

UV/IR phenomenon of Noncommutative Quantum Fields in Example

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Abstract: Noncommutative Quantum Field (NCQF) is a field defined over a space endowed with a noncommutative structure. In the last decade, the theory of NCQF has been studied intensively, and many qualitatively new phenomena have been discovered. In this article we study one of these phenomena known as UV/IR mixing.

Keywords: Noncommutative quantum field theory.

1. Introduction

Noncommutative quantum field theory (NC QFT) is the natural generalization of standard quantum field theory (QFT). It has been intensively developed during the past years, for reviews, see [1,2]. The idea of NC QFT was firstly suggested by Heisenberg and the first model of NC QFT was developed in Snyder's work [3]. The present development in NC QFT is very strongly connected with the development of noncommutative geometry in mathematics [4], string theory [5] and physical arguments of noncommutative space-time [6].

The simplest version of NC field theory is based on the following commutation relations between coordinates [7]:

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (1)$$

where $\theta^{\mu\nu}$ is a constant antisymmetric matrix.

Since the construction of NC QFT in a general case ($\theta^{0i} \neq 0$) has serious difficulties with unitarity and causality [8-10], we consider a simpler version with $\theta^{0i} = 0$ (thus space-space noncommutativity only), in which there do not appear such difficulties. This case is also a low-energy limit of the string theory [1, 2].

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2. Moyal Product

We introduce d -dimensional noncommutative space-time by assuming that time and position are not c -numbers but self-adjoint operators defined in a Hilbert space and obeying the commutation algebra

$$[\hat{x}^\mu, \hat{x}^\nu] = i\theta^{\mu\nu}, \quad (2)$$

where the $\theta^{\mu\nu}$ are the elements of a real constant $d \times d$ antisymmetric matrix θ . Then we define the *Moyal star product*

$$\begin{aligned} f(x) \star g(x) &= f(x)g(x) + \sum_{n=1}^{\infty} \left(\frac{i}{2}\right)^n \frac{1}{n!} \theta^{\mu_1\nu_1} \dots \theta^{\mu_n\nu_n} [\partial_{\mu_1} \dots \partial_{\mu_n} f(x)] [\partial_{\nu_1} \dots \partial_{\nu_n} g(x)] \\ &= f(x) \exp \left[\frac{i}{2} \frac{\bar{\partial}}{\partial x^\mu} \theta^{\mu\nu} \frac{\bar{\partial}}{\partial x^\nu} \right] g(x). \end{aligned} \quad (3)$$

In particular we have:

$$e^{ip_\mu x^\mu} \star e^{iq_\nu x^\nu} = \exp\left(-\frac{i}{2} p \wedge q\right) e^{i(p+q)_\mu x^\mu}, \quad (4)$$

where we have defined the *wedge product*

$$p \wedge q = \sum_{\mu, \nu} p_\mu \theta^{\mu\nu} q_\nu. \quad (5)$$

The natural generalization of the star product (3) follows:

$$f_1(x_1) \star f_2(x_2) \star \dots \star f_n(x_n) = \prod_{a < b} \exp\left(\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x_a^\mu} \frac{\partial}{\partial x_b^\nu}\right) f_1(x_1) \dots f_n(x_n), \text{ for } a, b = 1, \dots, n. \quad (6)$$

A simple prescription to construct NC FT is to replace ordinary products by (Moyal) star products \star all over the place. For example, the action for a noncommutative Φ^4 real-valued scalar field

$$S[\Phi] = \int d^d x \left[\frac{1}{2} \partial_\mu \Phi \star \partial^\mu \Phi - \frac{m^2}{2} \Phi \star \Phi - \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi \right]. \quad (7)$$

For $\theta^{0i} = 0$ we can construct NC quantum fields by canonically quantizing NC classical fields. This can be done by applying formal canonical quantization method. Alternatively, we can quantize NC classical fields by path integral method. Thus

$$Z[J] = \int D\mu[\Phi] e^{iS[\Phi]} e^{i \int d^d x (J \star \Phi)}, \quad (8)$$

with some specification of the integral measure.

3. Noncommutative Perturbative Quantization

Now we will restrict ourselves to the perturbative evaluation of $Z[J]$. The first important observation is that the free approximation is locally θ -independent

$$S[\Phi]_{\text{free}} = \frac{1}{2} \int d^d x \left[\partial_\mu \Phi \star \partial^\mu \Phi - m^2 \Phi \star \Phi \right] = \frac{1}{2} \int d^d x \left[\partial_\mu \Phi \partial^\mu \Phi - m^2 \Phi^2 \right]. \tag{9}$$

The Fourier transform of the Feynman propagator is the same as for commutative scalar field

$$\tilde{G}(p) = \frac{i}{p^2 - m^2 + i0}. \tag{10}$$

Upon Fourier transformation

$$\int d^d x \Phi(x) \star \dots \star \Phi(x) = \int d^d p (2\pi)^d \delta\left(\sum_{k=1}^n p_k\right) \tilde{\Phi}(p_1) \dots \tilde{\Phi}(p_n) W(p_1, \dots, p_n), \tag{11}$$

$$\text{where } W(p_1, \dots, p_n) = \exp\left[-\frac{i}{2} \sum_{i < j} p_i \wedge p_j\right], \tag{12}$$

is the Moyal phase. Thus we get a simple Feynman rule for the interactions:

$$-i\lambda_n \rightarrow -i\lambda_n W(p_1, \dots, p_n), \tag{13}$$

i.e. the standard Feynman vertex is mapped into itself times the Moyal phase.

Hence, the Feynman rules in momentum space of noncommutative field theory are similar to those of commutative ones except that the vertices of the NC theory are modified by the Moyal phase factor.

4. The UV/IR mixing of NC QFT

The phenomenon of UV/IR mixing is the most radical feature of NC QFT that significantly differs from those of ordinary QFT. It occurs in perturbation theory, so we can study this phenomenon in details. We analyze the UV/IR mixing in the case of real-valued Φ^4 scalar field.

The NC real-valued Φ^4 theory in the four-dimensional space-time, is described by

$$L = \frac{1}{2} \partial_\mu \Phi \star \partial^\mu \Phi - \frac{m^2}{2} \Phi \star \Phi - \frac{\lambda}{4!} \Phi \star \Phi \star \Phi \star \Phi. \tag{14}$$

As we have seen in Eqs (9), (13), under the integration the star product of the fields does not affect the quadratic parts of the Lagrangians, whereas it makes the interaction parts become nonlocal by the Moyal phase (12).

For the Lagrangian (14), the Feynman rule for the noncommutative vertex is

$$\begin{aligned} & \frac{-i\lambda}{3} \left[\cos\left(\frac{1}{2}(p_1 \wedge p_2 + p_1 \wedge p_3 + p_2 \wedge p_3)\right) + \cos\left(\frac{1}{2}(p_1 \wedge p_2 + p_1 \wedge p_3 - p_2 \wedge p_3)\right) + \right. \\ & \left. \cos\left(\frac{1}{2}(p_1 \wedge p_2 - p_1 \wedge p_3 - p_2 \wedge p_3)\right) \right], \tag{15} \end{aligned}$$

where p_i , $i = 1, \dots, 4$, are momenta coming out of the vertex and $p_i \wedge p_j = p_{i\mu} \theta^{\mu\nu} p_{j\nu}$.

In the commutative Φ^4 model the leading mass renormalization comes from the normal-ordering diagram contribution to the self energy [11]:

$$\Sigma_{\text{commutative}} = -\frac{\lambda}{2} \int \frac{d^4 k}{i(2\pi)^4} \frac{1}{k^2 + m^2} = \frac{-\lambda}{32\pi^2} \left(\Lambda^2 - m^2 \ln \left(\frac{\Lambda^2}{m^2} \right) + O \left(\frac{m^4}{\Lambda^2} \right) \right), \quad (16)$$

where Λ is the ultraviolet *cutoff*.

In the noncommutative Φ^4 model, we have two contributions, planar and nonplanar Feynman diagrams. The planar diagram gives almost the same contribution (16), except the factor $1/3$ instead of $1/2$, which is responsible for different symmetry of the diagram. Thus

$$\Sigma_{NCP} = \Sigma_{\text{nc planar}} = -\frac{\lambda}{3} \int \frac{d^4 k}{i(2\pi)^4} \frac{1}{k^2 + m^2} = \frac{-\lambda}{48\pi^2} \left(\Lambda^2 - m^2 \ln \left(\frac{\Lambda^2}{m^2} \right) + \dots \right). \quad (17)$$

and the nonplanar diagram gives

$$\Sigma_{NCP} = \Sigma_{\text{nc nonplanar}} = -\frac{\lambda}{6} \int \frac{d^4 k}{i(2\pi)^4} \frac{\cos(p \wedge k)}{k^2 + m^2} = -\frac{\lambda}{96\pi^2} \left(\Lambda_{\text{eff}}^2 - m^2 \ln \left(\frac{\Lambda_{\text{eff}}^2}{m^2} \right) + \dots \right) \quad (18)$$

where

$$\tilde{p} = p_\mu \theta^{\mu\nu} \quad \text{and} \quad \Lambda_{\text{eff}}^2 = \frac{1}{\tilde{p}^2 + 1/\Lambda^2} \quad (19)$$

is the *effective cutoff*, which shows the mixing of UV divergence and IR singularity.

Note that the nonplanar contribution is one half of the planar one. We computed all above integrations by using dimensional regularization method [11]. So we can normalize the theory at fixed p and fixed θ by subtracting the planar divergence in the limit when the cutoff Λ tends to infinity

$$m^2 \rightarrow M^2 = m^2 - \frac{\lambda}{48\pi^2} \left(\Lambda^2 - m^2 \ln \left(\frac{\Lambda^2}{m^2} \right) \right). \quad (20)$$

Finally, we obtain one particle irreducible (or 1PI) effective action

$$\Gamma_{1PI} = \int d^4 p \Phi(-p) \Gamma^{(2)}(p) \Phi(p) + \dots \quad (21)$$

$$\Gamma^{(2)}(p) = p^2 + M^2 - \frac{\lambda}{96\pi^2} \frac{1}{\tilde{p}^2} + \frac{\lambda M^2}{48\pi^2} \ln \left(\frac{1}{M^2 \tilde{p}^2} \right) + \dots \quad (22)$$

Thus the effective action has a singularity at $\tilde{p} = 0$ that can be interpreted either as a non-analytic function of θ at fixed p , or an IR singularity at fixed θ .

In the case that Φ is a complex scalar field, there are two ways of ordering the fields Φ and Φ^* in the quartic interaction $(\Phi^* \Phi)^2$. So, the most general potential of the NC complex scalar field action is

$$V(\Phi) = A \Phi^* \star \Phi \star \Phi^* \star \Phi + B \Phi^* \star \Phi^* \star \Phi \star \Phi. \quad (23)$$

It was shown in [12] that the theory is not generally renormalizable for arbitrary values of A and B and is renormalizable at one-loop level only when $B = 0$ or $A = B$.

5. Conclusion

Our main focus in this article is to point out several important aspects of NC field theories, especially noncommutative perturbative path-integral quantization and the renormalization problem of NC QFT. We have figured out significant analogies and radical differences between the perturbative description of NC QFT and that of the ordinary QFT. We successfully calculated noncommutative vertex, one-loop renormalized mass and 1PI effective action for noncommutative real-valued scalar field. We found that UV/IR mixing terms, as a direct consequence of phase factors induced in the vertex, generally appear in all perturbative quantum calculations. The analysis and computing techniques used here are very useful and applicable for other models of NC QFT.

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References

- [1] M. R. Douglas and N. A. Nekrasov, Noncommutative Field Theory, *Rev. Mod. Phys.* 73, 977-1029 (2001).
- [2] R. J. Szabo, Quantum Field Theory on Noncommutative Spaces, *Phys. Rept.* 378, 207 (2003).
- [3] H. S. Snyder, Quantized Space-Time, *Phys. Rev.* 71, 38 (1947).
- [4] A. Connes, Noncommutative Geometry, Academic Press, New York (1994).
- [5] N. Seiberg and E. Witten, String Theory and Noncommutative Geometry, *JHEP* 9909, 32 (1999).
- [6] S. Doplicher, K. Fredenhagen and J. E. Roberts, Spacetime quantization induced by classical gravity, *Phys. Lett. B* 331, 39 (1994); The quantum structure of spacetime at the Planck scale and quantum fields, *Comm. Math. Phys.* 172, 187 (1995).
- [7] T. Filk, Divergencies in a field theory on quantum space, *Phys. Lett. B* 376, 53 (1996).
- [8] J. Gomis and T. Mehen, Space-Time Noncommutative Field Theories And Unitarity, *Nucl. Phys. B* 591, 265 (2000).
- [9] M. Chaichian, K. Nishijima and A. Tureanu, Spin-Statistics and CPT Theorems in Noncommutative Field Theory, *Phys. Lett. B* 568, 146 (2003).
- [10] M. Chaichian, P. Prešnajder and A. Tureanu, New Concept of Relativistic Invariance in Noncommutative Space-Time: Twisted Poincaré Symmetry and Its Implications, *Phys. Rev. Lett.* 94, 151602 (2005).
- [11] M. E. Peskin, D. V. Schroeder, *Introduction to Quantum Field Theory*, 1995.
- [12] I. Ya. Aref'eva, D. M. Belov, A. S. Koshelev, Two-Loop Diagrams in Noncommutative ϕ_4^4 theory, *Phys. Lett. B* 476, 431 (2000).