

On The Convergence of Stability Domain on Time Scales

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Abstract: This paper studies convergence of the stability domains for a sequence of time scales. It is proved that if the sequence of time scales (\mathbb{T}_n) converges to a time scale \mathbb{T} in Hausdorff topology then their stability domains $\mathcal{U}\mathbb{T}_n$ will converge to the stability domain $\mathcal{U}\mathbb{T}$ of \mathbb{T} .

Keywords: Implicit dynamic equations, time scales, convergence, stability domain, stability radius.

1. Introduction

It is known that the linear system $\dot{x}(t) = Ax(t); t \in \mathbb{R}^+ = [0, \infty)$ is exponentially stable if and only if the spectrum $\sigma(A)$ lies within the half plan $\subset \mathbb{C}_-$ of the complex numbers \mathbb{C} . Also, if we consider the different system $x_{n+1} = x_n + hAx_n, n \in h\mathbb{Z}_+$, then this system is exponentially stable $\Leftrightarrow \sigma(A)$ lies in the disk $\mathcal{U}\mathbb{T}_h := \{z : |z + \frac{1}{h}| < \frac{1}{h}\}$. Where $h > 0$ and $h\mathbb{Z}_+^+ = \{0, h, 2h, \dots\}$.

In view of theory of time scales, the sets $\mathbb{R}^+, \mathbb{Z}_h^+$ are time scales and \mathbb{C}_- and $\mathcal{U}\mathbb{T}_h$ are respectively their stability domains. Further, when $h \rightarrow 0$, the sequence of time scales (\mathbb{Z}_h^+) is “close” to \mathbb{R}^+ and the set of the disks $\mathcal{U}\mathbb{T}_h$ will “enlarge” to the set \mathbb{C}_- in some sense. Indeed, the Hausdorff distance $d(\mathbb{R}^+, \mathbb{Z}_h^+) = \sup\{d(x, \mathbb{Z}_h^+) : x \in \mathbb{R}^+\}$ of \mathbb{R}^+ and \mathbb{Z}_h^+ is h and $\bigcup_{h>0} \mathcal{U}\mathbb{T}_h = \mathbb{R}^+$.

The question rises here if we can generalize this idea to an arbitrary set of time scales (\mathbb{T}_n) ? That is if the sequence (\mathbb{T}_n) tends to the time scale \mathbb{T} in Hausdorff topology, can we conclude that their respective stability domain converges to one of \mathbb{T} .

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This paper concerns with such problem. Firstly, we consider the continuous dependence of the exponential functions to time scales. Then, we show that the stability domain $\mathcal{U}\mathbb{T}_n$ corresponding to the time scale \mathbb{T}_n converges when \mathbb{T}_n tends to \mathbb{T} .

The paper is organized as follows. Section 2 summarizes some preliminary results on time scales, property of exponential functions on time scales and characterizes the stability domain of a time scale. The main results of the paper are derived in Section 3. We study the convergence of the stability domains here. The last section deals with some conclusions and open problems.

2. Preliminaries

2.1. Time scales

Let \mathbb{T} be a closed subset of \mathbb{R} , enclosed with the topology inherited from the standard topology on \mathbb{R} . Let $\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}$, $\mu(t) = \sigma(t) - t$ and $\rho(t) = \sup\{s \in \mathbb{T} : s < t\}$, $\nu(t) = t - \rho(t)$ (supplemented by $\sup\emptyset = \inf\mathbb{T}$, $\inf\emptyset = \sup\mathbb{T}$). A point $t \in \mathbb{T}$ is said to be *right-dense* if $\sigma(t) = t$, *right-scattered* if $\sigma(t) > t$, *left-dense* if $\rho(t) = t$, *left-scattered* if $\rho(t) < t$ and *isolated* if t is simultaneously right-scattered and left-scattered. A function f defined on \mathbb{T} is *regulated* if there exist the left-sided limit at every left-dense point and right-sided limit at every right-dense point. A regulated function is called *rd-continuous* if it is continuous at every right-dense point, and *ld-continuous* if it is continuous at every left-dense point. It is easy to see that a function is continuous if and only if it is both rd-continuous and ld-continuous. A function f from \mathbb{T} to \mathbb{R} is *positively regressive* if $1 + \mu(t)f(t) > 0$ for every $t \in \mathbb{T}$. We denote by \mathcal{R}^+ the set of positively regressive functions from \mathbb{T} to \mathbb{R} .

Definition 2.1 (Delta Derivative). A function $f : \mathbb{T} \rightarrow \mathbb{R}^d$ is called delta differentiable at t if there exists a vector $f^\Delta(t)$ such that for all $\epsilon > 0$

$$\|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\| \leq \epsilon |\sigma(t) - s|$$

for all $s \in (t - \delta, t + \delta) \cap \mathbb{T}$ and for some $\delta > 0$. The vector $f^\Delta(t)$ is called the delta derivative of f at t .

If $\mathbb{T} = \mathbb{R}$ then the delta derivative is $f'(t)$ from continuous calculus; if $\mathbb{T} = \mathbb{Z}$ then the delta derivative is the forward difference, $\Delta f(t) = f(t+1) - f(t)$, from discrete calculus.

Let f be a rd-continuous function and $a, b \in \mathbb{T}$. Then, the Riemann integral $\int_a^b f(s) \Delta_{\mathbb{T}} s$ exists (see [1]). In case $a, b \notin \mathbb{T}$, writing $\int_a^b f(s) \Delta_{\mathbb{T}} s$ means $\int_{\bar{a}}^{\bar{b}} f(s) \Delta_{\mathbb{T}} s$, where $\bar{a} = \min\{t > a : t \in \mathbb{T}\}$; $\bar{b} = \max\{t < b : t \in \mathbb{T}\}$. If there is no confusion, we write simply $\int_a^b f(s) \Delta s$ (resp. $\int_a^b f(s) \Delta_n s$) for $\int_a^b f(s) \Delta_{\mathbb{T}} s$ (resp. $\int_a^b f(s) \Delta_{\mathbb{T}_n} s$).

Fix $t_0 \in \mathbb{R}$. Let \mathcal{T} be the set of all time scales with bounded graininess such that $t_0 \in \mathbb{T}$ for all $\mathbb{T} \in \mathcal{T}$. We endow \mathcal{T} with the Hausdorff distance, that is Hausdorff distance between two time scales \mathbb{T}_1 and \mathbb{T}_2 is defined by

$$d_H(\mathbb{T}_1, \mathbb{T}_2) := \max \left\{ \sup_{t_1 \in \mathbb{T}_1} d(t_1, \mathbb{T}_2), \sup_{t_2 \in \mathbb{T}_2} d(t_2, \mathbb{T}_1) \right\}, \tag{2.1}$$

where

$$d(t_1, \mathbb{T}_2) = \inf_{t_2 \in \mathbb{T}_2} |t_1 - t_2| \text{ and } d(t_2, \mathbb{T}_1) = \inf_{t_1 \in \mathbb{T}_1} |t_2 - t_1|.$$

For properties of the Hausdorff distance, we refer the interested readers to [2, 3].

2.2. Exponential Function

Let \mathbb{T} be an unbounded above time scale, that is $\sup \mathbb{T} = \infty$.

Definition 2.2 (Exponential stability). Let $p: \mathbb{T} \rightarrow \mathbb{R}$ is regressive, we define the exponential function by

$$e_p(t, t_0) = \exp \left\{ \int_{t_0}^t \lim_{h \rightarrow \mu(s)} \frac{\text{Ln}(1 + hp(s))}{h} \Delta s \right\},$$

where $\text{Ln} a$ is the principal logarithm of the number a .

Theorem 2.3 (see [4]). If p is regressive and $t_0 \in \mathbb{T}$, then $e_p(\cdot, t_0)$ is a unique solution of the initial value problem

$$y^\Delta(t) = p(t)y(t), y(t_0) = 1.$$

When $p(t) = \lambda$, where λ is a constant in \mathbb{C} , we write $e_\lambda(t, s)$ for $e_{p(\cdot)}(t, s)$

Theorem 2.4 (Properties of the Exponential Function). If $p, q: \mathbb{T} \rightarrow \mathbb{R}$ are regressive, rd-continuous functions and $t, r, s \in \mathbb{T}$ then the following hold:

1. $e_0(t, s) = 1$, and $e_p(t, t) = 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
3. $\frac{1}{e_p(t, s)} = e_{-p}(t, s) = e_p(s, t)$
4. $e_p(t, s)e_q(t, s) = e_{p+q}(t, s)$
5. $e_p(t, s)e_p(s, r) = e_p(t, r)$
6. $\frac{e_p(t, s)}{e_q(t, s)} = e_{p-q}(t, s)$

Lemma 2.5 (Gronwall-Bellman lemma, see [4]). Let $f(t)$ be a positive continuous function and $k > 0, f(t_0) \in \mathbb{R}$. Assume that $f(t)$ satisfies the inequality

$$f(t) \leq f_0 + k \int_{t_0}^t f(s) \Delta s, \text{ for all } t \in \mathbb{T}, t \geq t_0. \tag{2.2}$$

Then, the following relation holds,

$$f(t) \leq f_0 \cdot e_{\lambda}(t, t_0), \text{ for all } t \in \mathbb{T}, t \geq t_0. \quad (2.3)$$

We see that

$$|e_{\lambda}(t, t_0)| = \exp \left\{ \int_{t_0}^t \lim_{h \succ \mu(s)} \frac{\ln |1 + h\lambda|}{h} \Delta s \right\}.$$

By using the notation

$$\zeta_{\lambda}(s) = \lim_{h \succ s} \frac{\ln |1 + h\lambda|}{h} = \begin{cases} \Re \lambda & \text{if } s = 0 \\ \frac{\ln |1 + s\lambda|}{s} & \text{if } s \neq 0, \end{cases}$$

we can rewrite

$$|e_{\lambda}(t, t_0)| = \exp \left\{ \int_{t_0}^t \zeta_{\lambda}(\mu(s)) \Delta s \right\}. \quad (2.4)$$

We note that $|e_{\lambda}(t, s)| = |e_{\bar{\lambda}}(t, s)|$ for any $\lambda \in \mathbb{C}$. Further, it is easy to see that $\zeta_{\lambda}(x) \leq |\lambda|$ for all $x \geq 0$. For the properties of exponential function $e_{\lambda}(t, s)$ the interested readers can refer to [5].

2.3. Exponential stability

Let $\mathbb{T}_{t_0} = \{t \in \mathbb{T} : t \geq t_0\}$. Consider a dynamic equation

$$x^{\Delta} = f(t, x), \quad t \geq t_0. \quad (2.5)$$

We assume that the function $f : \mathbb{T}_{t_0} \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ satisfies conditions such that Equation (2.5) has a unique solution $x(t, s, x_0), t \geq s$ with the initial condition $x(s, s, x_0) = x_0$ for any $s \in \mathbb{T}_{t_0}$ and $x_0 \in \mathbb{R}^m$.

Definition 2.1 (Exponential stability). The dynamic equation (2.5) is called uniformly exponentially stable if there exist constants $\alpha > 0$ with $-\alpha \in \mathcal{R}^+$ and $K > 0$ such that for every $s < t, s, t \in \mathbb{T}_{t_0}$, the inequality

$$\|x(t, s, x_0)\| \leq K \|x_0\| e_{-\alpha}(t, s) \quad (2.6)$$

holds for any $x_0 \in \mathbb{R}^m$.

In the linear homogeneous case, i.e., $f(t, x) = Ax$ we have the equation

$$x^{\Delta}(t) = Ax(t), \quad t \geq t_0. \quad (2.7)$$

It is known that Equation (2.7) is uniformly exponentially stable if and only if so is for the scalar equation

$$x^{\Delta}(t) = \lambda x(t), \quad t \geq t_0, \quad (2.8)$$

for any $\lambda \in \sigma(A)$ (see [6, 7]).

Denote by $\mathcal{U}\mathbb{T}$ the set of complex values λ such that (2.8) is uniformly exponentially stable. We call $\mathcal{U}\mathbb{T}$ the domain of uniform exponential stability (or stability domain for short) of the time scale \mathbb{T} . Denote

$$L(\lambda) := \limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau < 0. \tag{2.9}$$

Propotion 2.7. Let $\lambda \in \mathbb{C}$, then $\lambda \in \mathcal{U}\mathbb{T}$ if and only if

$$L(\lambda) := \limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(s)) \Delta s < 0. \tag{2.10}$$

Proof. Assume that (2.7) is uniformly exponentially stable. It implies that there exist constants $\alpha > 0, K > 0$ such that

$$|e_\lambda(t, s)| \leq \exp \left\{ \int_s^t \lim_{h \searrow \mu(\tau)} \frac{\ln |1 + h\lambda|}{h} \Delta \tau \right\} \leq Ke^{-\alpha(t-s)}, \text{ for all } t \geq s.$$

$$\int_s^t \lim_{h \searrow \mu(\tau)} \frac{\ln |1 + h\lambda|}{h} \Delta \tau \leq -\alpha(t-s) + \ln K, \text{ for all } t \geq s.$$

Therefore, we have

$$\limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(s)) \Delta s < -\alpha < 0.$$

Assume that $L(\lambda) := -\alpha < 0$. Then there is an integer number N large enough such that

$$\frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(s)) \Delta s < \frac{-\alpha}{2}, \text{ for all } t-s > N.$$

Thus $|e_\lambda(t, s)| < |e^{\frac{-\alpha}{2}(t-s)}|, \forall t-s > N$.

By virtue of the inequality $\zeta_\lambda(x) \leq \lambda |$ for any $x \geq 0$, we have

$$|e_\lambda(t, s)| \leq \exp \left\{ \int_s^t \zeta_\lambda(\mu(s)) \Delta s \right\} \leq e^{|\lambda|N}, \forall t-s < N.$$

Hence

$$|e_\lambda(t, s)| \leq Ke^{\frac{-\alpha(t-s)}{2}}, \forall t \geq s.$$

with $K := e^{|\lambda|N + \frac{\alpha N}{2}} > 1$. The proposition is proved. □

Further, it is easy to verify that if $|b_1| \geq |b_2|$ then $\zeta_{a+ib_1}(x) \geq \zeta_{a+ib_2}(x)$ for any $a \in \mathbb{R}$ and $x \geq 0$. So $\mathcal{U}\mathbb{T}$ is symmetric with respect to the real axis of the complex plan \mathbb{C} . This means that $\lambda \in \mathcal{U}\mathbb{T}$ implies the segment $[\lambda, \bar{\lambda}] \subset \mathcal{U}\mathbb{T}$.

Propotion 2.8. Let \mathbb{T} be a time scale with bounded graininess, then stability domain $\mathcal{U}\mathbb{T}$ is an open set in \mathbb{C} .

Proof. Let $\lambda \in \mathcal{U}\mathbb{T}$, then there are $K > 0$ and $\lambda \in \mathcal{R}^+$ such that

$$|e_\lambda(t, s)| \leq Ke^{-\alpha(t-s)}, \text{ for all } t \geq s. \tag{2.11}$$

We now proof that there exists $\epsilon > 0$ such that the equation

$$x^\Delta = \beta x \tag{2.12}$$

is also uniformly exponentially stable for all $\beta \in \mathbb{C}$ with $|\beta - \lambda| \leq \epsilon$. We rewrite Equation (2.12) under the form $x^\Delta = \lambda x + (\beta - \lambda)x$. Using variation of constants formula yields

$$e_\beta(t, s) = e_\lambda(t, s) + \int_s^t e_\lambda(t, \sigma(u))(\beta - \lambda)e_\beta(u, s)\Delta u. \quad (2.13)$$

This implies that

$$|e_\beta(t, s)| \leq Ke^{-\alpha(t-s)} + K|\beta - \lambda| \int_s^t |e_\lambda(t, \sigma(u))e_\beta(u, s)| \Delta u.$$

Hence,

$$e^{\alpha(t-s)} |e_\beta(t, s)| \leq K + K|\beta - \lambda| \int_s^t e^{\alpha u(u)} e^{\alpha(u-s)} |e_\beta(u, s)| \Delta u.$$

Let $f(t) = e^{\alpha(t-s)} |e_\beta(t, s)|$, we have

$$f(t) \leq K + K|\beta - \lambda| e^{\alpha H} \int_s^t f(u) \Delta u.$$

Using Gronwall's inequality (with $f(s) = 1$) obtains

$$f(t) \leq Ke^M(t, s), \text{ for all } t \geq s,$$

where $M = K|\beta - \lambda| e^{\alpha H}$. Thus

$$|e_\beta(t, s)| \leq Ke^{(-\alpha+M)(t-s)}, \text{ for all } t \geq s.$$

By choosing $\epsilon = \frac{\alpha}{2Ke^{\alpha H}}$ we get that $|e_\beta(t, s)| \leq Ke^{-\frac{\alpha}{2}(t-s)}$, for all $t \geq s$, for any β such that $|\beta - \lambda| \leq \epsilon$. The proof is complete. \square

3. Main results

In this section, we consider a sequence $\{\mathbb{T}_n\}_{n \in \mathbb{N}} \subset \mathcal{T}$ of time scales satisfying:

$$\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}.$$

Denote by $\mu_n(t)$ (resp. $\mu(t)$) the graininess of \mathbb{T}_n (resp. \mathbb{T}) at time t . Since $\mathbb{T} \in \mathcal{T}$, $\sup\{\mu(t) : t \in \mathbb{T}\} < \infty$. Therefore, it is easy to prove that if $\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}$ then $\sup\{\mu_n(t) : t \in \mathbb{T}_n, n \in \mathbb{N}\} < \infty$, $\{\sup\{\mu(t) : t \in \mathbb{T}, n \in \mathbb{N}\} < \infty$. Denote

$$\mu^* = \max\{\sup\{\mu(t) : t \in \mathbb{T}\}, \sup\{\mu_n(t) : t \in \mathbb{T}_n, n \in \mathbb{N}\}\}.$$

First, we need the following lemmas to derive some characteristics of stability domains $\mathcal{U}\mathbb{T}_n$ when \mathbb{T}_n tends to \mathbb{T} .

Lemma 3.1. For any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ we have

$$L(\lambda) \leq 0 \quad \text{if and only if } \lambda \in \overline{\mathcal{U}\mathbb{T}}.$$

Proof. Denote $\mu_* = \sup\{\mu(t) : t \in \mathbb{T}\}$ and let $\lambda \in \overline{\mathcal{UT}} \setminus \mathbb{R}$. Then, there is a sequence $\{\lambda_n\} \subset \mathcal{UT}$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$. Let $\lambda = a + ib$ with $b \neq 0$ and $\lambda_n = a_n + ib_n$. Using Lagrange finite increments formula, for all $x > 0$ we have

$$\zeta_{\lambda_n}(x) - \zeta_\lambda(x) = \frac{x(|\lambda_n|^2 - |\lambda|^2) + 2(a_n - a)}{2(1 + 2x(a + \theta(a_n - a)) + x^2(|\lambda|^2 + \theta(|\lambda_n|^2 - |\lambda|^2)))}, \theta \in (0, 1).$$

Since $0 < 1 + 2xa + x^2|\lambda|^2$ for all $x \geq 0$, we can choose a $n_0 \in \mathbb{N}$ and a constant $c_1 > 0$ such that $c_1 < 1 + 2x(a + \theta(a_n - a)) + x^2(|\lambda|^2 + \theta(|\lambda_n|^2 - |\lambda|^2))$ for all $0 \leq x \leq \mu_*$ and $n > n_0$. Thus, for any $\epsilon > 0$, there exists $n_1 > n_0$ satisfying

$$\zeta_\lambda(x) - \zeta_{\lambda_n}(x) < \epsilon, \forall 0 \leq x \leq \mu_*, \forall n > n_1.$$

This implies that

$$\int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau < \int_s^t \zeta_{\lambda_n}(\mu(\tau)) \Delta \tau + \epsilon(t - s) < \epsilon(t - s), \quad \forall t_0 \leq s \leq t, \forall n > n_1.$$

Hence

$$\limsup_{t \rightarrow \infty} \frac{1}{t - s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau < \epsilon, \forall \epsilon > 0.$$

Thus

$$\limsup_{t \rightarrow \infty} \frac{1}{t - s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau \leq 0.$$

Conversely, let $\lambda = a + ib \in \mathbb{C} \setminus \mathbb{R}$ such that

$$\limsup_{t \rightarrow \infty} \frac{1}{t - s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau \leq 0.$$

For any $\epsilon > 0$, let $\lambda_\epsilon = a_\epsilon + ib_\epsilon$ be chosen such that $0 < |b_\epsilon| < |b|$; $a_\epsilon < a$ and $|\lambda| > |\lambda_\epsilon| > |\lambda| - \epsilon$. Since $0 < 1 + 2xa + x^2|\lambda|^2$, we can choose a_ϵ and b_ϵ such that

$$0 < 2(1 + 2x(a + \theta(a_\epsilon - a)) + x^2(|\lambda|^2 + \theta(|\lambda_\epsilon|^2 - |\lambda|^2))) < c_2 \text{ for all } 0 \leq x \leq \mu_*.$$

Thus,

$$\zeta_{\lambda_\epsilon}(x) - \zeta_\lambda(x) < \frac{a_\epsilon - a}{c_2}, \forall 0 \leq x \leq \mu_*.$$

This implies that

$$\int_s^t \zeta_{\lambda_\epsilon}(\mu(\tau)) \Delta \tau < \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau + \frac{a_\epsilon - a}{c_2}(t - s), \forall t_0 \leq s \leq t.$$

Hence,

$$\limsup_{t \rightarrow \infty} \frac{1}{t - s} \int_s^t \zeta_{\lambda_\epsilon}(\mu(\tau)) \Delta \tau \leq \frac{a_\epsilon - a}{c_2} < 0.$$

which follows that $\lambda_\epsilon \in \mathcal{UT}$. This means that $\lambda \in \overline{\mathcal{UT}}$. The proof is completed. □

Lemma 3.2. Let $K \subset \mathbb{C} \setminus \mathbb{R}$ be a compact set. Suppose that $\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}$, then for any $\epsilon > 0$, there are $\delta > 0, n_0 \in \mathbb{N}$ such that

$$\left| \int_s^t \zeta_\lambda(\mu_n(h)) \Delta_n h - \int_s^t \zeta_\lambda(\mu(h)) \Delta h \right| < 2\epsilon(t-s) + 8M \frac{t-s}{\delta} d_H(\mathbb{T}, \mathbb{T}_n), \forall s < t, \quad (3.1)$$

for $n > n_0, \lambda \in K$ and $M = \sup_{\lambda \in K, x \in [0, \mu^*]} |\zeta_\lambda(x)|$.

Proof. Since $K \subset \mathbb{C} \setminus \mathbb{R}$ is a compact set, $M < \infty$. First, assume that $\mathbb{T}_n \subset \mathbb{T}$. We see that the function

$$\frac{d\zeta_\lambda(x)}{dx} = \begin{cases} \frac{\Re \lambda + x |\lambda|^2}{(1 + 2x \Re \lambda + x^2 |\lambda|^2)x} - \frac{\ln(1 + 2x \Re \lambda + x^2 |\lambda|^2)}{2x^2} & \text{if } x > 0, \\ \frac{1}{2} |\lambda|^2 - (\Re \lambda)^2 & \text{if } x = 0, \end{cases}$$

is continuous in (x, λ) , provided $\Im \lambda \neq 0$. Therefore, the family of functions $(\zeta_\lambda(u))_{\lambda \in K}$ is equicontinuous in u on $[0, \mu^*]$, i.e., for any $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that if $|u - v| < \delta$ then $|\zeta_\lambda(u) - \zeta_\lambda(v)| < \epsilon$ for any $\lambda \in K$. Since $\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}$, we can choose n_0 such that $d_H(\mathbb{T}, \mathbb{T}_n) < \frac{\delta}{2}$ when $n > n_0$.

Fix $t_0 \leq s < t; s, t \in [0, \infty)$ and $n > n_0$. Denote

$$A_1 = \{h \in \mathbb{T}_n \cap [s, t] : \mu_n(h) \geq \delta\}, \quad A_2 = \{h \in \mathbb{T}_n \cap [s, t] : \mu_n(h) < \delta\}.$$

The assumption $\mathbb{T}_n \subset \mathbb{T}$ implies that $0 \leq \mu(h) \leq \mu_n(h)$ for all $h \in \mathbb{T}_n$. If $h \in A_2$ then $\mu(h) \leq \mu_n(h) < \delta$, which implies $|\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(h))| < \epsilon$. On the other hand, the cardinal of A_1 , say r , is finite and $r \leq \left\lceil \frac{t-s}{\delta} \right\rceil$. Thus, we can write $A_1 = \{s_1 < s_2 < \dots < s_r\}$.

Denote sequence τ_i by

$$\tau_i = \max \left\{ h \in \mathbb{T} : h \leq \frac{s_i + \sigma_n(s_i)}{2} \right\}; \text{ and } \bar{\tau}_i = \sigma(\tau_i), i = 1, 2, \dots, r.$$

Since $d_H(\mathbb{T}, \mathbb{T}_n) \geq \max\{|\tau_i - s_i|, |\sigma_n(s_i) - \bar{\tau}_i|\}$, it follows that

$$|\tau_i - s_i| < \frac{\delta}{2}, |\sigma_n(s_i) - \bar{\tau}_i| < \frac{\delta}{2}.$$

Therefore,

$$|\mu(\tau_i) - \mu_n(s_i)| = |\mu_n(s_i) - \tau_i - \bar{\tau}_i| = |\tau_i - s_i| + |\sigma_n(s_i) - \bar{\tau}_i| < \delta,$$

which implies

$$|\zeta_\lambda(\mu(\tau_i)) - \zeta_\lambda(\mu_n(s_i))| < \epsilon.$$

For any $h \in \mathbb{T}$, there exists a unique $u \in \mathbb{T}_n$, say $u = \gamma^{\mathbb{T}, \mathbb{T}_n}(h)$, such that either $h = u$ or $h \in (u, \sigma_n(u))$. It is easy to check that the function $\gamma^{\mathbb{T}, \mathbb{T}_n}(h)$ is rd-continuous on \mathbb{T} . By definition of integral on time scales we have

$$\int_s^t \zeta_\lambda(\mu_n(h)) \Delta_n h = \int_s^t \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h))) \Delta h.$$

Therefore,

$$\begin{aligned} & \left| \int_s^t \zeta_\lambda(\mu_n(h)) \Delta_n h - \int_s^t \zeta_\lambda(\mu(h)) \Delta h \right| \leq \int_s^t |\zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h))) - \zeta_\lambda(\mu(h))| \Delta h \\ &= \int_s^{s_1} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h + \sum_{i=1}^r \left(\int_{s_i}^{t \wedge \bar{\tau}_i} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h \right. \\ &= \int_s^{s_1} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h + \sum_{i=1}^r \left(\int_{s_i}^{t \wedge \bar{\tau}_i} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h \right. \\ &+ \int_{t \wedge \bar{\tau}_i}^{t \wedge \sigma_n(s_i)} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h + \int_{t \wedge \bar{\tau}_i}^{t \wedge \sigma_n(s_i)} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h \Big) \\ &+ \sum_{i=1}^{r-1} \int_{\sigma_n(s_i)}^{s_{i+1}} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h, \end{aligned}$$

where $a \wedge b = \min\{a, b\}$.

Since $|\mu(h) - \mu(\gamma^{\mathbb{T}, \mathbb{T}_n}(h))| < \delta$ for all $h \in [t_0, s_1] \cup [\sigma(s_i), s_{i+1} \wedge t], 1 \leq i \leq r-1$,

$$\int_s^{s_1} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h < \epsilon(\tau_1 - s),$$

$$\int_{\sigma_n(s_i)}^{s_{i+1}} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h < \epsilon(s_{i+1} - \sigma_n(s_i)).$$

On the other hand, for $i = 1, 2, \dots, r$ we have

$$\begin{aligned} & \int_{t \wedge \bar{\tau}_i}^{t \wedge \tau_i} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h = (t \wedge \bar{\tau}_i - t \wedge \tau_i) |\zeta_\lambda(\mu(\tau_i)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(\tau_i)))| \\ &= (t \wedge \bar{\tau}_i - t \wedge \tau_i) |\zeta_\lambda(\mu(\tau_i)) - \zeta_\lambda(\mu_n(s_i))| < \epsilon(t \wedge \bar{\tau}_i - t \wedge \tau_i), \end{aligned}$$

and

$$\int_{s_i}^{t \wedge \bar{\tau}_i} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h \leq 2M(t \wedge \tau_i - s_i) \leq 2Md_H(\mathbb{T}, \mathbb{T}_n)$$

$$\int_{t \wedge \bar{\tau}_i}^{t \wedge \sigma_n(s_i)} |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h \leq 2M(t \wedge \sigma_n(s_i) - t \wedge \bar{\tau}_i) \leq 2Md_H(\mathbb{T}, \mathbb{T}_n)$$

Thus, we obtain

$$\begin{aligned} & \int_s^t |\zeta_\lambda(\mu(h)) - \zeta_\lambda(\mu_n(\gamma^{\mathbb{T}, \mathbb{T}_n}(h)))| \Delta h < \epsilon((\tau_1 - s) + \sum_{i=1}^r (t \wedge \bar{\tau}_i - t \wedge \tau_i) + \sum_{i=1}^{r-1} (s_{i+1} - \sigma_n(s_i))) \\ &+ 4M \sum_{i=1}^r d_H(\mathbb{T}, \mathbb{T}_n) < \epsilon(t - s) + 4Mrd_H(\mathbb{T}, \mathbb{T}_n) < \epsilon(t - s) + 4M \frac{t - s}{\delta} d_H(\mathbb{T}, \mathbb{T}_n). \end{aligned}$$

Therefore,

$$\left| \int_s^t \zeta_\lambda(\mu_n(h))\Delta_n h - \int_s^t \zeta_\lambda(\mu(h))\Delta h \right| < \epsilon(t-s) + 4M \frac{t-s}{\delta} d_H(\mathbb{T}, \mathbb{T}_n).$$

If $\mathbb{T}_n \not\subset \mathbb{T}$, we put $\widehat{\mathbb{T}}_n = \mathbb{T}_n \cup \mathbb{T}$. It is easy to see that

$$d_H(\mathbb{T}, \mathbb{T}_n) = \max\{d_H(\widehat{\mathbb{T}}_n, \mathbb{T}), d_H(\widehat{\mathbb{T}}_n, \mathbb{T}_n)\}. \tag{3.2}$$

By the above proof, we have

$$\left| \int_s^t \zeta_\lambda(\mu_n(h))\Delta_n h - \int_s^t \zeta_\lambda(\mu(h))\Delta_{\widehat{\mathbb{T}}_n} h \right| < \epsilon(t-s) + 4M \frac{t-s}{\delta} d_H(\mathbb{T}, \mathbb{T}_n),$$

$$\left| \int_s^t \zeta_\lambda(\mu(h))\Delta h - \int_s^t \zeta_\lambda(\mu(h))\Delta_{\widehat{\mathbb{T}}_n} h \right| < \epsilon(t-s) + 4M \frac{t-s}{\delta} d_H(\mathbb{T}, \mathbb{T}_n).$$

This implies that

$$\left| \int_s^t \zeta_\lambda(\mu_n(h))\Delta_n h - \int_s^t \zeta_\lambda(\mu(h))\Delta h \right| < 2\epsilon(t-s) + 8M \frac{t-s}{\delta} d_H(\mathbb{T}, \mathbb{T}_n).$$

The proof is complete. □

Denote by $\mathcal{U}\mathbb{T}_n$ (resp. $\mathcal{U}\mathbb{T}$) the domain of stability of the time scale \mathbb{T}_n (resp. \mathbb{T}).

Proposition 3.3. *Suppose that $\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}$. Then, for any $\lambda \in \mathcal{U}\mathbb{T}$ we can find a neighborhood $B(\lambda, \delta)$ of λ and $n_\lambda > 0$ such that $B(\lambda, \delta) \subset \mathcal{U}\mathbb{T} \bigcap_{n > n_\lambda} \mathcal{U}\mathbb{T}_n$.*

Proof. Firstly, we prove the proposition with $\lambda \in \mathcal{U}\mathbb{T} \setminus \mathbb{R}$. Following the proof of Lemma 3.1 and by Proposition 2.8, there exists a $\delta_1 > 0$ satisfying $B(\lambda, \delta_1) \subset \mathcal{U}\mathbb{T}$ and

$$\zeta_{\bar{\lambda}}(x) - \zeta_\lambda(x) < \frac{-L(\lambda)}{4}; \forall 0 \leq x \leq \mu_*, \forall \bar{\lambda} \in B(\lambda, \delta_1).$$

Hence, by (2.10)

$$L(\bar{\lambda}) \leq L(\lambda) + \frac{-L(\lambda)}{4} = \frac{3L(\lambda)}{4} \text{ for any } \bar{\lambda} \in B(\lambda, \delta_1). \tag{3.3}$$

By choosing $\delta_\lambda := \min\{\delta_1, \frac{|\Im \lambda|}{3}\} > 0$ we see that $\overline{B(\lambda, \delta_\lambda)} \subset \mathcal{U}\mathbb{T} \setminus \mathbb{R}$. Using Lemma 3.2 with

$K = \overline{B(\lambda, \delta_\lambda)}$ and $\epsilon = \frac{-L(\lambda)}{8}$ we can find a $\delta_2 > 0$ and n_0 such that

$$\int_s^t \zeta_{\bar{\lambda}}(\mu_n(h))\Delta_n h < \int_s^t \zeta_{\bar{\lambda}}(\mu(h))\Delta h + \frac{-L(\lambda)}{4}(t-s) + 8M \frac{t-s}{\delta_2} d_H(\mathbb{T}, \mathbb{T}_n), \forall s < t, \tag{3.4}$$

for $n > n_0$ and $\bar{\lambda} \in \overline{B(\lambda, \delta_\lambda)}$. We choose $n_\lambda > n_0$ such that $d_H(\mathbb{T}, \mathbb{T}_n) < \frac{-\delta_2 L(\lambda)}{32M}$ for any $n > n_\lambda$.

From (3.4) we get

$$\limsup_{t \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_{\bar{\lambda}}(\mu_n(h))\Delta_n h \leq \limsup_{t \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_{\bar{\lambda}}(\mu(h))\Delta h - \frac{L(\lambda)}{4} - \frac{L(\lambda)}{4}$$

$$\leq \frac{3L(\lambda)}{4} - \frac{L(\lambda)}{4} - \frac{L(\lambda)}{4} = \frac{L(\lambda)}{4} < 0, \forall n > n_0, \tilde{\lambda} \in B(\lambda, \delta_\lambda).$$

This means that $B(\lambda, \delta_\lambda) \subset \mathcal{U}\mathbb{T}_n$ for all $n > n_\lambda$.

We now consider the case $\lambda \in \mathcal{U}\mathbb{T} \cap \mathbb{R}$. Since $\mathcal{U}\mathbb{T}$ is open set, there exists $\delta_3 > 0$ such that $B(\lambda, \delta_3) \subset \mathcal{U}\mathbb{T}$. Let $\lambda_1 = \lambda + i\frac{\delta_3}{2}$. Following above argument, there exist $n_{\lambda_1} > 0$ and $0 < \delta_{\lambda_1} < \frac{\delta_3}{2}$ such that $B(\lambda_1, \delta_{\lambda_1}) \subset \mathcal{U}\mathbb{T} \cap \mathcal{U}\mathbb{T}_n$ for all $n > n_{\lambda_1}$. Since $\mathcal{U}\mathbb{T}_n$ is symmetric with respect to the real axis, the segment $[\lambda', \bar{\lambda}'] \subset \mathcal{U}\mathbb{T}_n$, for all $\lambda' \in B(\lambda_1, \delta_{\lambda_1})$. Thus $B(\lambda, \delta_{\lambda_1}) \subset \mathcal{U}\mathbb{T} \cap \mathcal{U}\mathbb{T}_n$ for all $n > n_{\lambda_1}$. The proposition is proved. \square

Theorem 3.4. *If $\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}$ then*

$$\mathcal{U}\mathbb{T} \subset \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} \mathcal{U}\mathbb{T}_m \text{ and } \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} \mathcal{U}\mathbb{T}_m \setminus \mathbb{R} \subset \overline{\mathcal{U}\mathbb{T}} \setminus \mathbb{R}. \quad (3.5)$$

Proof. The first relation follows immediately from Proposition 3.3.

To prove the second one, let $\lambda \in \bigcap_{n=1}^{\infty} \bigcup_{m \geq n} \mathcal{U}\mathbb{T}_m \setminus \mathbb{R}$. By definition, there is a sequence $\{n_k\} \rightarrow \infty$ such that $\lambda \in \mathcal{U}\mathbb{T}_{n_k} \setminus \mathbb{R}$ for all k . Using again inequality (3.1), for any $\epsilon > 0$, yields

$$\limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau \leq \limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_{\lambda}(\mu_{n_k}(\tau)) \Delta_{n_k} \tau + 2\epsilon + \frac{8M}{\delta} d_H(\mathbb{T}_{n_k}, \mathbb{T}). \quad (3.6)$$

Taking limit as $n_k \rightarrow \infty$ we obtain

$$\limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau < 2\epsilon, \forall \epsilon > 0.$$

This implies that

$$\limsup_{t-s \rightarrow \infty} \frac{1}{t-s} \int_s^t \zeta_\lambda(\mu(\tau)) \Delta \tau \leq 0.$$

Thus, $\lambda \in \overline{\mathcal{U}\mathbb{T}} \setminus \mathbb{R}$ by Lemma 3.1. The proof is complete. \square

4. Conclusion

In this paper, we study the convergence of the stability domains on the time scales in the sense of Hausdorff topology. We prove that if $\lim_{n \rightarrow \infty} \mathbb{T}_n = \mathbb{T}$ in Hausdorff distance then

$$\mathcal{U}\mathbb{T} \subset \liminf_{n \rightarrow \infty} \mathcal{U}\mathbb{T}_n \text{ and } \limsup_{n \rightarrow \infty} (\mathcal{U}\mathbb{T}_n \setminus \mathbb{R}) \subset \overline{\mathcal{U}\mathbb{T}} \setminus \mathbb{R}.$$

So far the question whenever $\lim_{n \rightarrow \infty} \mathcal{U}\mathbb{T}_n = \mathcal{U}\mathbb{T}$ is still an open problem.

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