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Original Article

Tail Distribution Estimates of Fractional CIR Model

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Abstract: The aim of this work is to study the tail distribution of the Cox–Ingersoll–Ross (CIR) model driven by fractional Brownian motion. We first prove the existence and uniqueness of the solution. Then based on the techniques of Malliavin calculus and a result established recently in [1], we obtain an explicit estimate for tail distributions.

Keywords: CIR model, Fractional Brownian motion, Malliavin calculus.

1. Introduction

It is well-known that the Cox–Ingersoll–Ross (CIR) model, one of very popular models in finance, describes the evolution of interest rates. It was introduced in 1985 by John C. Cox, Jonathan E. Ingersoll and Stephen A. Ross as an extension of the Vasicek model. The dynamic of this model is described by the following Itô stochastic differential equation

$$X_t = X_0 + \int_0^t (a - bX_s)ds + \int_0^t \sigma\sqrt{X_t}\,dB_s, \quad 0 \le t \le T, \tag{1}$$

where X_t is instantaneous interest rate at time t; a, b, σ are positive constants, a is a mean reversion speed, b is a long-run mean, σ is a volatility rate; $B = (B_t)_{0\le t\le T}$ is a standard Brownian motion and $X_0 > 0$.

The solution $(X_t)_{0\le t\le T}$ to the model (1) is a Markov process without memory. However, the real financial models are often characterized by the so-called “memory phenomenon” [2]. It is well-known that the CIR process is ergodic and has a stationary distribution. Therefore, the standard Brownian motion should be replaced by other more suitable processes. Many studies have pointed out that the

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dynamics driven by fractional Brownian motion are a good choice to model such objects. Some applications of fractional Brownian motion can be found in [3, 4]. We recall that a fractional Brownian motion (fBm) of Hurst parameter $H \in (0,1)$ is a centered Gaussian process $B^H = (B_t^H)_{0\le t\le T}$ with covariance function

$$R_H(t,s) := E[B_t^H B_s^H] = \frac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

For $H > 1/2$, B_t^H admits the so-called Volterra representation [5].

$$B_t^H = \int_0^t K(t,s)dW_s,$$

where $(W_t)_{t\ge 0}$ is a standard Brownian motion,

$$K(t,s) := c_H s^{\frac{1}{2}-H} \int_s^t (u-s)^{H-\frac{3}{2}} u^{H-\frac{1}{2}} du, \quad s \le t$$

and $c_H = \sqrt{\dfrac{H(2H-1)}{\beta(2-2H, H-\frac{1}{2})}}$, where β is the Beta function.

In this work, we consider the fractional CIR model that is defined as the stochastic differential equations of the form

$$X_t = X_0 + \int_0^t (a - bX_s)ds + \int_0^t \sigma\sqrt{X_t}\,dB_s^H, \quad 0 \le t \le T, \tag{2}$$

where the initial condition X_0 and a, b, σ are positive constants, B_t^H is fBm with $H > \frac{1}{2}$. The stochastic integral with respect to B^H is interpreted as a pathwise Stieltjes integral, which has been frequently used in the studies related to fBm. We refer readers to Zähle′s paper [6] for a detailed presentation of this integral.

Some research results on the existence and convergence of solutions of Eq. (2) can be found in [7, 8]. In this work, our aim is to study the tail distribution of solutions to Eq. (2). This problem is important because the probability distribution function is one of the most natural features for any random variable. In fact, in the last decade, the tail distribution and density estimates for various random variables have been investigated by many authors. We can find more results about tail distribution estimates in [1, 9] about density estimates in [10-12] and references therein. In the present paper, we will focus on providing explicit estimates for the probability distribution $P(X_t \ge x)$, where X_t is the solution of Eq. (2).

Interestingly, the volatility coefficient of the model (2) violates the Lipschitz, only $\frac{1}{2}-$Hölder continuous, which are traditionally imposed in the study of stochastic differential equations. To deal with these difficulties, we use the techniques of Malliavin calculus and a result established recently in [1] as tools. In Section 2, we recall some fundamental concepts of Malliavin calculus. The main results of the paper are stated and proved in Section 3.

2. Preliminaries

Firstly, we recall some elements of Malliavin calculus. We suppose that $(B_t)_{t\in[0,T]}$ is defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, where $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ is a natural filtration generated by the Brownian motion B. For $h \in L^2[0,T]$, we denote by $B(h)$ the Wiener integral

$$B(h) = \int_0^T h(t) dB_t.$$

Let $\mathcal{S}$ denote a dense subset of $L^2(\Omega, \mathcal{F}, P)$ that consists of smooth random variables of the form

$$F = f\left(B(h_1), B(h_2), ..., B(h_n)\right), \tag{3}$$

where $n \in \mathbb{N}$, $f \in C_0^\infty(\mathbb{R}^n), h_1, h_2, ..., h_n \in L^2[0,T]$. If F has the form (3), we define its Malliavin derivative as the process $DF := D_t F, t \in [0,T]$ given by

$$D_t F = \sum_{k=1}^n \frac{\partial f}{\partial x_k}(B(h_1), B(h_2), ..., B(h_n)).h_k(t)$$

More generally, for each $k \geq 1$, we can define the iterated derivative operator on a cylindrical random variable by setting

$$D^k_{t_1,...,t_k} F = D_{t_1} ... D_{t_k} F.$$

For any $1 \leq p, k < \infty$, we denote by $\mathbb{D}^{k,p}$ the closure of $\mathcal{S}$ with respect to the norm

$$\| F \|^p_{k,p} := E | F |^p + E\left[\int_0^t |D_u F|^p \, du\right] + ... + E\left[\int_0^T ... \int_0^T |D^k_{t_1,...,t_k} F|^p \, dt_1 ... dt_k\right].$$

A random variable F is said Malliavin differentiable if it belongs to $\mathbb{D}^{1,2}$.

To get the result in Theorem 3.1 (see below), we use the following general estimate for tail probabilities:

Lemma 2.1. Let $F \in \mathbb{D}^{1,2}$ is a centered random variable. Assume there exists a non-random constant M such that

$$\int_0^T E[D_r F | \mathcal{F}_r]^2 dr \leq M^2 \quad \text{a.s.}$$

Then following estimate for tail probabilities holds

$$P(F \geq x) \leq e^{-\frac{x^2}{2M^2}}, \quad x > 0.$$

Proof. For the reader's convenience, we recall here the proof provided in [1]. By Clark-Ocone formula, we have

$$F = \int_0^T E[D_r F | \mathcal{F}_r |] dW_r.$$

Hence, for any $\lambda \in \mathbb{R}$, it holds that

$$Ee^{\lambda F} = E \exp\left(\lambda \int_0^T E[D_r F | \mathcal{F}_r |] dW_r - \frac{\lambda^2}{2}\int_0^T (E[D_r F | \mathcal{F}_r |])^2 dr + \frac{\lambda^2}{2}\int_0^T (E[D_r F | \mathcal{F}_r |])^2 dr\right)$$

$$\leq e^{\frac{\lambda^2}{2}M^2} E \exp\left(\lambda \int_0^T E[D_r F | \mathcal{F}_r |] dW_r - \frac{\lambda^2}{2}\int_0^T (E[D_r F | \mathcal{F}_r |])^2 dr\right).$$

By Itô formula, the stochastic process

$$N_T = \exp\left(\lambda \int_0^T E[D_r F | \mathcal{F}_r |] dW_r - \frac{\lambda^2}{2}\int_0^T (E[D_r F | \mathcal{F}_r |])^2 dr\right)$$

satisfies

$$N_T = 1 + \lambda \int_0^T N_r E[D_r F \mid \mathcal{F}_r]dW_r.$$

Thus, $EN_T = 1.$ We can get

$$Ee^{\lambda F} \le e^{\frac{\lambda^2}{2}M^2} EN_T = e^{\frac{\lambda^2}{2}M^2}.$$

This, together with Markov's inequality, gives us

$$P(F \le x) = P(e^{\lambda F} \le e^{\lambda x}) \le e^{\frac{\lambda^2}{2}M^2 - \lambda x}, \quad \lambda > 0, x \in \mathbb{R}.$$

When $x > 0,$ the function $\lambda \to e^{\frac{\lambda^2}{2}M^2 - \lambda x}$ attains its minimum value at $\lambda_0 = \frac{x}{M^2}.$ Choosing $\lambda = \lambda_0$ we obtain

$$P(F \le x) \le e^{-\frac{x^2}{2M^2}}, \quad x > 0.$$

3. Results and Discussion

We first show that Eq. (2) has a unique solution. Some authors proved this problem, see e.g. [8]. Following the method used in [13], in this paper, we consider the following equation

$$dY_t = \frac{1}{2}\left(\frac{a}{Y_t} - bY_t\right)dt + \frac{\sigma}{2}dB_t^H, \quad 0 \le t \le T, \tag{4}$$

the initial value $Y_0 = \sqrt{X_0}.$ We put

$$g(x) = \frac{1}{2}\left(\frac{a}{x} - bx\right), \quad x > 0,$$

and rewrite Eq. (4) as follows

$$Y_t = Y_0 + \int_0^t g(Y_s)ds + \frac{\sigma}{2}B_t^H.$$

Proposition 3.1. Eq. (4) admits a unique positive solution. Moreover, $Y_t > 0$ a.s. for any $t \ge 0.$

Proof. We observe that the function $g(x) = \frac{1}{2}\left(\frac{a}{x} - bx\right)$ is Lipschitz continuous and linear growth on the neighborhood of $Y_0 > 0.$ Hence, there exists a local solution Y_t on the interval $[0, \tau],$ where τ is the stopping time such that

$$\tau = \inf\{t > 0 : Y_t = 0\}.$$

Assume that $\tau < \infty.$ For all $t \in [0, \tau)$ we have

$$0 = Y_\tau = Y_t + \int_t^\tau g(Y_s)ds + \frac{\sigma}{2}\left(B_\tau^H - B_t^H\right). \tag{5}$$

We note that

$$\lim_{x \to 0^+} xg(x) = \frac{a}{2} > 0,$$

hence, there exists $\varepsilon > 0$ such that $g(x) > \frac{a}{4x}, \forall x \in (0, \varepsilon)$.

Since Y_t is continuous and $Y_\tau = 0$ there exists t_0 such that $Y_t \in (0,\varepsilon)\ \forall t \in [t_0, \tau)$. which implies that

$$g(Y_t) > \frac{a}{4Y_t}, \forall t \in [t_0, \tau). \tag{6}$$

Notice that, the paths of fractional Brownian motion are $\beta -$ Hölder continuous for any $\beta < H$ [5]. Because $H \in \left(\frac{1}{2}, 1\right)$, we can fix $\beta > \frac{1}{2}$. Then, there exists a finite random variable $C_\beta(\omega)$ such that

$$\sigma \left| B_\tau^H - B_t^H \right| \le C_\beta(\omega) \,|\, \tau - t \,|^\beta .$$

This combined with (5) gives following:

$$Y_t = -\int_t^\tau g(Y_s)ds - \frac{\sigma}{2}\left(B_\tau^H - B_t^H\right) < \frac{\sigma}{2}\left|B_\tau^H - B_t^H\right| \le C_\beta(\omega)(\tau - t)^\beta, \forall t \in [t_0, \tau),$$

and

$$\int_t^\tau g(Y_s)ds < C_\beta(\omega)(\tau - t)^\beta.$$

As a consequence, it follows from (6) that

$$\begin{aligned} C_\beta(\omega)(\tau - t)^\beta &> \int_t^\tau g(Y_s)ds > \int_t^\tau \frac{a}{4Y_s}ds \\ &> \int_t^\tau \frac{a}{4C_\beta(\omega)(\tau - s)^\beta}ds = \frac{a}{4C_\beta(\omega)(1-\beta)}(\tau - t)^{1-\beta}. \end{aligned}$$

We obtain

$$4C_\beta^2(\omega)(1-\beta) > a(\tau - t)^{1-2\beta}. \tag{7}$$

Since $\beta > \frac{1}{2}$ then $1 - 2\beta < 0$. When $t \to \tau$, the right hand of (7) tends to ∞ while the left hand is bounded. We get a contradiction. Thus, Eq. (4) exists global solution with $Y_0 > 0$.

The uniqueness of the solutions can be verified as follows. Let Y_t and Y_t^* be two solutions of (4) with the same initial condition Y_0. We have

$$Y_t - Y_t^* = \int_0^t \left[g(Y_s) - g(Y_s^*) \right] ds, \quad 0 \le t \le T.$$

It implies that

$$(Y_t - Y_t^*)^2 = 2\int_0^t (Y_s - Y_s^*)\left[g(Y_s) - g(Y_s^*) \right] ds.$$

By using Lagrange's theorem, there exists a random variable ζ lying between 0 and 1 such that

$$g(Y_t) - g(Y_t^*) = g'\left(Y_t + \zeta(Y_t^* - Y_t)\right)(Y_t - Y_t^*), \quad t \ge 0.$$

We deduce

$$(Y_t - Y_t^*)^2 = 2\int_0^t g'\left(Y_s + \zeta(Y_s^* - Y_s)\right)(Y_s - Y_s^*)^2 ds, \quad 0 \le t \le T. \tag{8}$$

Combining $g'(x) = \dfrac{-a}{2x^2} - \dfrac{b}{2} < 0 \quad \forall x \ne 0$, $a, b > 0$ with (8), we have $Y_t = Y_t^*, \forall t \in [0,T]$. The proof of Proposition is completed.

Proposition 3.2. Eq. (2) has a unique solution X_t given by $X_t = Y_t^2, 0 \le t \le T$, where Y_t is the solution of Eq. (4).

Proof. Assume Y_t - is the unique solution of Eq. (4). Put $X_t = Y_t^2$ then $Y_t = \sqrt{X_t}$. By Itô formula, we obtain

$$dX_t = (a - bX_t)dt + \sigma\sqrt{X_t}dB_t^H.$$

On the one hand, because Y_t is the solution of Eq. (4), Y_t is $\beta-$ H$\ddot{o}$lder continuous $\left(\beta > \frac{1}{2}\right)$. Thus $\sqrt{X_t}$ is $\beta-$ H$\ddot{o}$lder continuous. Additionally, B_t^H is $\gamma-$ H$\ddot{o}$lder continuous for any $\gamma < H$. Notice that, $H > \frac{1}{2}$ then we can choice $\gamma > \frac{1}{2}$. Therefore, $\beta + \gamma > 1$ leads to X_t, solution of (2) given by

$$X_t = X_0 + \int_0^t (a - bX_s)ds + \sigma\int_0^t \sqrt{X_s}dB_s^H,$$

where $X_0 > 0$ is the initial (see theorem 4.2.1 in [6]).

On the other hand, if X_t is the solution of (1.2), $Y_t = \sqrt{X_t}$ is the solution of (4). Because Eq. (4) has the unique solution, Eq. (1.2) has the unique solution too.

Next, we will prove the solution (4) is Malliavin differentiable. By Volterra expression of fBm, we can rewrite (4) by the following equation

$$Y_t = Y_0 + \int_0^t g(Y_s)ds + \sigma\int_0^t K(t,s)dW_s. \tag{9}$$

Proposition 3.3. Let $(Y_t)_{0\le t\le T}$ be the solution of the Eq. (4). Then, for each $t \in (0,T]$, the random variable Y_t is Malliavin differentiable. Moreover, we have

$$D_s Y_t = \sigma\int_s^t K_1(v,s)\exp\left(\int_v^t g'(Y_r)dr\right)dv \mathbf{1}_{[0,t]}(s),$$

where $K_1(v,s) = \dfrac{\partial}{\partial v}K(v,s) = c_H (v-s)^{H-\frac{3}{2}} v^{H-\frac{1}{2}} r^{\frac{1}{2}-H}$.

Proof. Fix $t \in (0,T]$, we compute the directional derivative $\langle DY_t, h\rangle_{L^2[0,T]}$ with $h \in L^2[0,T]$:

$$\langle DY_t, h\rangle_{L^2[0,T]} = \frac{dY_t^\varepsilon}{d\varepsilon}\Big|_{\varepsilon=0},$$

where Y_t^ε solves the following equation

$$\begin{aligned} Y_t^\varepsilon &= Y_0 + \int_0^t g(Y_s)ds + \sigma\int_0^t K(t,s)d\left(W_s + \varepsilon\int_0^s h_u du\right) \\ &= Y_0 + \int_0^t g(Y_s)ds + \sigma\int_0^t K(t,s)(dW_s + \varepsilon h_s ds), \varepsilon \in [0,1). \end{aligned}$$

By using Lagrange's theorem, have some random variables lying between 0 and 1 such that

$$Y_t^\varepsilon - Y_t = \int_0^t g'\Big(Y_s + \xi_s(Y_s^\varepsilon - Y_s)\Big)(Y_s^\varepsilon - Y_s)ds + \sigma\varepsilon\int_0^t K(t,s)h_s ds. \quad (10)$$

The linear Eq. (10) has solution given by

$$Y_t^\varepsilon - Y_t = \sigma\varepsilon\int_0^t\Big(\int_0^s K_1(s,u)h_u du\Big)\exp\Big(\int_s^t g'(Y_r + \xi_r(Y_r^\varepsilon - Y_r))dr\Big)ds.$$

Hence,

$$\lim_{\varepsilon\to 0^+}\frac{Y_t^\varepsilon - Y_t}{\varepsilon} = \sigma\int_0^t\Big(\int_0^s K_1(s,u)h_u du\Big)\exp\Big(\int_s^t g'(Y_r + \xi_r(Y_r^\varepsilon - Y_r))dr\Big)ds$$
$$= \Big\langle h_s, \sigma\int_s^t K_1(v,s)\exp\Big(\int_v^t g'(Y_r)dr\Big)dv 1_{[0,t]}(s)\Big\rangle_{L^2[0,T]}.$$

Thus, Y_t is Malliavin differentiable and we have

$$D_s Y_t = \sigma\int_s^t K_1(v,s)\exp\Big(\int_v^t g'(Y_r)dr\Big)dv 1_{[0,t]}(s).$$

Theorem 3.1. Let X_t be the unique solution of Eq. (1.2). Then, for each $t\in(0,T]$, the tail distribution of X_t satisfies

$$P(X_t \ge x) \le \exp\left(-\frac{(\sqrt{x}-\sqrt{\mu_t})^2 e^{bt}}{2\sigma^2 t^{2H}}\right),\ \ x > \mu_t,$$

where $\mu_t := E[X_t]$.

Proof. We have $g'(x) = \frac{-a}{2x^2} - \frac{b}{2} < -\frac{b}{2} = M$, $\forall x > 0.$ Since Proposition 3.3 we get

$$0 \le D_r Y_t = \sigma\int_r^t K_1(v,r)e^{\int_v^t g'(Y_r)dr}dv$$
$$\le \sigma\int_r^t K_1(v,r)e^{M(t-v)}dv = \sigma\Big(K(v,r)e^{M(t-v)}\Big|_r^t + M\int_r^t K(v,r)e^{M(t-v)}dv\Big)$$
$$= \sigma\Big(K(t,r) + M\int_r^t K(v,r)e^{M(t-v)}dv\Big).$$

Because the function $v \to K(v,r)$ is non-decreasing, this implies

$$D_r Y_t \le \sigma\Big(K(t,r) + MK(t,r)\int_r^t e^{M(t-v)}dv\Big)$$
$$= \sigma K(t,r)e^{M(t-r)}, 0 \le r \le t \le T.$$

We deduce

$$\int_0^T\big(E\big[D_r Y_t \,|\, \mathrm{F}_r\big]\big)^2 dr = \int_0^t\big(E\big[D_r Y_t \,|\, \mathrm{F}_r\big]\big)^2 dr$$
$$\le \int_0^t E[|D_r Y_t|^2 | \mathrm{F}_r]dr \le \int_0^t\Big(\sigma K(t,r)e^{M(t-r)}\Big)^2 dr$$
$$\le \sigma^2 e^{2Mt}\int_0^t K^2(t,r)dr,\ \ 0 \le r \le t \le T.$$

Moreover, $\int_0^t K^2(t,s)ds = E|B_t^H|^2 = t^{2H}.$ Thus,

$$\int_0^t \left(E\left[D_r Y_t \mid \mathrm{F}_r\right]\right)^2 dr \le \sigma^2 e^{2Mt} t^{2H}.$$

Fixed $t \in (0,T]$, put $F = Y_t - E[Y_t]$ then $EF = 0$ and $D_s F = D_s Y_t$. We obtain the following estimates

$$\int_0^T \left(E\left[D_r F \mid \mathrm{F}_r\right]\right)^2 dr = \int_0^T \left(E\left[D_r Y_t \mid \mathrm{F}_r\right]\right)^2 dr \le \sigma^2 e^{2Mt} t^{2H}.$$

We observe that, by Lyapunov's inequality, $E([\sqrt{X_t}] \le \sqrt{E[X_t]} = \sqrt{\mu_t}$. We deduce

$$\begin{aligned} P(X_t \ge x) &= P(\sqrt{X_t} \ge \sqrt{x}) = P(Y_t \ge \sqrt{x}) \\ &= P(Y_t - E[Y_t] \ge \sqrt{x} - E[Y_t]) \\ &= P(F \ge \sqrt{x} - E([\sqrt{X_t}]) \le P(F \ge \sqrt{x} - \sqrt{\mu_t}). \end{aligned}$$

By applying Lemma 2.1 we get

$$P(X_t \ge x) \le \exp\left(-\frac{(\sqrt{x} - \sqrt{\mu_t})^2}{2\sigma^2 e^{2Mt} t^{2H}}\right), \quad x > \mu_t.$$

Replace $M = \frac{-b}{2}$ we obtain

$$P(X_t \ge x) \le \exp\left(-\frac{(\sqrt{x} - \sqrt{\mu_t})^2 e^{bt}}{2\sigma^2 t^{2H}}\right), \quad x > \mu_t.$$

4. Conclusion

In this work, we recalled a method to prove the the existence and uniqueness of the solution of some SDEs. This method is shown during our proof of the existence and uniqueness of the solution of the fractional CIR model. Next, we used the techniques of Malliavin calculus to estimate the tail distribution of the frational CIR model. The contribution of this work is to provide ability to obtain an explicit estimate for the tail distributions. Our work also provides one more fundamental property of CIR models. In this sense, we partly enrich the knowledge of CEV models.

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