Original Article

Dynamic Analysis of Hexagon Honeycomb Sandwich Plate Resting on Elastic Foundations

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Abstract: Dynamic analysis of hexagon honeycomb sandwich plate resting on elastic foundations are presented. Modeling is formed by 3 layers, in which core layer is hexagon honeycomb and facesheet layers is isotropic material. The Reddy’s first order shear deformation plate theory and Galerken method are used to investigate the influence of elastic foundations, geometrical parameters and material properties of hexagon honeycomb on the deflection time curve. The value of the natural frequency was compared to the results reported in other works for evaluating the accuracy of used method.

Keywords: Dynamic analysis; sandwich plate; hexagon honeycomb; composite material; elastic foundations.

1. Introduction

Materials with negative poisson coefficients have many outstanding properties such as mechanical properties and special designs such as: 2D auxetic, 3D auxetic, hexagon honeycomb, etc. Because of such outstanding properties, this material attracts the attention of researchers and manufacturers in recent years. Research is carried out using numerical and analytical methods such as Zang et al., [1] performed experiments to test compressive properties and failure mechanisms of materials reinforced by auxetic structures. To evaluate the contribution of auxetic core on the relationship between force and deflection of sandwich plate, the study are investigated by Yolcu and Baba [2]. Li et al., [3] proposed a new auxetic structure with mechanical properties that are determined by the finite element method and experiments. By using the first order shear deformation theory, the vibration behavior and load capacity of sandwich plates with honeycomb auxetic core and porous materials facesheets are evaluated by Fu et al., [4]. Lieu

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et al., [5] used the third-order shear deformation theory to investigate the influence of geometric characteristics of the honeycomb core layer on the static bending and buckling of nanobeams. A novel 3D hybrid auxetic-honeycomb lattice structures were developed by Hu et al., [6] to demonstrate that the compression resistance of the new structure is 84 – 122 times better than the corresponding structure. To solve the problem of low stiffness due to high porosity, Zhu et al., [7] proposed a new elliptical auxetic structure, the results indicated that a novel structure has a higher the average stress and absorption capacity.

Vibration plays an important role in structural evaluation and analysis because it represents the relationship between load and deflection. This evaluation provides important information about the working capacity and bearing capacity of structures subjected to loads such as wind, earthquake, and impact loads. To minimize peeling between the data board surface and the core layer, the authors of [8] used foam. Besides, the authors also evaluated the influence of foam on the mechanical properties of the auxetic structure. Zanjanchi et al., [9] used the first shear deformation plate theory to investigate the effect of elastic foundation, geometric parameter, temperature increment, distribution of CNTs on the nonlinear vibration of sandwich plate with auxetic honeycomb core. Liu et al., [10] considered the effective material properties of auxetic honeycombs with Gibson function and used analytical method to determine the response curves with the influence of important parameters. The authors of [11] studied the auxetic beams reinforced by graphene origami layer reinforcement with distribution, weight fraction, variable thickness in fluid. Ni et al., [12] proposed a novel auxetic structure combined from three structures: original re-entrant hexagonal honeycomb and cross-chiral honeycomb. The novel structure has better stiffness and stability than the component materials. Burlayenko and Sadowski [13] used finite element code ABAQUS to investigate dynamic behavior of skin/core debonding on sandwich plates.

In this work we investigate the nonlinear vibration and dynamic of hexagon honeycomb sandwich plate resting on elastic foundation. The galerkin method and the first order shear theory are used to evaluate the influence of geometrical parameter, elastic foundation, structure of hexagon honeycomb on the deflection - time curves.

2. Problem Statement

![Figure 1. Modeling of the sandwich plate on Pasternak’s elastic foundation.](image)

Considering a sandwich plate composed of a hexagon honeycomb core layer and isotropic material face sheets in the $xyz$ coordinate system with length $a$, width $b$, and thickness $h$. The $xy$ plane is placed
in the middle of the plate and the z-axis is set according to the thickness of the plate. The thickness of plate \( h \) is sum of the thickness of a core layer \( h_c \) and two face sheet layers \( h_f \). Observing in Figure 1, the layer of plate with the hexagon honeycomb special shape made from isotropic homogeneous aluminum, the material properties are shown as:

\[
E_1^{(c)} = E_c \left( \frac{t_c}{l_c} \right)^3 \frac{\cos \theta_c}{1 + \sin \theta_c} \sin^2 \theta_c \left[ 1 - \cot^2 \theta_c \left( \frac{t_c}{l_c} \right)^2 \right],
\]

\[
E_2^{(c)} = E_c \left( \frac{t_c}{l_c} \right)^3 \frac{1 + \sin \theta_c}{\cos^3 \theta_c} \left[ 1 - \left( \sec^2 \theta_c + \tan^2 \theta_c \right) \left( \frac{t_c}{l_c} \right)^2 \right],
\]

\[
G_{12}^{(c)} = G_c \left( \frac{t_c}{l_c} \right)^3 \frac{1 + \sin \theta_c}{3 \cos \theta_c},
\]

\[
G_{13}^{(c)} = G_c \left( \frac{t_c}{l_c} \right) \frac{\cos \theta_c}{1 + \sin \theta_c},
\]

\[
G_{23}^{(c)} = \frac{1}{2} G_c \left( \frac{t_c}{l_c} \right) \frac{1 + \sin \theta_c}{2 \cos \theta_c} \left[ 1 + \left( \frac{1 + \sin \theta_c}{2(1 + \sin \theta_c) \cos \theta_c} \right) \right],
\]

\[
V_{12}^{(c)} = \frac{\cos^2 \theta_c}{1 + \sin \theta_c} \sin \theta_c \left[ 1 - \csc^2 \theta_c \left( \frac{t_c}{l_c} \right)^2 \right],
\]

\[
V_{21}^{(c)} = \frac{(1 + \sin \theta_c) \sin \theta_c}{\cos^2 \theta_c} \left[ 1 - 2 \sec^2 \theta_c \left( \frac{t_c}{l_c} \right)^2 \right],
\]

\[
\rho^{(c)} = \rho_c \left( \frac{t_c}{l_c} \right) \frac{2}{1 + \sin \theta_c} \cos \theta_c.
\]

in which \( l_c \) and \( t_c \) are the cell wall length and cell wall thickness.

The Young’s modulus, Poisson’s ratio, density and thermal expansion coefficient of material Aluminum with three types are shown in Table 1 [14].

<table>
<thead>
<tr>
<th>Material</th>
<th>PmPV</th>
<th>PMMA</th>
<th>Ti-6Al-4V</th>
</tr>
</thead>
<tbody>
<tr>
<td>E (GPa)</td>
<td>(3.51 – 0.0047T)</td>
<td>(3.52 – 0.00347)</td>
<td>122.56(1 – 0.004568T)</td>
</tr>
<tr>
<td>( \rho (Kg/m^3) )</td>
<td>1150</td>
<td>1150</td>
<td>4429</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.3</td>
<td>0.34</td>
<td>0.29</td>
</tr>
<tr>
<td>( \alpha / K )</td>
<td>45(1 + 0.0005ΔT)×10^-6</td>
<td>45(1 + 0.0005ΔT)×10^-6</td>
<td>7.5788{6.638×10^-4T – 3.147×10^-6T^2}</td>
</tr>
</tbody>
</table>

3. Governing Equations

The force and moment resultants of the sandwich plate are defined as
\[ N_i = \sum_{k=1}^{n} \int \left( \sigma_i \right)_k dz, i = x, y, xy, \]
\[ M_j = \sum_{k=1}^{n} \int \left( \sigma_j \right)_k zdz, i = x, y, xy, \]
\[ Q = K \sum_{k=1}^{n} \int \left( \sigma_j \right)_k dz, i = xz, yz. \]

The compatibility equation of the imperfect sandwich plate as
\[ \left( N_x, N_y, N_{xy} \right) = \left( B_{11}, B_{12}, B_{16} \right) \varepsilon^0_x + \left( B_{12}, B_{22}, B_{26} \right) \varepsilon^0_y + \left( B_{16}, B_{26}, B_{66} \right) \gamma^0_{xy} \]
\[ + \left( C_{11}, C_{12}, C_{16} \right) k_x + \left( C_{12}, C_{22}, C_{26} \right) k_y + \left( C_{16}, C_{26}, C_{66} \right) k_{xy} - \left( \Omega^x, \Omega^y, \Omega^T \right) \Delta T, \]
\[ \left( M_x, M_y, M_{xy} \right) = \left( C_{11}, C_{12}, C_{16} \right) \varepsilon^0_x + \left( C_{12}, C_{22}, C_{26} \right) \varepsilon^0_y + \left( C_{16}, C_{26}, C_{66} \right) \gamma^0_{xy} \]
\[ + \left( E_{11}, E_{12}, E_{16} \right) k_x + \left( E_{12}, E_{22}, E_{26} \right) k_y + \left( E_{16}, E_{26}, E_{66} \right) k_{xy} - \left( N^x, N^y, N^T \right) \Delta T, \]
\[ \left( Q_x, Q_y \right) = \left( I_{45}, I_{44} \right) \frac{\partial}{\partial x} + \left( I_{55}, I_{45} \right) \frac{\partial}{\partial y}. \]

where
\[ \varepsilon^0_x = B_{11} f_{xy} + B_{12} f_{xx} - B_{16} f_{xy} + C_{11} k_x + C_{12} k_y + C_{16} k_{xy} + \Phi^x \Delta T, \]
\[ \varepsilon^0_y = B_{12} f_{xy} + B_{22} f_{xx} - B_{26} f_{xy} + C_{21} k_x + C_{22} k_y + C_{26} k_{xy} + \Phi^y \Delta T, \]
\[ \gamma^0_{xy} = B_{16} f_{xy} + B_{26} f_{xx} - B_{66} f_{xy} + C_{36} k_x + B_{46} k_y + C_{66} k_{xy} + \Phi^T \Delta T. \]

The motion equations for the sandwich plates are
\[ M_{i1}(w) + M_{i2}(\phi_i) + M_{i3}(\phi_i) + Q_i(w, f) \]
\[ + M_{i4}(w^*, f) + q = \int_0^\sigma \frac{\partial^2 w}{\partial t^2}, \]

\[ M_{i1}(w) + M_{i2}(\phi_i) + M_{i3}(\phi_i) + M_{i4}(f) + M_{i1}(w^*) = \left( \begin{array}{c} \int_0^\sigma \frac{\partial^2 \phi_i}{\partial t^2} \\
\int_0^\sigma \frac{\partial^2 \phi_i}{\partial t^2} \\
\int_0^\sigma \frac{\partial^2 \phi_i}{\partial t^2} \end{array} \right), \]

\[ M_{i2}(w) + M_{i3}(\phi_i) + M_{i4}(f) + M_{i1}(w^*) = \left( \begin{array}{c} \int_0^\sigma \frac{\partial^2 \phi_i}{\partial t^2} \\
\int_0^\sigma \frac{\partial^2 \phi_i}{\partial t^2} \\
\int_0^\sigma \frac{\partial^2 \phi_i}{\partial t^2} \end{array} \right). \]
in which the parameters \( M_{i1}, M_{i2}, M_{i3}, (i = 1, 4), i = 0, 1, 2 \) are expressed in Appendix A.

The compatibility equation of the imperfect sandwich plate as
\[ B_{11}' f_{,yy} + \left( 2 B_{12}' + B_{16}' \right) f_{,xy} - 2 B_{16}' f_{,yy} - 2 B_{26}' f_{,xy} + B_{22}' f_{,xx} \]
\[ + \left( C_{11}' - C_{16}' \right) \phi_{,xx} + C_{16}' \phi_{,xy} + C_{26}' \phi_{,xx} + \left( C_{12}' - C_{16}' \right) \phi_{,xy} \]
\[ + C_{16}' \phi_{,yy} + \left( C_{16}' - C_{26}' \right) \phi_{,xy} + \left( C_{22}' - C_{66}' \right) \phi_{,yy} + C_{26}' \phi_{,xx} \]
\[ = w_{,x}^2 - w_{,x} w_{,y} + 2 w_{,x} w_{,y} - w_{,xx} w_{,yy} - w_{,yy} w_{,xx}. \]
in which
The boundary conditions assume that the four edges of the imperfect sandwich plate are simply supported as
\[ w = N_y = M_{x} = 0, \quad N_x = N_y^0 \quad \text{at} \quad x = 0, \ a, \]
\[ w = N_y = M_{y} = 0, \quad N_y = N_y^0 \quad \text{at} \quad y = 0, \ b. \] (8)

The problem’s solutions consist of a double trigonometric function that meets the given boundary condition as follows
\[
\begin{align*}
    w(x,y,t) &= \left( \frac{W(t)}{W_0} \right) \sin \lambda_n x \sin \delta_n y \\
    w'(x,y,t) &= \left( \frac{W(t)}{W_0} \right) \sin \lambda_n x \sin \delta_n y \\
    \phi_x(x,y,t) &= \left( \frac{W(t)}{W_0} \right) \cos \lambda_n x \cos \delta_n y \\
    \phi_y(x,y,t) &= \left( \frac{W(t)}{W_0} \right) \cos \lambda_n x \cos \delta_n y \\
    f(x,y,t) &= T_1(t) \cos 2\lambda_n x + T_2(t) \cos 2\delta_n y + T_3(t) \sin \lambda_n x \sin \delta_n y \\
    + T_4(t) \cos \lambda_n x \cos \delta_n y + \frac{1}{2} N_{x\theta} y^2 + \frac{1}{2} N_{y\theta} x^2.
\end{align*}
\] (9)

where
\[
T_1 = \frac{1}{32} \lambda_n^2 \omega^2 W(W + 2W_0), \quad T_2 = \frac{1}{32} \lambda_n^2 \omega^2 \frac{\sin \lambda_n x \sin \delta_n y}{W(W + 2W_0)}, \]
\[
T_3 = T_1 \phi_x + T_2 \phi_y, \quad T_4 = T_1 \phi_x + T_2 \phi_y. \] (11)

with
\[
Y_1 = \left[ \lambda_n^2 \delta^4 + \left( 2B_{12} + B_{10} \right) \lambda_n^2 \delta^2 + B_{22} \right], \quad Y_2 = \left[ 2B_{x\theta} \lambda_n^2 \delta^3 + 2B_{y\theta} \lambda_n^2 \delta^2 \right], \]
\[
Y_3 = \left[ (C_{16} - C_{10}) \lambda_n^2 \delta^2 + C_{26} \right], \quad Y_4 = \left[ (C_{22} - C_{16}) \lambda_n^2 \delta^2 \right], \]
\[
Y_5 = \left[ -C_{16} \delta^3 + (C_{26} - C_{10}) \lambda_n^2 \delta \right], \quad Y_6 = \left[ (C_{22} - C_{16}) \lambda_n^2 \delta^2 \right].
\]
Applying the Bubnov-Galerkin method to Eq. (5) after replacing Eq. (9) results in

\[
\begin{aligned}
&\left[ m_{11} - \left( N_{m0} \lambda^2 + N_{m0} \delta^2 \right) \right] W + m_{12} \Phi_x + m_{13} \Phi_y + m_{14} \left( W + W_0 \right) \Phi_x \\
&+ m_{15} \left( W + W_0 \right) \Phi_y + m_{16} W_0 + m_{17} W \left( W + W_0 \right) (W + 2W_0) + m_{18} q = \partial^2 W / \partial t^2,
\end{aligned}
\]

(12)

\[
\begin{aligned}
&\left[ m_{21} W + m_{22} \Phi_x + m_{23} \Phi_y + m_{24} W \left( W + 2W_0 \right) + m_{25} W_0 = \left( i \frac{2}{2} - \frac{1}{i} \right) \partial^2 \Phi_x / \partial t^2,
\end{aligned}
\]

\[
\begin{aligned}
&\left[ m_{31} W + m_{32} \Phi_x + m_{33} \Phi_y + m_{34} W \left( W + 2W_0 \right) + m_{35} W_0 = \left( i \frac{2}{2} - \frac{1}{i} \right) \partial^2 \Phi_y / \partial t^2.
\end{aligned}
\]

where the detail of coefficients \( m_{1i} (i = 1,8), m_{jk} (j = 2,3, k = 1,5) \) may be found in Appendix B.

The reaction forces on two sides \( y = 0, b \) is determined as

\[
\begin{aligned}
&\left[ m_1 + \left( P_{11} \lambda^2 + P_{11} \delta^2 \right) h \right] W + m_{12} \Phi_x + m_{13} \Phi_y + m_{14} \left( W + W_0 \right) \Phi_x \\
&+ m_{15} \left( W + W_0 \right) \Phi_y + m_{16} W_0 + m_{17} W \left( W + W_0 \right) (W + 2W_0) + m_{18} q = \partial^2 W / \partial t^2,
\end{aligned}
\]

(13)

\[
\begin{aligned}
&\left[ m_{21} W + m_{22} \Phi_x + m_{23} \Phi_y + m_{24} W \left( W + 2W_0 \right) + m_{25} W_0 = \left( i \frac{2}{2} - \frac{1}{i} \right) \partial^2 \Phi_x / \partial t^2,
\end{aligned}
\]

\[
\begin{aligned}
&\left[ m_{31} W + m_{32} \Phi_x + m_{33} \Phi_y + m_{34} W \left( W + 2W_0 \right) + m_{35} W_0 = \left( i \frac{2}{2} - \frac{1}{i} \right) \partial^2 \Phi_y / \partial t^2.
\end{aligned}
\]

In order to obtain the natural frequencies, the system equations (15) are used with \( q = 0 \)

\[
\begin{bmatrix}
 m_{11} + I_0 \omega^2 & m_{12} & m_{13} \\
 m_{21} & m_{22} \rho \omega^2 & m_{23} \\
 m_{31} & m_{32} & m_{33} + \rho \omega^2
\end{bmatrix}
= 0
\]

(14)

4. Discussion

To evaluate the accuracy of the method used, the value of the natural oscillation frequency was compared with the results of another similar paper. Table 2 is the result of comparing the dimensionless frequency with the results reported in [16]. The compared results indicated that the used methodology in this work is completely appropriate.

<p>| Table 2. Comparison of the natural frequencies of isotropic plate |
|-----------------------------|-----------------------------|-----------------------------|</p>
<table>
<thead>
<tr>
<th>((m,n))</th>
<th>((a/b = 2; h/a = 1/12))</th>
<th>((a/b = 2; h/a = 1/20))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1,1))</td>
<td>0.1576</td>
<td>0.1563</td>
</tr>
<tr>
<td>((2,1))</td>
<td>0.2444</td>
<td>0.2409</td>
</tr>
<tr>
<td>((3,1))</td>
<td>0.3788</td>
<td>0.3977</td>
</tr>
<tr>
<td>((4,1))</td>
<td>0.5497</td>
<td>0.6253</td>
</tr>
</tbody>
</table>
In order to evaluate the effect of geometrical parameter, elastic foundation, structure of hexagon honeycomb on the deflection – time curves, the titanium alloy (Ti-6Al-4V) is chosen. Table 3 shows the effect of mode \((m,n)\) and elastic foundations on the natural frequencies of sandwich plate. As can see that the mode \((m,n)\) and elastic foundations have a positive influence. The natural frequencies increase when mode \((m,n)\) and elastic foundations increase.

Table 3. The natural frequencies of sandwich plate with the influence of mode \((m,n)\) and elastic foundations

<table>
<thead>
<tr>
<th>((k_1, k_2))((\text{GPa/m; GPa} \cdot \text{m}))</th>
<th>((m, n))</th>
<th>((1,1))</th>
<th>((1,2))</th>
<th>((1,3))</th>
<th>((3,1))</th>
<th>((3,3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>0,0</td>
<td>((1,1))</td>
<td>1050.62</td>
<td>2585.41</td>
<td>5043.47</td>
<td>5053.65</td>
<td>8766.17</td>
</tr>
<tr>
<td>0.2;0.02</td>
<td>((1,1))</td>
<td>1476.4</td>
<td>2969.52</td>
<td>5413.78</td>
<td>5423.21</td>
<td>9135.15</td>
</tr>
<tr>
<td>0.2;0.04</td>
<td>((1,1))</td>
<td>1701.15</td>
<td>3254.58</td>
<td>5729.32</td>
<td>5738.18</td>
<td>9471.23</td>
</tr>
<tr>
<td>0,5;0.05</td>
<td>((1,1))</td>
<td>1947.73</td>
<td>3466.79</td>
<td>5926.01</td>
<td>5934.54</td>
<td>9662.23</td>
</tr>
</tbody>
</table>

Figure 2. Effect of the \(a/b\) ratio on the time - deflection curves.

Figure 3. Effect of the \(a/h\) ratio on the time - deflection curves.
Figs. 2 and 3 show the impact of $a/b$ ratio and $a/h$ ratio on deflection - time curves of sandwich plate. In Fig. 2, three values of $a/b$ ratio (1, 2, 3) are considered. It is evident that as the $a/b$ ratio increases, the amplitude deflection also increases. This it can be shown that the $a/b$ ratio has a negative effect. The effect assessment of the $a/h$ ratio in the Fig. 3 similar with the $a/b$ ratio.

To explore the nonlinear vibration of the plate with the influence of the geometric characteristics of the hexagonal honeycomb core layer, the influence of $l_c/t_c$ ratio on the deflection – time curves of the sandwich plate in Fig. 4. It can be see that the deflection amplitude uptrends when the $l_c/t_c$ ratio increases. This behavior is explained because increasing the $l_c/t_c$ ratio leads to the unit cells of the hexagonal honeycomb core larger, in other words the wall thickness of the hexagonal edge smaller.

Figure 4. Effect of $l_c/t_c$ ratio on the time - deflection curves.

Besides, the influence of the honeycomb characteristic angle on the time - deflection curves of the sandwich plate are plotted in Fig. 5. It can be observed that the deflection amplitude reduces when the honeycomb characteristic angle decreases.

Figure 5. Effect of the honeycomb characteristic angle on the deflection – time curves of the sandwich plate.
Figs. 6 and 7 display the effect of the Winkler and Pasternak foundation on the deflection – time curves on the sandwich plate. The value of Winkler and Pasternak foundation have positive effect on the decreasing the deflection amplitude. This behavior can be explained by elastic foundation is considered as reinforcement layer with aim to improve mechanical strength results in enhancement load – carrying capability of sandwich plate.

![Figure 6](image.png)

Figure 6. Effect of Winkler foundation on the time-deflection curves.

![Figure 7](image.png)

Figure 7. Effect of Pasternak foundation on the time-deflection curves.

5. Conclusions

In this work we investigated the nonlinear vibration and dynamic of hexagon honeycomb sandwich plate resting on elastic foundation. The galerkin method and the first order shear theory were used to evaluate the influence of geometrical parameter, elastic foundation, structure of hexagon honeycomb on the deflection – time curves. Some outstanding results as:
- The accuracy of used method is verified with the results of previously published papers.
- The fluctuation amplitude of sandwich plate reduced if sandwich plate was reinforced by elastic foundation.
- The structure of core layer is evaluated in details with teta and lc/tc.
- The geometric parameters have a substantial influence on the time - deflection curves.

Acknowledgments

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References


Appendixes

Appendix A

\[ M_{11}(w) = 2I_{45}w_{,xx} + I_{55}w_{,xx} + I_{44}w_{,yy} - k_1w + k_2\left(w_{,xx} + w_{,yy}\right) \]

\[ M_{12}(\phi_x) = I_{55}\phi_{x,x} + I_{45}\phi_{x,y} \]

\[ M_{13}(\phi_y) = I_{45}\phi_{y,x} + I_{44}\phi_{y,y} \]

\[ M_{14}(w, f) = f_{,yy}w_{,xx} - 2f_{,xy}w_{,xy} + f_{,x}w_{,yy} \]

\[ M_{21}(w) = -I_{55}w_{,x} - I_{45}w_{,y} \]

\[ M_{22}(\phi_x) = L_{45}\phi_{x,x,xx} + (L_{34} + L_{35})\phi_{x,xy} + L_{36}\phi_{x,yy} - I_{55}\phi_x \]

\[ M_{23}(\phi_y) = L_{45}\phi_{y,x,xx} + (L_{26} + L_{15})\phi_{y,xy} + (L_{36} + L_{45})\phi_{y,yy} - I_{45}\phi_y \]

\[ M_{24}(f) = L_{11}f_{,x,yy} + L_{12}f_{,x,xxx} + L_{13}f_{,x,xy} + L_{21}f_{,y,yy} + L_{32}f_{,x,xy} + L_{33}f_{,xy} \]

\[ M_{31}(w) = -I_{45}w_{,x} - I_{44}w_{,y} \]

\[ M_{32}(\phi_x) = (L_{24} + L_{56})\phi_{x,x,xx} + L_{56}\phi_{x,xy} + L_{45}\phi_{x,xx} - I_{45}\phi_x \]

\[ M_{33}(\phi_y) = L_{25}\phi_{y,x,xx} + (L_{26} + L_{55})\phi_{y,xy} + L_{46}\phi_{y,xx} - I_{44}\phi_y \]

\[ M_{34}(f) = L_{21}f_{,y,yy} + L_{22}f_{,y,xxx} + L_{23}f_{,y,xy} + L_{31}f_{,x,yy} + L_{32}f_{,x,xy} + L_{33}f_{,xy} \]

Appendix B

\[ m_{11} = -I_{55}\lambda_m^2 - I_{44}\delta_n^{2} - k_1 - k_2\left(\lambda_m^2 + \delta_n^2\right), m_{12} = -I_{55}\lambda_m^2, \]

\[ m_{13} = -I_{44}\delta_n, m_{14} = \frac{32}{3\pi n \omega T^2}, m_{15} = \frac{32}{3\pi n \omega T^2}, m_{16} = -I_{55}\lambda_m^2 - I_{44}\delta_n^{2} - k_1 - k_2\left(\lambda_m^2 + \delta_n^2\right), \]

\[ m_{17} = -\frac{1}{16} B_{22}^{3}, m_{18} = \frac{16}{16 B_{22}^{3}}, m_{18} = \frac{16}{mn\pi^2}, \]
\[ m_{21} = -I_{55} \lambda_n, m_{25} = -I_{55} \lambda_n, \]
\[ m_{22} = -\left( -L_{34} \lambda_n^2 - L_{56} \delta_n^2 - I_{55} \right) + \left[ \left( L_{43} + L_{53} \right) \lambda_n \delta_n^2 - L_{42} \lambda_n^3 \right] T_3^2 \]
\[ + \left[ \left( L_{43} + L_{52} \right) \lambda_n \delta_n^2 + L_{33} \delta_n^3 \right] T_4^2, \]
\[ m_{23} = -\left( L_{56} + L_{35} \right) \lambda_n \delta_n + \left[ \left( L_{41} + L_{51} \right) \lambda_n \delta_n^2 - L_{42} \lambda_n^3 \right] T_3^2 \]
\[ + \left[ \left( L_{43} + L_{52} \right) \lambda_n \delta_n^2 + L_{33} \delta_n^3 \right] T_4^2, \]
\[ m_{24} = \frac{8 \lambda_n \delta_n}{3 m \pi^2} \left( \frac{L_{22}}{B_{22}} \delta_n + \frac{L_{51}}{B_{51}} \lambda_n \right), \]
\[ m_{31} = -I_{44} \delta_n, m_{35} = -I_{44} \delta_n \]
\[ m_{32} = -\left( L_{24} + L_{56} \right) \lambda_n \delta_n + \left[ \left( L_{22} + L_{52} \right) \lambda_n \delta_n^2 - L_{21} \delta_n^3 \right] T_3^2 \]
\[ + \left[ \left( L_{23} + L_{51} \right) \lambda_n \delta_n^2 + L_{22} \lambda_n^3 \right] T_4^2, \]
\[ m_{33} = -L_{56} \lambda_n^2 - I_{44} - L_{22} \delta_n^2 + \left[ \left( L_{22} + L_{52} \right) \lambda_n \delta_n^2 - L_{21} \delta_n^3 \right] T_3^2 \]
\[ + \left[ \left( L_{23} + L_{51} \right) \lambda_n \delta_n^2 + L_{22} \lambda_n^3 \right] T_4^2, \]
\[ m_{34} = \frac{8 \lambda_n \delta_n}{3 m \pi^2} \left( \frac{L_{21}}{B_{21}} \lambda_n + \frac{L_{52}}{B_{52}} \delta_n \right). \]