end{aligned}$$

and

$$E_w\|e(t)\|^2 \le 3\left[E_w e^2(t_k)+(\bar{\tau}+1)K\int_{t_k}^t E_w\|\xi(s)\|^2 ds\right]. \tag{23}$$

As a result

$$E_w\|\xi\|^2 = E_w\|(x,e)\|^2 = E_w\|x(t)\|^2+E_w\|e(t)\|^2 \le 3\left[E_w\|\xi(t_k)\|^2+2(\bar{\tau}+1)K\int_{t_k}^t E_w\|\xi(s)\|^2 ds\right].$$

Hence

$$\begin{aligned} E_b\left(\sup_{t_k\le t\le t_{k+1}} E_w\|\xi(t,b)\|^2\right) &\le 3\left[E_bE_w\|\xi(t_k,b)\|^2+2(\bar{\tau}+1)K\int_{t_k}^{t_{k+1}} E_bE_w\|\xi(s,b)\|^2 ds\right] \\ &\le 3\left[\frac{M}{m}E_w\|\xi_0\|^2 e^{-b_3t_k}+2(\bar{\tau}+1)K\int_{t_k}^{t_{k+1}} E_w\|\xi_0\|^2 e^{-b_3s}ds\right] \\ &\le 3\left[1-\frac{2}{b_3}K(\bar{\tau}+1)(1-e^{b_3\bar{\tau}})\right]\frac{M}{m}E_w\|\xi_0\|^2 e^{-b_3t_{k+1}} \\ &\le Ce^{-b_3t_{k+1}}, \end{aligned}$$

where

$$C = 3\left[1-\frac{2}{b_3}K(\bar{\tau}+1)(1-e^{b_3\bar{\tau}})\right]\frac{M}{m}E_w\|\xi_0\|^2.$$

Applying Chebyshev's inequality, we get

$$\begin{aligned} P\left(b:\sup_{t_k\le t\le t_{k+1}} E_w\|\xi(t,b)\|^2 > e^{-\frac{b_3}{2}t_{k+1}}\right) &\le \frac{E_b\left(\sup_{t_k\le t\le t_{k+1}} E_w\|\xi(t,b)\|^2\right)}{e^{-\frac{b_3}{2}t_{k+1}}} \\ &\le Ce^{-\frac{b_3}{2}t_{k+1}}. \end{aligned}$$

Since $t_0=0$ and $0<\delta<t_{k+1}-t_k\le\bar{\tau}$, it is clear that

$$\begin{aligned} \sum_{k=0}^{\infty} e^{-\frac{b_3}{2}t_{k+1}} &= e^{-\frac{b_3}{2}t_1}+e^{-\frac{b_3}{2}t_2}+\cdots+ \\ &= e^{-\frac{b_3}{2}(t_1-t_0)}+e^{-\frac{b_3}{2}(t_2-t_1+t_1-t_0)}+\cdots<+\infty. \end{aligned}$$

Using Borel-Cantelli's lemma argument (see Mao [7]) to conclude that there exist a set Ω_1 with $P(\Omega_1)=1$ and an integer-value random variable k_0 such that for every $b\in\Omega_1$ we have

$$\sup_{t_k\le t\le t_{k+1}} E_w\|\xi(t,b)\|^2 \le e^{-\frac{b_3}{2}t_{k+1}}, \forall k\ge k_0(b). \tag{24}$$

That means

$$E_w\|\xi(t,b)\|^2 \le e^{-\frac{b_3}{2}t_{k+1}}, \forall t\in(t_k,t_{k+1}), \forall k\ge k_0(b).$$

Similarly to argument as above, using Borel-Cantelli's lemma again, there exist a set Ω_2 with $P(\Omega_2)=1$ and an integer-value random variable k_1 such that for every $w\in\Omega_2$ we have

$$\|\xi(t,b)\|^2 \le e^{-\frac{b_3}{2}t_{k+1}}, \forall t\in(t_k,t_{k+1}), \forall k\ge k_1(w). \tag{25}$$

Let $k_c=\max\{k_0,k_1\}$, $\Omega_0=\Omega_1\cap\Omega_2$, we have $P(\Omega_0)=1$ and

$$\|\xi(t,b)\|^2 \le e^{-\frac{b_3}{2}t_{k+1}}, \forall t\in(t_k,t_{k+1}), \forall k\ge k_c(w), (b,w)\in\Omega_0. \tag{26}$$

Consequently

$$\limsup_{t\to\infty}\frac{1}{t}\log\|(\xi(t),b)\| \le -\frac{b_3}{8} < 0. \tag{27}$$

The proof is completed.

Remark 3 The inequalities (5) and (10) are existent. In fact, we choose $g_1(x,e)=g_2(x,e)=e$, $W(e)=\|e\|=\left(e_1^2+e_2^2\right)^{1/2}$ and $\beta=0$. Then we have

$$\frac{\partial W}{\partial e}.g_1(x,e)=\left\langle\frac{\partial W}{\partial e}, g_1^T(x,e)\right\rangle=\left(e_1^2+e_2^2\right)^{1/2}\le 2\alpha W(e), \forall\alpha\ge 0,$$

Moreover,

$$g_2^T.\frac{\partial^2 W(e)}{\partial e^2}.g_2=0\le 2(2\gamma\eta-b_3)W(e),$$

as long as $2\gamma\eta-b_3\ge 0$.

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