VNU Journal of Science: Mathematics – Physics, Vol. 41, No. 2 (2025) 42-55

VNU Journal of Science: Mathematics - Physics

Journal homepage: https://js.vnu.edu.vn/MaP

Original Article

e-Idempotent Matrices over Commutative Semirings

Ha Chi Cong*

University of Finance and Accountance, 2 Le Quy Don, Tu Nghia, Quang Ngai, Vietnam

Received 19th November 2024
Revised 18th April 2025; Accepted 20th June 2025

Abstract: In this work, we introduce the notion of e-idempotent matrices over a commutative semiring, this is a generalization of the idempotent matrices. We investigate some characteristic properties of e-idempotent matrices over general semirings. We provide a formula to calculate the number of e-idempotent triangular matrices over finite commutative semirings without zero divisors.

Keywords: Semiring, e-idempotent matrix, Triangular matrix, Idempotent matrix, e-invertible matrix.

1. Introduction

In 1934, the concept of a semiring was first introduced by Vandiver. Semirings were studied as a generalization of rings by removing the requirement for the existence of additive inverses. Semirings have appeared in coding theory and fuzzy logic, as well as in problems of dynamic programming and optimization. By the 21st century, semirings have been widely used in various fields such as computer science, optimization and graph theory, algebraic modeling, game theory, and several other areas. Some semirings that have attracted research interest in recent years include the tropical semiring, the boolean semiring, and the min-plus semiring.

In semiring theory, idempotent matrices play an impotant role and are known as matrices E that satisfy the condition $E^2 = E$. The structure and characteristic properties of idempotent matrices over specific semirings have been considered. Bapat et al., [1] described the structure of nonnegative idempotent matrices with a given rank r (see [1, Theorem 2]). Kang et al., [2] provided the characteristic properties of idempotent matrices over general Boolean algebras and chain semirings (see [2, Theorems 2.11 and 3.1]). Beasley et al., [3] provided a new structural characterization of idempotent Boolean matrices to describe all Boolean matrices that are majorized by a given

* Corresponding author.
E-mail address: hachicong@tckt.edu.vn

https//doi.org/10.25073/2588-1124/vnumap.4979

idempotent (see [3, Theorem 4.16]). Dolžan and Oblak [4] given a characterization of idempotent matrices by digraphs over a commutative antiring. Theorem 3.8 in [5] described the structure of triangular idempotent matrices over a given commutative semiring.

To enrich the study of idempotent matrices over semirings, in this work, we provide several characteristic properties of e-idempotent matrices over commutative semirings. The paper is organized as follows: In Section 2, we present some defintions and related results concering matrices over semirings. In Section 3, we investigate several characteristic properties of e-idempotent matrices over commutative semirings and establish a formula to calculate the number of e-idempotent triangular matrices over fintite commutative semirings.

2. Preliminaries

Semiring [6] R is a non-empty set equipped with two operations, addition (+) and multiplication (.), such that:

i) $(R,+)$ is a commutative monoid with the identity element 0.

ii) $(R,.)$ is a monoid with the identity element 1.

iii) $(u+v).w = u.w + v.w$; $w.(u+v) = w.u + w.v$ for all $u,v,w \in R$.

iv) $u.0 = 0.u = 0$ for all $u \in R$.

v) $0 \neq 1$.

In this work, we write uv instead of $u.v$ for all $u,v \in R$. The semiring R is called *commutative* if $uv = vu$ for all $u,v \in R$. The semiring R is said to be *without zero divisors* if it satisfies the condition: If $a \neq 0$ and $b \neq 0$ then $ab \neq 0$ for all $a,b \in R$. An element $u \in R$ is called *additively idempotent* if $u+u=u$, we denote by $E_a(R)$ the set of additively idempotent elements of R, R is called *additively idempotent semiring* if $E_a(R) = R$. An element $e \in R$ is called *multiplicatively idempotent* if $e^2 = e$, we denote by $E_m(R)$ the set of multiplicatively idempotent elements of R, R is called *multiplicatively idempotent semiring* if $E_m(R) = R$. An element $u \in R$ is called *invertible* if there exists an element $v \in R$ such that $uv = vu = 1$; u is called *additively invertible* if there exists an element $s \in R$ such that $u + s = 0$. We denote by $U(R)$ the set of invertible elements and by $V(R)$ the set of additively invertible elements of the semiring R. Note that if $V(R) = R$, then R is called a *ring*.

Let R be a semiring, we denote by $M_{m\times n}(R)$ the set of all $m \times n$ matrices over R, and by $M_n(R)$ the set of all $n \times n$ square matrices over R. For a matrix $A \in M_{m\times n}(R)$, we denote the entry in the i-th row and j-th column of A by $a_{i,j}$ or a_{ij}.

Now, we define an e-idempotent matrix as follows: Let R be a commutative semiring and $e \in E_m(R)\setminus\{0\}$, a matrix $A \in M_n(R)$ is called *e-idempotent* if $A^2 = eA$.

Thus, every idempotent matrix is a 1-idempotent matrix. If A is an e-idempotent matrix, then $(eA)^2 = eAeA = eA^2 = e(eA) = eA$, implying that eA is both an idempotent matrix and an e-idempotent matrix. However, if eA is an idempotent matrix, A is not necessarily idempotent.

Furthermore, if A is an e-idempotent matrix, A is not necessarily idempotent. For example, consider the semiring $R=[0,1]\subset\mathbb{R}$ with addition and multiplication defined by $a+b=\max\{a,b\}$ and $ab=\min\{a,b\}$ for all $a,b\in R$. For the matrix $A=\begin{pmatrix}0.5 & 1\\ 0.3 & 0.8\end{pmatrix}$ and $e=0.8$, we see that $eA=\begin{pmatrix}0.5 & 0.8\\ 0.3 & 0.8\end{pmatrix}$ satisfies $(eA)^2=\begin{pmatrix}0.5 & 0.8\\ 0.3 & 0.8\end{pmatrix}=eA$, so eA is an idempotent matrix. However, since $A^2=\begin{pmatrix}0{,}5 & 0{,}8\\ 0{,}3 & 0{,}8\end{pmatrix}=eA\neq A$, A is an e-idempotent matrix but not an idempotent matrix.

Let R be a semiring and a matrix $A=(a_{ij})\in M_n(R)$. We denote by $[A^i]$ the i-th column of the matrix A and by $[A_j]$ the j-th row of the matrix A. The matrices (a_{11}), $\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$,..., $\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \cdots & \vdots\\ a_{n1} & a_{n2} & \cdots & a_{nn}\end{pmatrix}$ are called *the principal submatrices* of A. The matrix A is called *(upper) triangular matrix* if $a_{ij}=0$ for all $i,j\in\{1,2,...,n\}$ with $i>j$. The set of $n\times n$ triangular matrices over the semiring R is denoted by $TM_n(R)$, if $a_{ij}=0$ for all $i,j\in\{1,2,...,n\}, i\neq j$ then A is called a *diagonal matrix*, denoted as $diag(a_{11},a_{22},...,a_{nn})$. Furthermore, if the diagonal matrix $diag(a_{11},a_{22},...,a_{nn})$ has $a_{jj}=0$ for all $j\in\{1,2,...,n\}, j\neq i$, we write it as $diag_i(0,...,0,a_{ii},0,...,0)$. The matrix A is called *invertible* if there exists a matrix $B\in M_n(R)$ such that $AB=BA=I_n$.

Proposition 2.1 ([7, Lemma 2.2]). Let R be a semiring, if a matrix $A=(a_{ij})\in M_n(R)$ is invertible, then $a_{ik}a_{jk}, a_{ki}a_{kj}\in V(R)$ for all $i,j,k\in\{1,2,...,n\}$ with $i\neq j$.

Recall from [8] that for a semiring R and $e\in E_m(R)\setminus\{0\}$, a matrix $A\in M_n(R)$ is called e-*invertible* if there exists a matrix $B\in M_n(R)$ such that $AB=BA=eI_n$. Note that every invertible matrix is a 1-invertible matrix. However, if $A\in M_n(R)$ is e-invertible, it is not necessarily invertible.

Recall from [6, p. 8] that a non-empty set S is totally ordered by the relation "$\ll$" such that $0\in S$ is the smallest element and $1\in S$ is the largest element. Then S, together with the operations of addition and multiplication defined as $a+b=\max\{a,b\}, ab=\min\{a,b\}$ for all $a,b\in S$, forms a semiring called a *bottleneck algebra*.

Example 2.2. Given the set $S=\{0,0.5,1\}\subset\mathbb{R}$ with the total order relation "$\leq$" on $\mathbb{R}$, S together with the operations $a+b=\max\{a,b\}, ab=\min\{a,b\}$ forms a bottleneck algebra.

3. Main Results

In this section, we investigate several characteristic properties of e-idempotent matrices and e-idempotent triangular matrices over general semirings. We provide a formula to determine the number of e-idempotent triangular matrices over finite semirings with no zero divisors. Note that the semirings discussed in this section are commutative semirings.

Proposition 3.1. Let R be a semiring, $e, f \in E_m(R) \setminus \{0\}$, and let $A \in M_n(R)$ be an e-idempotent matrix. Then, the following statements hold:

i) If the matrix eA is invertible, then $A = I_n$.

ii) The matrix feA is idempotent.

iii) If $ef = 0$ and A is an f-idempotent matrix, then the matrix $(e+f)A$ is idempotent. Furthermore, if R is an additively idempotent semiring, then A is an $(e+f)$-idempotent matrix.

iv) $A^k = A^2$ for all $k \in \mathbb{N}, k \geq 2$.

v) If R is a semiring without zero divisors and A is an f-idempotent matrix, then A is an ef-idempotent matrix.

vi) Let $SE_e(R)$ denote the set of e-idempotent matrices over R, and $SE_f(R)$ denote the set of f-idempotent matrices over R. Then, if R is an additively idempotent semiring without zero divisors, $SE_e(R) \cap SE_f(R) \subset SE_{ef}(R)$, equality occurs if and only if $e = f$.

Proof.

i) Since eA is an invertible matrix, there exists a matrix $B \in M_n(R)$ such that $(eA)B = B(eA) = e(AB) = e(BA) = I_n$. Then, $eI_n = ee(AB) = e(AB) = I_n$, which implies that $e = 1$. Therefore, A is an idempotent matrix. Furthermore, $I_n = BA = BA^2 = (BA)A = I_nA = A$.

ii) Since A is an e-idempotent matrix, we have $(feA)^2 = feA^2 = feeA = feA$. Thus, the matrix feA is idempotent.

iii) Since $ef = 0$, we have $(e+f)^2 = e^2 + ef + ef + f^2 = e + f$, this implies that $e + f \in E_m(R)$. Then, $((e+f)A)^2 = (e+f)A^2 = eA^2 + fA^2 = eeA + ffA = eA + fA = (e+f)A$. Therefore, the matrix $(e+f)A$ is idempotent. Furthermore, if R is an additively idempotent semiring then $A^2 = A^2 + A^2 = eA + fA = (e+f)A$. Moreover, if $e + f = 0$, then $e = e + 0 = e + e + f = e + f = 0$, which contradicts $e \neq 0$. Therefore, $e + f \neq 0$. Hence, A is an $(e+f)$-idempotent matrix.

iv) Suppose $A^k = A^2$ for $k \in \mathbb{N}, k \geq 2$. Then, we have $A^{k+1} = AA^k = AA^2 = AeA = eA^2 = e(eA) = eA = A^2$. Therefore, by induction, $A^n = A^2$ for all $n \in \mathbb{N}, n \geq 2$.

v) Since R is a semiring without zero divisors, we have $ef \neq 0$. By iv), we obtain $A^2 = A^3 = A^2A = eAA = eA^2 = efA$. Thus, A is an ef-idempotent matrix.

vi) Since R is a semiring without zero divisors, we have $ef \neq 0$. For any matrix $B \in SE_e(R) \cap SE_f(R)$, by v), we have that B is an ef-idempotent matrix, which means $B \in SE_{ef}(R)$. Thus, $SE_e(R) \cap SE_f(R) \subset SE_{ef}(R)$.

If $e = f$, then clearly $SE_e(R) \cap SE_f(R) = SE_f(R) = SE_{ef}(R)$. On the other hand, if $SE_e(R) \cap SE_f(R) = SE_{ef}(R)$, consider the matrix $A = \begin{pmatrix} ef & 1 \\ 0 & ef \end{pmatrix}$ which satisfies $A^2 = \begin{pmatrix} ef & 1 \\ 0 & ef \end{pmatrix}\begin{pmatrix} ef & 1 \\ 0 & ef \end{pmatrix} = ef\begin{pmatrix} ef & 1 \\ 0 & ef \end{pmatrix} = efA$ (since R is an additively idempotent semiring). This implies that $A \in SE_{ef}(R) = SE_e(R) \cap SE_f(R)$. Therefore, $eA = A^2 = fA$ or $e\begin{pmatrix} ef & 1 \\ 0 & ef \end{pmatrix} = f\begin{pmatrix} ef & 1 \\ 0 & ef \end{pmatrix}$, which implies that $e = f$. □

The following fact gives a criterion for every e-idempotent matrix to be an idempotent matrix.

Proposition 3.2. Let R be an additively idempotent semiring, $e \in E_m(R) \setminus \{0\}$. Every e-idempotent matrix is an idempotent matrix if and only if $e = 1$.

Proof.

If $e = 1$, then it is clear that every 1-idempotent matrix is an idempotent matrix. Conversely, consider the matrix $A = \begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix}$, which satisfies $A^2 = \begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix}\begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix} = \begin{pmatrix} e^2 & e+e \\ 0 & e^2 \end{pmatrix} = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix} = eA$, so A is an e-idempotent matrix. By assumption, A is an idempotent matrix, we have $eA = A$ or $\begin{pmatrix} e & 1 \\ 0 & e \end{pmatrix} = \begin{pmatrix} e & e \\ 0 & e \end{pmatrix}$, which implies that $e = 1$. □

Proposition 3.3. Let R be a semiring, $e \in E_m(R) \setminus \{0\}$. If $A \in M_n(R)$ is an e-invertible matrix, then there exists a matrix $B \in M_n(R)$ such that the matrix $\left[B^i\right]\left[A_i\right]$ is e-idempotent, and $A\left[B^i\right]\left[A_i\right]B = diag_i(0,...,0,e,0,...,0)$ for all $i \in \{1,2,...,n\}$.

Proof.

Since A is an e-invertible matrix, there exists a matrix $B \in M_n(R)$ such that $AB = BA = eI_n$, this implies that $\left[A_i\right]\left[B^i\right] = (e)$ for all $i \in \{1,2,...,n\}$. Let $U_i = \left[B^i\right]\left[A_i\right]$, we have $U_i^2 = \left(\left[B^i\right]\left[A_i\right]\right)\left(\left[B^i\right]\left[A_i\right]\right) = \left[B^i\right]\left(\left[A_i\right]\left[B^i\right]\right)\left[A_i\right] = \left[B^i\right]e\left[A_i\right] = eU_i$, which means U_i is an e-

idempotent matrix. Furthermore, from $A\left[B^i\right]=\begin{pmatrix}0\\ \vdots\\ 0\\ e\\ 0\\ \vdots\\ 0\end{pmatrix}$ and $\left[A_i\right]B=\begin{pmatrix}0 & \cdots & 0 & e & 0 & \cdots & 0\end{pmatrix}$, we deduce that $A\left[B^i\right]\left[A_i\right]B=\left(A\left[B^i\right]\right)\left(\left[A_i\right]B\right)=diag_i\left(0,...,0,e,0,...,0\right)$. □

Proposition 3.4. Let R be a semiring, $e\in E_m\left(R\right)\setminus\{0\}$ and $A=\left(a_{ij}\right)\in M_n\left(R\right)$ is an e-idempotent matrix. If $a_{kk}=0,\ \left(1\le k\le n\right)$, then the following statements hold:

i) There exist elements $\alpha_1,\alpha_2,...,\alpha_{k-1},\alpha_{k+1},...,\alpha_n\in R$ such that $e\left[A^k\right]=\sum_{\substack{i=1\\ i\neq k}}^{n}\alpha_i\left[A^i\right]$.

ii) There exist elements $\beta_1,\beta_2,...,\beta_{k-1},\beta_{k+1},...,\beta_n\in R$ such that $e\left[A_k\right]=\sum_{\substack{i=1\\ i\neq k}}^{n}\beta_i\left[A_i\right]$.

Proof.

i) Without loss of generality, assume $a_{nn}=0$. Then, the matrix A can be represented as follows: $A=\begin{pmatrix}B & C\\ D & 0\end{pmatrix}$, where $B=\begin{pmatrix}a_{11} & a_{12} & \cdots & a_{1,n-1}\\ a_{21} & a_{22} & \cdots & a_{2,n-1}\\ \vdots & \vdots & \vdots & \vdots\\ a_{n-1,1} & a_{n-1,2} & \cdots & a_{n-1,n-1}\end{pmatrix}$, $C=\begin{pmatrix}a_{1n}\\ a_{2n}\\ \vdots\\ a_{n-1,n}\end{pmatrix}$, and $D=\begin{pmatrix}a_{n1} & a_{n2} & \cdots & a_{n,n-1}\end{pmatrix}$.

We have $A^2=\begin{pmatrix}B^2+CD & BC\\ DB & DC\end{pmatrix}=eA=\begin{pmatrix}eB & eC\\ eD & 0\end{pmatrix}$ (since A is an e-idempotent matrix). This implies that $e\begin{pmatrix}C\\ 0\end{pmatrix}=\begin{pmatrix}eC\\ 0\end{pmatrix}=\begin{pmatrix}BC\\ DC\end{pmatrix}=\begin{pmatrix}B\\ D\end{pmatrix}C$. Therefore, $\left[A^n\right]=a_{1n}\left[A^1\right]+a_{2n}\left[A^2\right]+\cdots+a_{n-1,n}\left[A^{n-1}\right]$.

ii) This is proven similarly. □

Next, we investigate some characteristic properties of e-idempotent triangular matrices on semirings.

Lemma 3.5. Let R be a semiring, $e\in E_m\left(R\right)\setminus\{0\}$, $A\in M_k\left(R\right), B\in M_{k\times(n-k)}\left(R\right)$, and $D\in M_{(n-k)\times(n-k)}\left(R\right)$. If the matrix $\begin{pmatrix}A & B\\ 0 & D\end{pmatrix}$ is e-idempotent, then the matrices A and D are also e-idempotent.

Proof.

We have $\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}^2 = \begin{pmatrix} A^2 & AB+BD \\ 0 & D^2 \end{pmatrix} = e\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} eA & eB \\ 0 & eD \end{pmatrix}$, which implies that $A^2 = eA, D^2 = eD$. Thus, A and D are e-idempotent. □

Proposition 3.6. Let R be a semiring, $e \in E_m(R) \setminus \{0\}$ and $A \in TM_n(R)$. Then, A is an e-idempotent matrix if and only if all principal submatrices of A are e-idempotent.

Proof.

If all principal submatrices of A are e-idempotent matrices, then it is clear that A is an e-idempotent matrix. Conversely, if A is an e-idempotent matrix, then for any principal submatrix $B \in M_k(R), 1 \le k \le n$ of A, we can express A in the form $A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$, where $C \in M_{k\times(n-k)}(R)$, $D \in M_{(n-k)\times(n-k)}(R)$. By applying Lemma 3.5, we conclude that B and D are e-idempotent matrices. □

Proposition 3.7. Let R be a semiring without zero divisors, $e \in E_m(R) \setminus \{0\}$, and $A = (a_{ij}) \in TM_n(R)$ be an e-idempotent matrix with $a_{11} = a_{22} = \cdots = a_{nn} = 0$. Then $A = (0)$.

Proof.

Since A is an e-idempotent matrix, eA is idempotent. Given that $a_{11} = a_{22} = \cdots = a_{nn} = 0$, the matrix eA has all entries on its main diagonal equal to 0. By [5, Proposition 3.4], we conclude that $eA = (0)$. Since R is a semiring without zero divisors and $e \ne 0$, this implies that $A = (0)$. □

Lemma 3.8. Let R be a semiring satisfying $E_a(R) = R$, equipped with a partial order "$\ll$" defined as follows: $a \ll b \Leftrightarrow a + b = b$ for all $a, b \in R$. For any $u, v, t \in R$, the following statements hold:

i) If $u \ll v$, then $us \ll vs$ for all $s \in R$.

ii) If $u \ll t$ and $v \ll t$, then $u + v \ll t$.

Proof.

i) Since $u \ll v$, we have $u + v = v$, which implies that $us + vs = vs$ for all $s \in R$. Therefore, $us \ll vs$ for all $s \in R$.

ii) Since $u \ll t$ and $v \ll t$, we have $u + t = t$ and $v + t = t$. This implies $u + v + t + t = t + t$, or $u + v + t = t$ (since $E_a(R) = R$), which leads to $u + v \ll t$. □

Theorem 3.9. Let R be a semiring satisfying $E_a(R) = E_m(R) = R$, equipped with a partial order "$\ll$" defined as follows: $a \ll b \Leftrightarrow a + b = b$ for all $a, b \in R$. For any triangular matrix
$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$ with $n \ge 2$, satisfying the following conditions:

i) $a_{ij} \ll a_{i,j+1}$ for all $i \in \{1, 2, ..., n-1\}, j \in \{i, ..., n-1\}$.

ii) $a_{l-1,n} \ll a_{ll}$ for all $l \in \{2,3,...,n\}$.

Then, $(a_{11}a_{22}...a_{nn})A$ is an idempotent matrix.

Proof.

For $n=2$, consider the matrix $A=\begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ satisfying the condition $a_{11} \ll a_{12} \ll a_{22}$. Then, $A^2=\begin{pmatrix} a_{11} & a_{12}(a_{11}+a_{22}) \\ 0 & a_{22} \end{pmatrix}=\begin{pmatrix} a_{11} & a_{12}a_{22} \\ 0 & a_{22} \end{pmatrix}$ (since $a_{11} \ll a_{22}$), this implies that $(a_{11}a_{22}A)^2=a_{11}a_{22}A^2=\begin{pmatrix} a_{11}a_{22} & a_{11}a_{12}a_{22} \\ 0 & a_{11}a_{22} \end{pmatrix}$ (since $E_m(R)=R$). Moreover, $a_{11}a_{22}A=\begin{pmatrix} a_{11}a_{22} & a_{11}a_{12}a_{22} \\ 0 & a_{11}a_{22} \end{pmatrix}=(a_{11}a_{22}A)^2$. Thus, the matrix $(a_{11}a_{22})A$ is idempotent.

Assume the theorem holds for $n=k,(k\ge 1)$. For any matrix $A=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1,k+1} \\ 0 & a_{22} & \cdots & a_{2,k+1} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{k+1,k+1} \end{pmatrix}$ that satisfies the conditions: $a_{ij} \ll a_{i,j+1}$ for all $i \in \{1,2,...,k\}, j \in \{i,...,k\}$, and $a_{l-1,k+1} \ll a_{ll}$ for all $l \in \{2,3,...,k+1\}$. Let $B=\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ 0 & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{kk} \end{pmatrix}$ and $b=\begin{pmatrix} a_{1,k+1} \\ a_{2,k+1} \\ \vdots \\ a_{k,k+1} \end{pmatrix}$, we have $A=\begin{pmatrix} B & b \\ 0 & a_{k+1,k+1} \end{pmatrix}$, which implies $A^2=\begin{pmatrix} B^2 & Bb+ba_{k+1,k+1} \\ 0 & a_{k+1,k+1}^2 \end{pmatrix}=\begin{pmatrix} B^2 & Bb+ba_{k+1,k+1} \\ 0 & a_{k+1,k+1} \end{pmatrix}$ (since $E_m(R)=R$). The matrix B satisfies the conditions: $a_{ij} \ll a_{i,j+1}$ for all $i \in \{1,2,...,k-1\}, j \in \{i,i+1,...,k-1\}$, and $a_{l-1,k} \ll a_{ll}$ for all $l \in \{2,3,...,k\}$. Thus, the matrix $(a_{11}a_{22}...a_{kk})B$ is idempotent. For every $i \in \{1,2,...,k\}$, the entry in the i-th row of the matrix Bb is $a_{ii}a_{i,k+1}+a_{i,i+1}a_{i+1,k+1}+\cdots+a_{ik}a_{k,k+1}$. Since $a_{il} \ll a_{i,k+1}$ and $a_{l,k+1} \ll a_{k+1,k+1}$ for all $l \in \{i,i+1,...,k\}$, we have $a_{il}a_{l,k+1} \ll a_{i,k+1}a_{l,k+1}$ and $a_{i,k+1}a_{l,k+1} \ll a_{i,k+1}a_{k+1,k+1}$ (by the Lemma 3.8), which implies $a_{il}a_{l,k+1} \ll a_{i,k+1}a_{k+1,k+1}$ for all $l \in \{i,i+1,...,k\}$. Therefore, $a_{ii}a_{i,k+1}+a_{i,i+1}a_{i+1,k+1}+\cdots+a_{ik}a_{k,k+1} \ll a_{i,k+1}a_{k+1,k+1}$, this leads to $a_{ii}a_{i,k+1}+a_{i,i+1}a_{i+1,k+1}+\cdots+a_{ik}a_{k,k+1}+a_{i,k+1}a_{k+1,k+1}=a_{i,k+1}a_{k+1,k+1}$. Thus, $Bb+ba_{k+1,k+1}=ba_{k+1,k+1}$, this implies that $A^2=\begin{pmatrix} B^2 & ba_{k+1,k+1} \\ 0 & a_{k+1,k+1} \end{pmatrix}$. We have

$$\left(\left(a_{11}a_{22}...a_{k+1,k+1}\right)A\right)^2=\begin{pmatrix}\left(a_{11}a_{22}...a_{k+1,k+1}\right)^2B^2 & \left(a_{11}a_{22}...a_{k+1,k+1}\right)^2ba_{k+1,k+1}\\ 0 & \left(a_{11}a_{22}...a_{k+1,k+1}\right)^2a_{k+1,k+1}\end{pmatrix}.$$

Since $\left(a_{11}a_{22}...a_{kk}\right)B$ is an idempotent matrix and $E_m(R)=R$, we have $\left(a_{11}a_{22}...a_{k+1,k+1}\right)^2B^2=\left(a_{11}a_{22}...a_{k+1,k+1}\right)B$, $\left(a_{11}a_{22}...a_{k+1,k+1}\right)^2ba_{k+1,k+1}=\left(a_{11}a_{22}...a_{k+1,k+1}\right)b$ and $\left(a_{11}a_{22}...a_{k+1,k+1}\right)^2a_{k+1,k+1}=\left(a_{11}a_{22}...a_{k+1,k+1}\right)a_{k+1,k+1}$, wich implies that $\left(\left(a_{11}a_{22}...a_{k+1,k+1}\right)A\right)^2=\begin{pmatrix}\left(a_{11}a_{22}...a_{k+1,k+1}\right)B & \left(a_{11}a_{22}...a_{k+1,k+1}\right)b\\ 0 & \left(a_{11}a_{22}...a_{k+1,k+1}\right)a_{k+1,k+1}\end{pmatrix}=\left(a_{11}a_{22}...a_{k+1,k+1}\right)A$. Therefore, the matrix $\left(a_{11}a_{22}...a_{k+1,k+1}\right)A$ is idempotent. □

The following fact gives a sufficient condition for a semiring be to a ring.

Proposition 3.10. Let R be a semiring and $e\in E_m(R)\setminus\{0\}$. If every e-idempotent triangular matrix $A\in TM_2(R)$, which has one entry on the main diagonal equal to e and the other equal to 0, there always exist invertible matrices $U,V\in M_2(R)$ such that $UAV=\begin{pmatrix}e & 0\\ 0 & 0\end{pmatrix}$, then R is a ring.

Proof.

For any $a\in R$, consider the matrix $A=\begin{pmatrix}e & a\\ 0 & 0\end{pmatrix}\in TM_2(R)$ satisfying $A^2=\begin{pmatrix}e & a\\ 0 & 0\end{pmatrix}\begin{pmatrix}e & a\\ 0 & 0\end{pmatrix}=\begin{pmatrix}e & ea\\ 0 & 0\end{pmatrix}=eA$, so A is an e-idempotent matrix. By assumption, there exist invertible matrices U, $V\in M_2(R)$ such that $UAV=\begin{pmatrix}e & 0\\ 0 & 0\end{pmatrix}$, which implies $A=U^{-1}\begin{pmatrix}e & 0\\ 0 & 0\end{pmatrix}V^{-1}$. Let $U^{-1}=\begin{pmatrix}m & n\\ p & q\end{pmatrix}$, $V^{-1}=\begin{pmatrix}r & s\\ t & u\end{pmatrix}$. Then $U^{-1}\begin{pmatrix}e & 0\\ 0 & 0\end{pmatrix}V^{-1}=\begin{pmatrix}m & n\\ p & q\end{pmatrix}\begin{pmatrix}e & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}r & s\\ t & u\end{pmatrix}=\begin{pmatrix}mer & mes\\ per & pes\end{pmatrix}$, this implies that $\begin{cases}mer=e\\ mes=a\\ per=0\\ pes=0\end{cases}$. Since V^{-1} is an invertible matrix, by Proposition 2.1 we have $rs\in V(R)$, so there exists an element $w\in R$ such that $w+rs=0$, which implies $mew+mers=0$, or $mew+es=0$ (since $mer=e$). Therefore, $m^2ew+mes=0$, so $m^2ew+a=0$ (because $mes=a$). Hence, $a\in V(R)$, and this leads to the conclusion that R is a ring. □

Lemma 3.11. Let R be a semiring without zero divisors, and let $e\in E_m(R)\setminus\{0,1\}$. Suppose $A=\left(a_{ij}\right)\in TM_n(R)$, $A\neq(0)$, and satisfies the following conditions:

i) A is an e-idempotent matrix.

ii) $a_{ii} \in \{0,1,e\}$ for all $i \in \{1,2,...,n\}$.

iii) $a_{ii}a_{jj} = 0$ for all $i, j \in \{1,2,...,n\}, i \neq j$.

Then, the main diagonal of the matrix A contains one entry equal to e, and all other diagonal entries are 0. Moreover,

+ If $a_{11} = e$, then $A = \begin{pmatrix} e & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$.

+ If $a_{nn} = e$, then $A = \begin{pmatrix} 0 & \cdots & 0 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-1,n} \\ 0 & \cdots & 0 & e \end{pmatrix}$.

+ If $a_{kk} = e, \ (1 < k < n)$, then $A = \begin{pmatrix} 0 & M & D \\ 0 & e & N \\ 0 & 0 & 0 \end{pmatrix}$ satisfies the condition $a_{ik}a_{kj} = ea_{ij}$ for all $i \in \{1,...,k-1\}$, $j \in \{k+1,...,n\}$, where $M = \begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix} \in M_{(k-1)\times 1}(R)$, $N = \begin{pmatrix} a_{k,k+1} & \cdots & a_{kn} \end{pmatrix} \in M_{1\times(n-k)}(R)$

and $D = \begin{pmatrix} a_{1,k+1} & a_{1,k+2} & \cdots & a_{1n} \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n} \end{pmatrix} \in M_{(k-1)\times(n-k)}(R)$.

Proof.

Since $a_{ii}a_{jj} = 0$ for all $i, j \in \{1,2,...,n\}$, $i \neq j$ and R is a semiring without zero divisors, the main diagonal of matrix A has at most one entry different from 0, with the rest being 0.

+ If $a_{11} = a_{22} = \cdots = a_{nn} = 0$, since A is an e-idempotent matrix, applying Proposition 3.7 yields $A = (0)$, which contradicts the assumption $A \neq (0)$.

+ If the main diagonal of matrix A contains one entry diferent from 0, with the rest being 0. Assume $a_{kk} = 1$, $(1 \leq k \leq n)$, let $B = (b_{ij}) = A^2$, we have $b_{kk} = a_{kk}^2 = 1$. Since A is e-idempotent, $B = A^2 = eA$, which implies $1 = b_{kk} = ea_{kk} = e.1 = e$, contradicting the condition that $e \in E_m(R) \setminus \{0,1\}$.

Therefore, the main diagonal of matrix A must contain exactly one entry equal to e, with the remaining entries being 0 (since $a_{ii} \in \{0,1,e\}$ for all $i \in \{1,2,...,n\}$).

+ If $a_{11} = e$, then matrix A has the form $A = \begin{pmatrix} e & G \\ 0 & H \end{pmatrix}$, where $H \in TM_{(n-1)}(R)$ has all diagonal entries equal to 0. Since A is e-idempotent, by Propositions 3.6 and 3.7, we obtain $H=(0)$. Thus, $A=\begin{pmatrix} e & a_{12} & \cdots & a_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$. In this case, it is clear that $A^2=\begin{pmatrix} e & ea_{12} & \cdots & ea_{1n} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}=eA$.

+ If $a_{nn}=e$, then matrix A has the form $A=\begin{pmatrix} E & F \\ 0 & e \end{pmatrix}$, where $E\in TM_{(n-1)}(R)$ has all diagonal entries equal to 0. Since A is e-idempotent, by Propositions 3.6 and 3.7, we obtain $E=(0)$. Therefore, $A=\begin{pmatrix} 0 & \cdots & 0 & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_{n-1,n} \\ 0 & \cdots & 0 & e \end{pmatrix}$. Furthermore, it is clear that $A^2=\begin{pmatrix} 0 & \cdots & 0 & ea_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & ea_{n-1,n} \\ 0 & \cdots & 0 & e \end{pmatrix}=eA$.

+ If $a_{kk}=e,(1<k<n)$, then $A=\begin{pmatrix} P & M & D \\ 0 & e & N \\ 0 & 0 & Q \end{pmatrix}$, where $M=\begin{pmatrix} a_{1k} \\ \vdots \\ a_{k-1,k} \end{pmatrix}\in M_{(k-1)\times 1}(R)$, $N=\begin{pmatrix} a_{k,k+1} & \cdots & a_{kn} \end{pmatrix}\in M_{1\times(n-k)}(R)$, $D=\begin{pmatrix} a_{1,k+1} & a_{1,k+2} & \cdots & a_{1n} \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n} \end{pmatrix}\in M_{(k-1)\times(n-k)}(R)$. The matrices $P\in TM_{(k-1)}(R)$, $Q\in TM_{(n-k)}(R)$ have all diagonal entries equal to 0. Since A is an e-idempotent matrix, by Propositons 3.6 and 3.7, we have $P=(0)$, $Q=(0)$, which implies $A=\begin{pmatrix} 0 & M & D \\ 0 & e & N \\ 0 & 0 & 0 \end{pmatrix}$.

Moreover, $A^2=\begin{pmatrix} 0 & eM & MN \\ 0 & e & eN \\ 0 & 0 & 0 \end{pmatrix}=eA=\begin{pmatrix} 0 & eM & eD \\ 0 & e & eN \\ 0 & 0 & 0 \end{pmatrix}$. This implies that $MN=eD$, and so $a_{ik}a_{kj}=ea_{ij}$ for all $i\in\{1,...,k-1\}$, $j\in\{k+1,...,n\}$. □

The problem of counting idempotent triangular matrices over a finite commutative semiring was studied in [5]; see [5, Corollaries 3.10 and 3.11]. The following result provides a formula for calculating the number of e-idempotent triangular matrices over a finite semiring without zero divisors. Note that let R be a semiring and $e\in E_m(R)$, for every $x\in R$, we denote $\varepsilon_x^e(R)=\{y\in R \mid x=ey\}$.

Theorem 3.12. Let R be a finite semiring without zero divisors, $e\in E_m(R)\setminus\{0,1\}$. Let $|R|$ denote the number of elements of R, and let $\mu(e,n)$ be the number of e-idempotent triangular matrices $A=(a_{ij})\in TM_n(R)$ where $a_{ii}\in\{0,1,e\}$ and $a_{ii}a_{jj}=0$ for all $i,j\in\{1,2,...,n\}$, with $i\neq j$. Then, $\mu(e,n)$ is defined as follows:

+ If $n=1$, then $\mu(e,1)=2$.

+ If $n=2$, then $\mu(e,2)=2|R|+1$.

+ If $n\geq 3$, then $\mu(e,n)=\sum_{k=2}^{n-1}\left[\sum_{\substack{(a_{1k},a_{2k},...,a_{k-1,k},a_{k,k+1},...,a_{kn})\in R^{n-1},\\ \varepsilon^e_{a_{ik}a_{kj}}(R)\neq\varnothing,\forall(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}}\left(\prod_{(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}\left|\varepsilon^e_{a_{ik}a_{kj}}(R)\right|\right)\right]+2|R|^{n-1}+1$

where $\left|\varepsilon^e_{a_{ik}a_{kj}}(R)\right|$ is the number of elements in the set $\varepsilon^e_{a_{ik}a_{kj}}(R)$.

Proof.

+ If $n=1$, then $A=(0)$ or $A=(e)$, which implies $\mu(e,1)=2$.

+ If $n=2$, then $A=(0)$ or, according to Lemma 3.11, the matrix A takes one of the following forms: $A=\begin{pmatrix} e & a_{12}\\ 0 & 0\end{pmatrix}$ or $A=\begin{pmatrix} 0 & a_{12}\\ 0 & e\end{pmatrix}$, where $a_{12}\in R$. Therefore, $\mu(e,2)=2|R|+1$.

+ If $n\geq 3$, then $A=(0)$ or, according to Lemma 3.11, the matrix A takes one of the following forms:

- If $A=\begin{pmatrix} e & a_{12} & \cdots & a_{1n}\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & 0 & \cdots & 0\end{pmatrix}$, then since $a_{12},a_{13},...,a_{1n}\in R$, so there are $|R|^{n-1}$ ways to form the matrix A.

- If $A=\begin{pmatrix} 0 & \cdots & 0 & a_{1n}\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \cdots & 0 & a_{n-1,n}\\ 0 & \cdots & 0 & e\end{pmatrix}$, then since $a_{1n},a_{2n},...,a_{n-1,n}\in R$, so there are $|R|^{n-1}$ ways to form the matrix A.

- If the matrix $A=\begin{pmatrix} 0 & M & D\\ 0 & e & N\\ 0 & 0 & 0\end{pmatrix}$ satisfies the condition $a_{ik}a_{kj}=ea_{ij}$ for all $i\in\{1,...,k-1\}$, $j\in\{k+1,...,n\}$, where $1<k<n$, $M=\begin{pmatrix} a_{1k}\\ \vdots\\ a_{k-1,k}\end{pmatrix}\in M_{(k-1)\times 1}(R)$,

$N=\begin{pmatrix} a_{k,k+1} & \cdots & a_{kn} \end{pmatrix} \in M_{1\times(n-k)}(R)$ and $D=\begin{pmatrix} a_{1,k+1} & a_{1,k+2} & \cdots & a_{1n} \\ a_{2,k+1} & a_{2,k+2} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k-1,k+1} & a_{k-1,k+2} & \cdots & a_{k-1,n} \end{pmatrix} \in M_{(k-1)\times(n-k)}(R)$. For each $k\in\{2,3,...,n-1\}$, if there exists $i\in\{1,...,k-1\}, j\in\{k+1,...,n\}$ such that $\varepsilon^e_{a_{ik}a_{kj}}(R)=\varnothing$, then the number of ways to form matrix D is 0, implying that the number of ways to form matrix A is also 0. If $\varepsilon^e_{a_{ik}a_{kj}}(R)\neq\varnothing$ for all $i\in\{1,...,k-1\}, j\in\{k+1,...,n\}$, then for each choice of entries $a_{1k},a_{2k},...,a_{k-1,k},a_{k,k+1},a_{k,k+2},...,a_{kn}\in R$, there are $\prod_{(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}\left|\varepsilon^e_{a_{ik}a_{kj}}(R)\right|$ ways to form matrix D. Therefore, the number of ways to form matrix $A=\begin{pmatrix} 0 & M & D \\ 0 & e & N \\ 0 & 0 & 0 \end{pmatrix}$ that satisfies the condition $a_{ik}a_{kj}=ea_{ij}$ for all $i\in\{1,...,k-1\}, j\in\{k+1,...,n\}$ and $1<k<n$ is

$$\sum_{k=2}^{n-1}\left[\sum_{\substack{(a_{1k},a_{2k},...,a_{k-1,k},a_{k,k+1},...,a_{kn})\in R^{n-1},\\ \varepsilon^e_{a_{ik}a_{kj}}(R)\neq\varnothing,\forall(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}}\left(\prod_{(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}\left|\varepsilon^e_{a_{ik}a_{kj}}(R)\right|\right)\right].$$

Therefore, if $n\geq 3$, the number of e-idempotent triangular matrices $A=(a_{ij})\in TM_n(R)$ with $a_{ii}\in\{0,1,e\}$ and $a_{ii}a_{jj}=0$ for all $i,j\in\{1,2,...,n\}$ and $i\neq j$ is given by the formula:

$$\mu(e,n)=\sum_{k=2}^{n-1}\left[\sum_{\substack{(a_{1k},a_{2k},...,a_{k-1,k},a_{k,k+1},...,a_{kn})\in R^{n-1},\\ \varepsilon^e_{a_{ik}a_{kj}}(R)\neq\varnothing,\forall(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}}\left(\prod_{(i,j)\in\{1,...,k-1\}\times\{k+1,...,n\}}\left|\varepsilon^e_{a_{ik}a_{kj}}(R)\right|\right)\right]+2|R|^{n-1}+1.$$

□

Example 3.13. Given the bottleneck algebra $R=\{0,0.5,1\}$ and $e=0.5$, the triangular matrices in Theorem 3.12 are determined as follows:

+ If $n=1$, then the possible matrices are as follows: (0), (e).

+ If $n=2$, then the possible matrices are as follows: $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} e & 0 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} e & e \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} e & 1 \\ 0 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix}$, $\begin{pmatrix} 0 & e \\ 0 & e \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ 0 & e \end{pmatrix}$.

+ If $n=3$, since $\varepsilon^e_0(R)=\{0\}$, $\varepsilon^e_e(R)=\{e,1\}$ and $\varepsilon^e_1(R)=\varnothing$, the possible matrices are as follows:

$$\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&0&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&0&e\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&0&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&e&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&e&e\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&e&1\\0&0&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}e&1&0\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&1&e\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}e&1&1\\0&0&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&0\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&e\\0&0&0\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&0\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&e\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&e\\0&0&e\\0&0&e\end{pmatrix},$$

$$\begin{pmatrix}0&0&1\\0&0&e\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&0\\0&0&1\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&e\\0&0&1\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&1\\0&0&1\\0&0&e\end{pmatrix}, \begin{pmatrix}0&0&0\\0&e&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&e&e\\0&0&0\end{pmatrix}, \begin{pmatrix}0&0&0\\0&e&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&e&0\\0&e&0\\0&0&0\end{pmatrix},$$

$$\begin{pmatrix}0&e&e\\0&e&e\\0&0&0\end{pmatrix}, \begin{pmatrix}0&e&1\\0&e&e\\0&0&0\end{pmatrix}, \begin{pmatrix}0&e&e\\0&e&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&e&1\\0&e&1\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&0\\0&e&0\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&e\\0&e&e\\0&0&0\end{pmatrix}, \begin{pmatrix}0&1&1\\0&e&e\\0&0&0\end{pmatrix}.$$

Acknowledgments

The authors express their deep gratitude to the anonymous referee for the extremely careful reading and highly professional working with our manuscript.

References

[1] R. B. Bapat, S. K. Jain, L. E. Snyder, Nonnegative Idempotent Matrices and The Minus Partial Order, Linear Algebra Appl., Vol. 261, No. 1-3, 1997, pp. 143-154, https//doi.org/10.1016/S0024-3795(96)00364-3.

[2] K. T. Kang, S. Z. Song, Y. O. Yang, Structures of Idempotent Matrices over Chain Semirings, Bull. Korean Math. Soc., Vol. 44, No. 4, 2007, pp. 721-729, https//doi.org/10.4134/BKMS.2007.44.4.721.

[3] L. B. Beasley, A. E. Guterman, K. T. Kang, S. Z. Song, Idempotent Boolean Matrices and Majorization, J. Math. Sci., Vol. 152, No. 4, 2008, pp. 456-468, https//doi.org/10.1007/s10958-008-9086-3.

[4] D. Dolžan, P. Oblak, Idempotent Matrices over Antirings, Linear Algebra Appl., Vol. 431, No. 5-7, 2009, pp. 823-832, https//doi.org/10.1016/j.laa.2009.03.035.

[5] H. C. Cong, Idempotent Triangular Matrices over Commutative Semirings, TNU J. Sci. Technol., Vol. 229, No. 06, 2024, pp. 57-64, https//doi.org/10.34238/tnu-jst.8971.

[6] J. S. Golan, Semirings and Their Applications. Kluwer Academic Publishers, Dordrecht-Boston-London, 1999.

[7] Y. J. Tan, Diagonability of Matrices over Commutative Semirings, Linear Multilinear Algebr., Vol. 68, No. 9, 2020, pp. 1743-1752, https//doi.org/10.1080/03081087.2018.1556243.

[8] L. Zhang, Y. Shao, E-invertible Matrices over Commutative Semirings, Indian J. Pure Appl. Math., Vol. 49, No. 2, 2018, pp. 227-238, https//doi.org/10.1007/s13226-018-0265-8.