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Explicit Phase Space Transformations and Their Application in Noncommutative Quantum Mechanics

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Abstract: We study a problem of transformations mapping noncommutative phase spaces into commutative ones. We find a simple way to obtain explicit formulas of such transformations in 3D and indicate matrix equations for numerical computation in higher dimensions. Then we use these formulas to calculate the energy levels of the hydrogen-like atom with six noncommutative parameters. We also find and prove new relations between the hydrogen eigenfunctions corresponding to the *n*-th energy level.

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1. Introduction

Noncommutativity of space-time has long been suggested as a quantum effect of gravity and as a natural way to regularize quantum field theories [1]. Even such original works were not very successful, recently, motivated by the developments in string theory, noncommutative quantum field theory (NCQFT) [2-4], noncommutative geometry [5], noncommutative quantum mechanics [6-8], noncommutative general relativity [9], noncommutative gravity [10], noncommutative black holes [11], noncommutative inflation [12], and noncommutative approaches to cosmological constant problem [13] have been studied extensively.

In literature, noncommutativity can be introduced by either replacing the standard commutative multiplication of functions by the Moyal star product or replacing the usual commutative commutators (or canonical commutators of canonical conjugate operators) by noncommutative ones. Both approaches seem to be equivalent [14], but the latter showing more convenience in calculation, is chosen for this article. There are different types of noncommutative structures. One of them, inferred

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from the string theory, is characterized by $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$, where $\hat{x}$ are the coordinate operators and $\theta_{\mu\nu}$ is the noncommutativity parameter and is of dimension of length squared [2]. This characterizes a noncommutative quantum configuration space (NCQCS or shortly NCQS). Although in string theory, only noncommutative spaces emerge, several authors [15-19] have proposed and studied models, in which coordinates of whole phase space exhibit noncommutativity. In this article, we consider 2*n* dimensional noncommutative quantum phase space (NCQPS) with commutation relations of the form:

$$\left[\hat{x}_j, \hat{x}_k\right] = i\hbar\theta_{jk},\ \left[\hat{p}_j, \hat{p}_k\right] = i\hbar\beta_{jk},\ \left[\hat{x}_j, \hat{p}_k\right] = i\hbar\gamma_{jk} = i\hbar\left(\delta_{jk} + \sigma_{jk}\right),\ \text{for } j,k = 1,\dots,n. \quad (1)$$

The coefficients θ_{ij}, β_{ij}, and σ_{ij} measure the noncommutativity of coordinates, momenta, and coordinates-momenta, respectively. Grouping these coefficients into matrices we get three real constant $n\times n$ matrices θ, β, and σ (or $\gamma = I + \sigma$), of which the first two are skew-symmetric [1]. In the commutative limit, $(\theta, \beta, \sigma) \to 0$, the commutators (1) reduce to the commutative relations (or canonical commutators):

$$[x_j, x_k] = 0,\ \ [p_j, p_k] = 0,\ \ [x_j, p_k] = i\hbar\delta_{jk}. \quad (2)$$

Phase space noncommutativity is considered not only because of itself interesting, but also of several motivations. First, it is needed in algebraic description of dynamics of particles in a magnetic field. Second, it seems to be a requirement in order to maintain Bose-Einstein statistics for systems of identical Bosons described by deformed annihilation-creation operators [15]. Third, it also appears naturally after accepting noncommutativity of coordinates and definition of momenta as partial derivatives of the action. Last but not least, the problem of quantization of constrained systems often leads directly to different types of the phase space noncommutativity. Therefore, we think that phase space noncommutativity deserves systematic study.

The paper is divided into five sections and an Appendix. In the next section, we investigate a problem of linear transformations mapping noncommutative structures into commutative ones. This is also known as representation problem of noncommutative coordinates of NCQPS in terms of commutative ones [16-18]. We find that these transformations can be expressed in terms of two symmetric matrices *S* and *T*, which are computable analytically and numerically. In section 3, we give explicit representations of several models of NCQPS in low dimension and propose a matrix equation for numerical computation of explicit transformations in high dimension. Instead of making guesses, we derive new representations in 3D by analyzing the matrix equation containing *S* and *T*. One of our explicit formulas is a generalization of the formula obtained in isotropic case [16], while other formulas are new. In section 4, using standard perturbation method and the solutions obtained in the two previous sections, we compute the energy spectrum, up to the first (and second) order in noncommutative parameters, for hydrogen-like (H-like) atom in NCQPS. To our best knowledge, noncommutative (non-, and relativistic) H-like atom was studied in two very specific cases: (**A1**) in noncommutative phase space with $\sigma = -\frac{1}{4}(\theta \cdot \beta)$ [19]; and (**A2**) in noncommutative configuration space (i.e. $\beta = 0 = \sigma$) [20-22]. In this section, we perform detailed calculation of energy levels for a

[1] In literature, σ is assumed to be symmetric but we also consider a case of non-symmetric σ.

naive H-like atom in two different NCQPS with $\sigma = -\frac{1}{4}(\theta\cdot\beta)$ and $\sigma = 0$. The last is new based on the explicit representation found in section 3. As a result of these calculations, we find new relations between the hydrogen degenerate eigenfunctions corresponding to the same energy level. In section 5, we discuss the obtained results, limits of the used techniques and propose new problems. Finally in appendix A, we give a proof of new relations found in the section 4.

2. Linear transformations in Noncommutative Quantum Phase Space

Suppose that $(\hat{x}, \hat{p})$ are obtained from the canonical coordinates $(x,\ p)$ by

$$\begin{bmatrix} \hat{x} \\ \hat{p} \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix}. \tag{3}$$

where A, B, C, and D are real constant $n\times n$ matrices. Inserting Eq. (3) into Eq. (1) and using canonical relations Eq. (2), we get

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix}^T = \begin{bmatrix} \theta & \gamma \\ -\gamma^T & \beta \end{bmatrix}, \tag{4}$$

which can be expressed as the system of equations [16] for matrix elements of A, B, C, and D:

$$AB^T - BA^T = \theta, \tag{5a}$$

$$CD^T - DC^T = \beta, \tag{5b}$$

$$AD^T - BC^T = \gamma. \tag{5c}$$

Eqs. (5a)-(5c) form a system consisting of $2n^2 - n$ polynomial equations in $4n^2$ unknowns. The first two matrix equations (5a)-(5b) are solvable because they are reducible to a linear system, but the last one (5c) is *nonlinear* and its general solution is unknown for $n \geq 3$. We will present a simple method to find a solution for Eq. (5c) in a further publication.

If we require that phase space transformation is invertible, i.e. $\det\begin{bmatrix} A & B \\ C & D \end{bmatrix} \neq 0$, then it is easy to see that system (5) has such solutions if and only if $\det\begin{bmatrix} \theta & \gamma \\ -\gamma^T & \beta \end{bmatrix} \neq 0$.

The general solution for Eq. (5a) is of the form $AB^T = \frac{1}{2}(\theta + S)$, where S is a symmetric matrix. Similarly, the general solution for Eq. (5b) can be expressed as $CD^T = \frac{1}{2}(\beta + T)$, where T is a symmetric matrix. Consequently, B is a function of A and S, while C is a function of D and T:

$$B = \frac{1}{2}(S - \theta)(A^{-1})^T,\ C = \frac{1}{2}(\beta + T)(D^{-1})^T, \tag{6}$$

and we can represent the general solution for Eqs. (5) in terms of four matrices: A, D, S, and T, where the last two are symmetric. These matrices are related by Eq. (5c), or explicitly

$$AD^T - \frac{1}{4}(S-\theta)(DA^T)^{-1}(T-\beta) = I + \sigma. \tag{7}$$

We split the Eq. (7) into two equations:

$$Q^T - \frac{1}{4}(S-\theta)\cdot Q^{-1}\cdot(T-\beta) = I + \sigma, \quad \text{(8a)} \qquad \text{and} \qquad DA^T = Q. \quad \text{(8b)}$$

To solve Eq. (7), we first solve nonlinear Eq. (8a), next substitute Q into Eq. (8b), which obviously has infinite number of solutions, $A^T = D^{-1}Q$ for any given invertible matrix D. If we look for a particular solution Q=I, then Eq. (8a) simplifies to

$$(S-\theta)\cdot(T-\beta) = -4\sigma, \tag{9}$$

which consists of n^2 nonlinear equations in n^2+n unknowns S_{ij} and T_{ij} with $1 \le i \le j \le n$. Eq. (9) is analytically solvable for small n, and numerically solvable for every n.

3. Representations of special noncommutative structures

In this section, we use the technique presented in the previous section to find a variety of particular solutions of Eq. (5) in special cases with

Case 1: $\sigma = -\frac{1}{4}\theta\cdot\beta$, or $\sigma = -\frac{1}{4}(S_1-\theta)\cdot(S_1-\beta)$, where S_1 is a given symmetric matrix.

Case 2: $\sigma = 0$.

Now we study these cases in details.

Case 1:

In the case (A) the Eq. (9) becomes $(S-\theta)\cdot(T-\beta) = \theta\cdot\beta$, which has a particular solution $S = T = 0$. Therefore, the most simple solution is

$$A = D = I, B = -\frac{1}{2}\theta, \text{ and } C = \frac{1}{2}\beta. \tag{10}$$

In the case (B) the Eq. (9) becomes $(S-\theta)\cdot(T-\beta) = (S_1-\theta)\cdot(S_1-\beta)$, which has a particular solution $S = T = S_1$.

Case 2: $\sigma = 0$.

Notation: In 3D, instead of using matrices θ and β , we use the vectors $\vec{\theta}$ and $\vec{\beta}$ defined by $\theta_{ij} = \varepsilon_{ijk}\theta_k$, $\beta_{ij} = \varepsilon_{ijk}\beta_k$, $\vec{\theta} = (\theta_k)$, $\vec{\beta} = (\beta_k)$. In this notation, the matrices θ and β are

$$\theta = \begin{bmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{bmatrix}, \beta = \begin{bmatrix} 0 & \beta_3 & -\beta_2 \\ -\beta_3 & 0 & \beta_1 \\ \beta_2 & -\beta_1 & 0 \end{bmatrix}. \tag{11}$$

Let us introduce several new quantities

$$a=\theta_1\beta_1,\ b=\theta_2\beta_2,\ c=\theta_3\beta_3,\ \alpha=(a+b+c-1)^2-4abc. \tag{12}$$

Approach 2a. Solving Eq. (8a).

By requiring *S* and *T* to be off-diagonal and *Q* to be diagonal, then Eq. (8a) has four solutions, two of which correspond to *S* and *T* given below

$$S=\begin{bmatrix}0&\theta_3&\theta_2\\\theta_3&0&\theta_1\\\theta_2&\theta_1&0\end{bmatrix},\ T=-\begin{bmatrix}0&\beta_3&\beta_2\\\beta_3&0&\beta_1\\\beta_2&\beta_1&0\end{bmatrix}, \tag{13a}$$

$$A=I_3,\ B=\begin{bmatrix}0&0&\theta_2\\\theta_3&0&0\\0&\theta_1&0\end{bmatrix},\ C=-\begin{bmatrix}0&0&\frac{\beta_2}{q_3}\\\frac{\beta_3}{q_1}&0&0\\0&\frac{\beta_1}{q_2}&0\end{bmatrix},\ D=\begin{bmatrix}q_1&0&0\\0&q_2&0\\0&0&q_3\end{bmatrix}, \tag{13b}$$

$$\begin{aligned}&q_1^{\pm}=\frac{1-a-b-c+2bc\pm\sqrt{\alpha}}{1-a+b-c\pm\sqrt{\alpha}},\ q_2^{\pm}=\frac{-1-a+b+c\mp\sqrt{\alpha}}{2(-1+b)},\\&q_3^{\pm}=\frac{-1+a-b+c\mp\sqrt{\alpha}}{2(-1+c)},\end{aligned} \tag{13c}$$

and the other two correspond to:

$$S=-\begin{bmatrix}0&\theta_3&\theta_2\\\theta_3&0&\theta_1\\\theta_2&\theta_1&0\end{bmatrix},\ T=\begin{bmatrix}0&\beta_3&\beta_2\\\beta_3&0&\beta_1\\\beta_2&\beta_1&0\end{bmatrix}, \tag{14a}$$

$$A=I_3,\ B=-\begin{bmatrix}0&\theta_3&0\\0&0&\theta_1\\\theta_2&0&0\end{bmatrix},\ C=\begin{bmatrix}0&\frac{\beta_3}{q_2}&0\\0&0&\frac{\beta_1}{q_3}\\\frac{\beta_2}{q_1}&0&0\end{bmatrix},\ D=\begin{bmatrix}q_1&0&0\\0&q_2&0\\0&0&q_3\end{bmatrix}, \tag{14b}$$

$$\begin{aligned}&q_1^{\pm}=1-\frac{c(1+a-b-c\mp\sqrt{\alpha})}{1-a-b-c+2ac\mp\sqrt{\alpha}},\ q_2^{\pm}=\frac{1-a-b-c+2ac\mp\sqrt{\alpha}}{1+a-b-c\mp\sqrt{\alpha}},\\&q_3^{\pm}=\frac{-1-a+b+c\pm\sqrt{\alpha}}{2(-1+c)}\end{aligned} \tag{14c}$$

Approach 2b. Solving Eq. (9).

Alternatively, by requiring that $Q = I_3$ and S to be off-diagonal, we can find a lot of representations, one of which is described below

$$S_{12} = \frac{\theta_3(a+b+c)}{a+c}, \quad S_{23} = -\frac{\theta_1(\theta_1\beta_1 + \theta_2\beta_2 + \theta_3\beta_3)}{\theta_1\beta_1 + \theta_3\beta_3}, \tag{15a}$$

$$T_{11} = -\frac{2\theta_1\beta_2}{\theta_3}, \; T_{12} = \frac{2a+c}{\theta_3}, \; T_{13} = -\beta_2, \; T_{22} = -\frac{2\beta_1(a+c)}{\theta_3\beta_2}, \; T_{23} = \beta_1, \; T_{33} = 0, \tag{15b}$$

where matrices $A = D = I_3$, and matrices B and C are calculated from Eq. (6). These transformations with $A = D = I_3$ are interesting because noncommutative coordinate operators are obtained simply by $\hat{\vec{x}} = \vec{x} + B \cdot \vec{p}$ and $\hat{\vec{p}} = \vec{p} + C \cdot \vec{x}$.

4. Applications

Let us start with the general case of simple quantum mechanical systems with a Hamiltonian operator $H_0 = \frac{p^2}{2m} + V(x,p)$, where (x,p) satisfy Eq. (2). Suppose that in NCQPS, the Hamiltonian operator keeps its form. Then the change of Hamiltonian from commutative to noncommutative space is

$$\Delta H = H(\hat{x},\hat{p}) - H_0 = \Delta K + \Delta V, \text{ where } \Delta K = \frac{1}{2m}\left(\hat{p}^2 - p^2\right), \; \Delta V = V(\hat{x},\hat{p}) - V(x,p). \tag{16}$$

The perturbation ΔH modifies the energy eigenstates and shifts the energy levels of the quantum system. Furthermore, the perturbation $\Delta H = h(x, p, A, B, C, D)$ is a function of not only the phase space coordinate operators x and p, but also of auxiliary elements of the matrices A, B, C, and D. However, it can be shown that the corrections to energy levels depend only on noncommutative parameters.

For simplicity, we consider two NCQPS models of naive H-like atom, in which we disregard effects due to the spins of the nucleus or the electron. We regard H-like atom as one-particle system (electron) in an external Coulomb potential $V(r) = -\frac{1}{4\pi\varepsilon_0}\frac{Ze^2}{r}$ of the nucleus. Thus, commutative Hamiltonian of the naive H-like atom is $H_0 = \frac{p^2}{2m} + V(r)$, and its noncommutative counterpart defined by

$$H = \frac{\hat{p}^2}{2m_e} - \frac{Ze^2}{4\pi\varepsilon_0}\frac{1}{\hat{r}} = \frac{\hat{p}^2}{2m_e} - \frac{Z\hbar^2}{m_e a_0}\frac{1}{\hat{r}}, \text{where } a_0 = \frac{4\pi\varepsilon_0\hbar^2}{m_e e^2} \tag{17}$$

is the Bohr radius ($a_0 \approx 0.529\times 10^{-10} m$).

Now, let us discuss two simple cases: (A1) $\sigma = -\frac{1}{4}(\theta\cdot\beta)$ and (A2) $\sigma = 0$.

4.1. Case A1: $\sigma = -\frac{1}{4}(\theta\cdot\beta)$

In this case, the noncommutative structure is described by

$$\left[\hat{x}_j,\hat{x}_k\right]=i\hbar\theta_{jk},\ \left[\hat{p}_j,\hat{p}_k\right]=i\hbar\beta_{jk},\ \left[\hat{x}_j,\hat{p}_k\right]=i\hbar\left(\delta_{jk}-\frac{\theta_{ja}\beta_{ak}}{4}\right). \quad (18)$$

With solution described in Eq. (10), transformation Eq. (3) can be written in the form [19, 20, 21]

$$\hat{x}_j = x_j - \frac{1}{2}\theta_{jk}p_k,\ \hat{p}_j = p_j + \frac{1}{2}\beta_{jk}x_k. \quad (19)$$

In 3D, we can rewrite Eq. (1) by using the usual product of vectors

$$\hat{\vec{x}} = \vec{x} - \frac{1}{2}(\vec{p}\times\vec{\theta}),\ \hat{\vec{p}} = \vec{p} + \frac{1}{2}(\vec{x}\times\vec{\beta}). \quad (20)$$

Let us define $\vec{L}=\vec{x}\times\vec{p},\ U_1 = -(\vec{\theta}\cdot\vec{L}),\ U_2 = \frac{1}{4}\|\vec{p}\times\vec{\theta}\|^2$, and $U = U_1 + U_2$, then we have $\hat{r}^2 = r^2 + U$. Heuristically generalizing a Maclaurin series of a function, $(1+u)^{-\frac{1}{2}} = 1 - \frac{u}{2} + \frac{3}{8}u^2 + O(u^3)$, to a Maclaurin series of an operator, we get

$$\hat{r}^{-1} = r^{-1} - \frac{1}{2}r^{-\frac{3}{2}}\cdot U_1\cdot r^{-\frac{3}{2}} - \frac{1}{2}r^{-\frac{3}{2}}\cdot U_2\cdot r^{-\frac{3}{2}} + \frac{3}{8}r^{-\frac{3}{2}}\cdot U_1\cdot r^{-2}\cdot U_1\cdot r^{-\frac{3}{2}} + O(\theta^3). \quad (21)$$

Since $[\vec{\theta}\cdot\vec{L}, r^2] = 0$, the potential energy is shifted by

$$\Delta V = -\frac{Z\hbar^2}{m_e a_0}(\hat{r}^{-1} - r^{-1}) = \Delta V_1 + \Delta V_2, \text{ where} \quad (22a)$$

$$\Delta V_1 = -\frac{Z\hbar^2}{2m_e a_0}r^{-3}(\vec{\theta}\cdot\vec{L}),\quad \Delta V_2 = \frac{Z\hbar^2}{2m_e a_0}\left[\frac{1}{4}r^{-\frac{3}{2}}\|\vec{p}\times\vec{\theta}\|^2\, r^{-\frac{3}{2}} - \frac{3(\vec{\theta}\cdot\vec{L})^2}{4r^5}\right]. \quad (22b)$$

Thus, the first and second-order corrections in noncommutative parameters follow

$$\Delta H_1 = \Delta K_1 + \Delta V_1 = -\frac{1}{2m_e}\left(\vec{\beta} + \frac{Z\hbar^2}{a_0 r^3}\vec{\theta}\right)\cdot\vec{L}, \quad (23a)$$

$$\Delta H_2 = \Delta K_2 + \Delta V_2 = \frac{\|\vec{x}\times\vec{\beta}\|^2}{8m_e} + \Delta V_2. \quad (23b)$$

The matrix elements of L and $r^{-3}L$ can be easily calculated

$$\left\langle n',l',m'\right|L_z\left|n,l,m\right\rangle = m\hbar\,\delta_{m,m'}\,\delta_{l,l'}\delta_{n,n'}, \quad (24a)$$

$$\left\langle n',l',m'\right|L_x\left|n,l,m\right\rangle = \frac{\hbar}{2}[C_{l,m,m+1}\delta_{m+1,m'} + D_{l,m,m-1}\delta_{m-1,m'}]\delta_{l,l'}\delta_{n,n'}, \quad (24b)$$

$$\left\langle n',l',m'\right|L_y\left|n,l,m\right\rangle=\frac{\hbar}{2i}[C_{l,m,m+1}\delta_{m+1,m'}-D_{l,m,m-1}\delta_{m-1,m'}]\delta_{l,l'}\delta_{n,n'},\tag{24c}$$

where $C_{l,m,m+1}=\sqrt{(l-m)(l+m+1)}=D_{l,m+1,m}$, (24d)

and the matrix elements of r^{-3} follow [2]

$$\left\langle n',l',m'\right|r^{-3}\left|n,l,m\right\rangle=\begin{cases}2\left(Z\,a_0^{-1}\right)^3\dfrac{\delta_{n,n'}\delta_{l,l'}\delta_{m,m'}}{n^3l(l+1)(2l+1)} & \text{for } l^2+l'^2>0,\\ \\ \infty & \text{for } l=l'=0.\end{cases}\tag{25}$$

Thus

$$\Delta E^{(1)}_{n',l',m';n,l,m}=\langle n',l',m'\,|\,\Delta H_1\,|\,n,l,m\rangle=-\frac{1}{2}[E_\beta+E_\theta],\tag{26}$$

where E_β and E_θ, using Eqs. (24) and Eq. (25), follow

$$E_\beta=\left[C_{l,m,m+1}\beta^-\delta^{m'}_{m+1}+D_{l,m,m-1}\beta^+\delta^{m'}_{m-1}+2\beta_3 m\,\delta^{m'}_{m}\right]\frac{\hbar\delta_{l,l'}\delta_{n,n'}}{2m_e},\tag{27a}$$

$$E_\theta=\left[C_{l,m,m+1}\theta^-\delta^{m'}_{m+1}+D_{l,m,m-1}\theta^+\delta^{m'}_{m-1}+2\theta_3 m\,\delta^{m'}_{m}\right]\frac{2\hbar^2\left(Z\,a_0^{-1}\right)^4}{n^3l(l+1)(2l+1)}\frac{\hbar\delta_{l,l'}\delta_{n,n'}}{2m_e},\tag{27b}$$

where $\beta^\pm=\beta_1\pm i\beta_2$ and $\theta^\pm=\theta_1\pm i\theta_2$. (27c)

By using a right numeration of eigenstates, the matrix $\Delta E^{(1)}$ is of tridiagonal form. For the ground state n=1, the first order correction to its energy level is equal to zero because $\Delta E^{(1)}_{1,0,0;1,0,0}=0$. If we calculate the second order correction to the first energy level, then in accordance with the perturbation theory, we obtain

$$\Delta E_1^{(2)}=\sum_{n'>1,l,m}\frac{|\langle 1,0,0\,|\,\Delta H_1\,|\,n',l,m\rangle|^2}{E_1^{(0)}-E_{n'}^{(0)}}=0.\tag{28}$$

We conclude that, if we only consider ΔH_1, there are no first and second-order corrections to energy of the ground state. However, if we consider ΔH_2, then we obtain non-physical result, because of divergence of $(\Delta H_2)_{100;100}$. For the first excited state n=2, there are four states:

$$f_1=|\,2,0,0\rangle,\ f_2=|\,2,1,-1\rangle,\ f_3=|\,2,1,0\rangle,\ f_4=|\,2,1,1\rangle.\tag{29}$$

In order to calculate the first order correction to E_2, we have to solve the eigenvalue problem for 4×4 secular matrix:

$$G\vec{\alpha}=\lambda\vec{\alpha},\ \text{where}\ \vec{\alpha}=(\alpha_1,\alpha_2,\alpha_3,\alpha_4)^T,G=(g_{ij}),g_{ij}=\langle f_i\,|\,\Delta H_1\,|\,f_j\rangle,\ \text{and}\ \lambda=E_2^{(1)}-E_2^{(0)}.\tag{30}$$

[2] If we include the corrections of QED, the quantity $\left\langle n',l',m'\right|r^{-3}\left|n,l,m\right\rangle$ for l=l'=0, is finite.

Calculating g_{ij}, using Eqs. (23a), (26)-(27), we obtain

$$G=\begin{bmatrix}0&0&0&0\\0&g_{22}&g_{23}&0\\0&\overline{g_{23}}&0&g_{23}\\0&0&\overline{g_{23}}&-g_{22}\end{bmatrix},\text{ where }\quad \begin{aligned}g_{22}&=\frac{\hbar}{2m_e}\left[\beta_3+\frac{Z^4\hbar^2}{24a_0^4}\theta_3\right],\\ g_{23}&=\frac{-\hbar}{2\sqrt{2}m_e}\left[\beta^++\frac{Z^4\hbar^2}{24a_0^4}\theta^+\right].\end{aligned} \tag{31}$$

Therefore, $\Delta E_2^{(1)}$ are roots of the characteristic polynomial of G

$$p_G(\lambda)=\det(G-\lambda I)=\lambda^2[\lambda^2-(2|g_{23}|^2+g_{22}^2)]=0, \tag{32}$$

which has four roots:

$$\lambda_{1,2}=0,\text{ and }\ \lambda_{3,4}=\pm\sqrt{2|g_{23}|^2+g_{22}^2}=\pm\frac{\hbar}{2m_e}\left\|\vec{\beta}+\left(\frac{Z^4\hbar^2}{24a_0^4}\right)\vec{\theta}\right\|. \tag{33}$$

Thus two states, $|\psi_1\rangle=|2,0,0\rangle$ and $|\psi_2\rangle$ defined by

$$|\psi_2\rangle=\frac{|u_2\rangle}{\sqrt{\langle u_2|u_2\rangle}},\text{where }\ |u_2\rangle=-\frac{g_{23}}{g_{32}}f_2+\frac{g_{22}}{g_{32}}f_3+f_4, \tag{34}$$

are not shifted in first order.

Then, the degeneracy of the state E_2 is only partially lifted. Formula (33) gives us the shift of the first excited state corresponding to n=2 in first order perturbation theory

$$E_2^{(1)}=E_2^{(0)}\pm\frac{\hbar}{2m_e}\left\|\vec{\beta}+\left(\frac{Z^4\hbar^2}{24a_0^4}\right)\vec{\theta}\right\|. \tag{35}$$

The lower (or upper) of the two energies corresponds to the perturbed eigenstate $\phi_?$ (or ϕ_+ respectively) with $\psi_\mp=\frac{|u_\mp\rangle}{\sqrt{\langle u_\mp|u_\mp\rangle}}$ and

$$|u_\mp\rangle=\frac{g_{22}^2+|g_{23}|^2\mp g_{22}\sqrt{g_{22}^2+2|g_{23}|^2}}{g_{32}^2}f_2+\frac{g_{22}\mp\sqrt{g_{22}^2+2|g_{23}|^2}}{g_{32}}f_3+f_4. \tag{36}$$

Therefore, noncommutativity splits the fourfold degenerate level E_2 into three levels. One of these levels is twofold degenerate and the magnitude of the splitting of the levels is proportional to the norm $\|\vec{\beta}+\left(\frac{Z^4\hbar^2}{24a_0^4}\right)\vec{\theta}\|$. For the next excited state, n=3, there are nine states:

$$f_{l^2+l+m+1}=|3,l,m\rangle,\ 0\le l\le 2,\ -l\le m\le l. \tag{37}$$

Calculating elements of the secular matrix, which is hermitian $g_{ji}=\overline{g_{ij}}$, we get the following nonzero elements

$$g_{22}=\frac{\hbar}{2m_e}\left[\beta_3+\frac{Z^4\hbar^2}{81a_0^4}\theta_3\right],\ g_{23}=g_{34}=\frac{-\hbar}{2\sqrt{2}m_e}\left[\beta^++\frac{Z^4\hbar^2}{81a_0^4}\theta^+\right], \tag{38a}$$

$$g_{55} = \frac{\hbar}{m_e}\left[\beta_3 + \frac{Z^4\hbar^2}{405a_0^4}\theta_3\right],\ g_{56} = g_{89} = \frac{-\hbar}{2m_e}\left[\beta^+ + \frac{Z^4\hbar^2}{405a_0^4}\theta^+\right], \tag{38b}$$

$$g_{44} = -g_{22},\ g_{66} = -g_{88} = \frac{g_{55}}{2},\ g_{67} = g_{78} = \sqrt{\frac{3}{2}}g_{56},\ g_{99} = -g_{55}. \tag{38c}$$

The tridiagonal secular matrix G has three zero and six nonzero eigenvalues:

$$\lambda_{1,2,3} = 0,\ \lambda_{4,5} = \frac{\pm\hbar}{2m_e}\left\|\vec{\beta} + \frac{Z^4\hbar^2}{405a_0^4}\vec{\theta}\right\|,\ \lambda_{6,7} = 2\lambda_{4,5},\ \lambda_{8,9} = \frac{\pm\hbar}{2m_e}\left\|\vec{\beta} + \frac{Z^4\hbar^2}{81a_0^4}\vec{\theta}\right\|. \tag{39}$$

In summation, noncommutativity splits the ninefold degenerate level E_3 into seven levels. One of these levels is threefold degenerate and the magnitude of the splitting of the third level is proportional to either the norm $\| \vec{\beta} + \frac{Z^4\hbar^2}{405a_0^4}\vec{\theta} \|$, or the norm $\| \vec{\beta} + \frac{Z^4\hbar^2}{81a_0^4}\vec{\theta} \|$.

4.2. Case A2: $\sigma = 0$

In this case, the noncommutative structure is described by

$$\left[\hat{x}_j, \hat{x}_k\right] = i\hbar\theta_{jk},\ \left[\hat{x}_j, \hat{p}_k\right] = i\hbar\delta_{jk},\ \left[\hat{p}_j, \hat{p}_k\right] = i\hbar\beta_{jk}. \tag{40}$$

Now, we compute the NC correction of the Hamiltonian (16) using the transformation (13). In order to get the correct expansion of $\hat{r}^{-1}$ in a Taylor series, we need to take care of the order of all operators. First we note that

$$\hat{r}^2 = r^2 + 2U_1 + U_2,\quad U_1 = \theta_1 x_3 p_2 + \theta_2 x_1 p_3 + \theta_3 x_2 p_1,\quad U_2 = \theta_1^2 p_2^2 + \theta_2^2 p_3^2 + \theta_3^2 p_1^2. \tag{41}$$

It implies

$$\hat{r}^{-1} = r^{-1} - r^{-\frac{3}{2}}\cdot U_1 \cdot r^{-\frac{3}{2}} - \frac{1}{2}r^{-\frac{3}{2}}\cdot U_2 \cdot r^{-\frac{3}{2}} + \frac{3}{2}r^{-\frac{3}{2}}\cdot U_1 \cdot r^{-2}\cdot U_1 \cdot r^{-\frac{3}{2}} + O(\theta^3). \tag{42}$$

Therefore, $\Delta H = \Delta H_1^{(II)} + \Delta H_2^{(II)} + O(\beta^3, \theta^3)$, where the first and second-order corrections are

$$\Delta H_1^{(II)} = -\frac{1}{m_e}\left[\beta_1 x_2 p_3 + \beta_2 x_3 p_1 + \beta_3 x_1 p_2\right] + \frac{Z\hbar^2}{m_e a_0} r^{-\frac{3}{2}}\cdot U_1 \cdot r^{-\frac{3}{2}}, \tag{43a}$$

$$\Delta H_2^{(II)} = \frac{Z\hbar^2}{2m_e a_0}\left[r^{-\frac{3}{2}}U_2 r^{-\frac{3}{2}} - 3r^{-\frac{3}{2}}U_1 r^{-2}U_1 r^{-\frac{3}{2}}\right] + \frac{1}{m_e}\left[\frac{(\beta_1^2 x_2^2 + \beta_2^2 x_3^2 + \beta_3^2 x_1^2)}{2} - (bp_1^2 + cp_2^2 + ap_3^2)\right]. \tag{43b}$$

Since $[\Delta H_1^{(II)}, H_0] \neq 0$, which means that $\Delta H_1^{(II)}$ and H_0 do not have common eigenstates, the operators approach used in the previous case seems to be no longer applicable. Therefore, we temporarily switch to the analytical method. However, we will see that analytical tools are not enough to obtain first order corrections to all energy levels E_n and again algebraic techniques nicely show their applicability.

Using position representation of eigenstate corresponding to the ground state energy $|1,0,0\rangle = \psi_{100}(\vec{r}) = \frac{Z^3}{\pi a^3} e^{-Zr/a}$, the NC correction to the energy of the ground state can be calculated by direct integration,

$$\Delta E^{(1)}_{1,\sigma=0} = \left\langle 1,0,0 \middle| \Delta H_1^{(II)} \middle| 1,0,0 \right\rangle = 0 \quad (44)$$

For the first excited state n=2, in the position representation, there are four states:

$$f_1 = u_{2,0} Y_{0,0},\ f_2 = u_{2,1} Y_{1,0},\ f_3 = u_{2,1} Y_{1,1},\ f_4 = u_{2,1} Y_{1,-1},\ \text{where} \quad (45a)$$

$$u_{2,0} = 2\left(\frac{Z}{2a}\right)^{\frac{3}{2}} \left[1 - \frac{Zr}{2a}\right] \exp\left(-\frac{Zr}{2a}\right),\ u_{2,1} = \frac{1}{\sqrt{3}} \left(\frac{Z}{2a}\right)^{\frac{3}{2}} \left[\frac{Zr}{a}\right] \exp\left(-\frac{Zr}{2a}\right), \quad (45b)$$

$$Y_{0,0} = \left(\frac{1}{4\pi}\right)^{\frac{1}{2}},\ Y_{1,0} = \left(\frac{3}{4\pi}\right)^{\frac{1}{2}} \cos\Theta,\ Y_{1,1} = -\sqrt{\frac{3}{8\pi}} \sin\Theta e^{i\varphi},\ Y_{1,-1} = -\bar{Y}_{1,1}. \quad (45c)$$

Calculating elements of the secular matrix $g_{ij} = \langle f_i \,|\, \Delta H_1^{(II)} \,|\, f_j \rangle$ by direct integration, we get the same matrix *g* and subsequently same energy corrections to E_2, as in Eqs. (31) and (33). Also for the next excited state *n*=3, we obtain the same result as in Eq. (39). However, performance of integration seems to be impractical for big *n*. Therefore, using integration techniques, we can show only that first order energy corrections to E_n in two cases, $\sigma = 0$ and $\sigma = -\frac{1}{4}\,(\theta \cdot \beta)$, coincide for small *n*. To deal with big *n*, we need another indirect approach. In general, if we want to show that the first order energy corrections to E_n in two cases coincide for all *n*, we must prove the following

Proposition 1. For ΔH_1 defined in Eq. (4.1) and $\Delta H_1^{(II)}$ defined in Eq. (43a), we have

$$\langle n,l',m' \big| \Delta H_1^{(II)} - \Delta H_1 \big| n,l,m \rangle = 0. \quad (46)$$

Contrary to a naive thought of inapplicability of the operators approach, we are able to prove this identity by using only algebraic techniques. The detailed proof is presented in appendix A. This proposition and its generalization seem to be potentially applicable to other calculations.

5. Conclusion

In this paper, through extensive analysis of solutions of a system of complicated nonlinear algebraic equations, we have found new explicit representations of noncommutative quantum phase spaces. This opens new possibilities for analytical and perturbative quantum calculations in spaces with noncommutative structures other than the very special and explored case with $\sigma = -\frac{1}{4}\,(\theta \cdot \beta)$ [19], or very simple case with zero-beta and zero-sigma (i.e. only spatial noncommutativity) which has been studied extensively [6, 20-22].

As an example of applications, we have studied the spectral problem of the naive H-like atom living in NCQPS with $\sigma = 0$. As a result, phase space noncommutativity changes excited-state energy level E_n by an amount proportional to the norm of the vector which is a linear combination of noncommutative vectors, i.e. $\| \vec{\beta} + k(n)\vec{\theta} \|$, and makes no correction to the ground-state energy level. Moreover, we find no difference between energy levels of H-like atom in two noncommutative phase spaces corresponding to $\sigma = -\frac{1}{4}(\theta \cdot \beta)$ and $\sigma = 0$. Noncommutativity in considered models partially lift the degeneracy. It splits the fourfold degenerate level E_2 and ninefold degenerate level E_3 into three and seven levels respectively. In this paper, for simplicity, we neglect the spin of electron and nucleus in the naive model, but it is straightforward to gain the results for more realistic models including the spin and relativistic effects, or fully relativistic model. If we include the spin as well as relativistic corrections, the spectrum of this naive system splits into further lines (i.e. NCQPS hyperfine structure). For heuristic reasons, we will present relativistic quantum effects appearing in different models of NCQPS in a further publication as the extension of Ref. [21]. Let us note that assuming a particular type of NCQPS, based on data of spectroscopy, one can estimate the upper bounds for noncommutative parameters, see Ref. [20] for three-parameter model or here for six-parameter model. However, if we consider a full NCQPS model in 3D with fifteen noncommutative parameters ($\theta_i, \beta_j, \sigma_{kl}$), the estimation for the upper bounds of noncommutativity needs to be explained. There remain several problems which require further investigation, such as (a) find an explicit representation of 3D NCQPS with nonzero sigma (which we will present in a further publication) and (b) calculate energy corrections of second order in noncommutative parameters.

Finally, we believe that detailed calculation for different noncommutative models outlined here might lead to a better understanding of quantum systems in both commutative and noncommutative phase spaces and might nicely serve as a case-study on noncommutative models. In-depth investigation of the case $\sigma = 0$ leads to the formula (46) or (47), which seems to be new for the H-like atom.

Additional comments on the physical applications:

First, at atomic or macroscopic scales, the parameters θ_{ij} or β_{ij} admit close analogies with a constant magnetic field both from the algebraic and dynamical viewpoints. Indeed, for a free charged particle, the coupling of the particle to a magnetic field can be described elegantly by replacing the canonical momentum p in the free Hamiltonian $H = \frac{1}{2m}\left(\vec{p} - \frac{e}{c}\vec{A}\right)^2$ by the kinematical momentum $\vec{\Omega} = \left(\vec{p} - \frac{e}{c}\vec{A}\right)$, whose components have a non-vanishing commutators: $\left[\hat{x}_j, \hat{x}_k\right] = 0$, $\left[\Omega_j, \Omega_k\right] = i\hbar\frac{e}{c}B_{jk}$, $\left[\hat{x}_j, \Omega_k\right] = i\hbar\delta_{jk}$, where $B_{jk} = \partial_j A_k - \partial_k A_j$.

Second, since a cellular structure in configuration space is not observed at macroscopic scales, the noncommutativity parameters θ_{ij} should manifest themselves at a length scale which is small compared to some basic length scale like the Planck length $\sqrt{\hbar G / c^3}$. In quantum field theory, we often write $\theta_{ij} = \frac{1}{\Lambda_{nc}^2} \tilde{\theta}_{ij}$, where the $\tilde{\theta}_{ij}$ are dimensionless and of order 1 and Λ_{nc} represents a characteristic energy scale for the noncommutative theory which is necessarily quite large. Thus, noncommutativity of space should be related to quantum gravity at very short distances and NCQM may be regarded as a deformation of classical mechanics that is independent of the deformation by quantization.

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Appendix A. Proof for the proposition 1

In this Appendix, we prove the equivalence in first order approximation of the two considered models. In other words, we are going to prove the proposition (46), or emphasize it explicitly

$$\left\langle n,l',m'\left|\frac{1}{m_e}[\beta_1 x_2 p_3+\beta_2 x_3 p_1+\beta_3 x_1 p_2]-\frac{Z\hbar^2}{m_e a_0}r^{-\frac{3}{2}}U_1 r^{-\frac{3}{2}}\right|n,l,m\right\rangle$$
$$=\left\langle n,l',m'\left|\frac{1}{2m_e}\left(\vec{\beta}+\frac{Z\hbar^2}{a_0 r^3}\vec{\theta}\right)\cdot\vec{L}\right|n,l,m\right\rangle, \qquad (47)$$

for U_1 defined in Eq. (41). In order to make the proof more transparent and useful for other quantum mechanical calculations, we formulate and prove two following lemmas.

Lemma 1 a) Matrix elements of $x_j p_k + x_k p_j$ corresponding to the same energy level are zero

$$0=\langle n,l',m'\,|\,x_2p_3+x_3p_2\,|\,n,l,m\rangle=\langle n,l',m'\,|\,x_3p_1+x_1p_3\,|\,n,l,m\rangle=\langle n,l',m'\,|\,x_1p_2+x_2p_1\,|\,n,l,m\rangle. \quad (48)$$

b) Matrix elements of $(\beta_1 x_2 p_3+\beta_2 x_3 p_1+\beta_3 x_1 p_2)$ corresponding to the same energy level are algebraically computable

$$\langle n,l',m'\,|\,\beta_1 x_2 p_3+\beta_2 x_3 p_1+\beta_3 x_1 p_2\,|\,n,l,m\rangle=\left\langle n,l',m'\left|\frac{1}{2}\left(\vec{\beta}\cdot\vec{L}\right)\right|n,l,m\right\rangle. \qquad (49)$$

Proof. Since $x_j p_k + x_k p_j = \frac{1}{2}[x_j x_k, p^2] = m_e[x_j x_k, H]$, we deduce

$$\langle n',l',m'\,|\,x_j p_k + x_k p_j\,|\,n,l,m\rangle = m_e(E_n - E_{n'})\langle n',l',m'\,|\,x_j x_k\,|\,n,l,m\rangle. \qquad (50)$$

Putting $n'=n$, we obtain Eq. (48). Next, using $x_2p_3=\frac{1}{2}(x_2p_3-x_3p_2)+\frac{1}{2}(x_2p_3+x_3p_2)$ and cyclic identities, i.e. $x_3p_1=\frac{1}{2}(x_3p_1-x_1p_3)+\frac{1}{2}(x_1p_3+x_3p_1)$ and $x_1p_2=\frac{1}{2}(x_1p_2-x_2p_1)+\frac{1}{2}(x_1p_2+x_2p_1)$

Together with Eq. (48), we deduce Eq. (49). Its algebraic computability is a consequence of Eqs. (26)-(27).

Lemma 2 a) Matrix elements of $r^{-\frac{3}{2}}(x_j p_k + x_k p_j)r^{-\frac{3}{2}}$ corresponding to the same energy level are zero

$$\langle n,l',m'\,|\,r^{-\frac{3}{2}}(x_2p_3+x_3p_2)r^{-\frac{3}{2}}\,|\,n,l,m\rangle=\langle n,l',m'\,|\,r^{-\frac{3}{2}}(x_3p_1+x_1p_3)r^{-\frac{3}{2}}\,|\,n,l,m\rangle$$
$$=\langle n,l',m'\,|\,r^{-\frac{3}{2}}(x_1p_2+x_2p_1)r^{-\frac{3}{2}}\,|\,n,l,m\rangle=0. \qquad (51)$$

b) Matrix elements of $r^{-\frac{3}{2}}\cdot U_1\cdot r^{-\frac{3}{2}}$ corresponding to the same energy level are algebraically computable

$$\langle n,l',m'\left|r^{-\frac{3}{2}}U_1 r^{-\frac{3}{2}}\right|n,l,m\rangle=\left\langle n,l',m'\left|-\frac{1}{2r^3}\left(\vec{\theta}\cdot\vec{L}\right)\right|n,l,m\right\rangle. \quad (52)$$

Proof. Again using

$$r^{-\frac{3}{2}}(x_j p_k+x_k p_j)r^{-\frac{3}{2}}=m_e r^{-\frac{3}{2}}[x_j x_k,H]r^{-\frac{3}{2}} \text{ and } H(r^{-\frac{3}{2}})=-\frac{3\hbar^2}{8m_e}r^{-\frac{7}{2}}-\frac{Z\hbar^2}{m_e a_0}r^{-\frac{5}{2}}=f(r), \quad (53)$$

we deduce

$$\begin{aligned}&\langle n',l',m'\,|\,r^{-\frac{3}{2}}(x_j p_k+x_k p_j)r^{-\frac{3}{2}}\,|\,n,l,m\rangle=m_e\langle n',l',m'\,|\,r^{-\frac{3}{2}}[x_j x_k,H]r^{-\frac{3}{2}}\,|\,n,l,m\rangle\\&=m_e(E_n-E_{n'})\left\langle n',l',m'\left|\frac{x_j x_k}{r^3}\right|n,l,m\right\rangle.\end{aligned} \quad (54)$$

Putting $n'=n$, we obtain Eq. (51).

Similarly, using $x_3p_2=-\frac{1}{2}(x_2p_3-x_3p_2)+\frac{1}{2}(x_2p_3+x_3p_2)$ and cyclic identities, i.e.

$x_1p_3=-\frac{1}{2}(x_3p_1-x_1p_3)+\frac{1}{2}(x_1p_3+x_3p_1)$ and $x_2p_1=-\frac{1}{2}(x_1p_2-x_2p_1)+\frac{1}{2}(x_1p_2+x_2p_1)$

Together with Eq. (51), we deduce Eq. (52). Its algebraic computability is a consequence of Eqs. (26)-(27).