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On the solution of a class of function equation in plane geometry

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Abstract. We deal with a class of function equation in plane geometry. Let $\Gamma(\Delta)$ be the set of all triples of positive numbers (A,B,C) such that

$$A+B+C=\pi,$$

i.e. every triple $(A,B,C)\in\Gamma(\Delta)$ forms a triangle ΔABC with 3 angles A,B,C, and let $\Gamma(\Delta)$ be the set of all triples of positive numbers (a,b,c) such that

$$|b-c|<a<b+c,$$

i.e. every triple $(a,b,c)\in\Gamma(\Delta)$ forms a triangle ΔABC with 3 side-lengths being a,b,c:

The main our purpose is to describe the general solutions of the following functional equation in plane geometry:

- *Determine all function* $f:(0,\infty)\to(0,\infty)$ *such that* $(f(A),f(B),f(C))\in\Gamma(\Delta)$ *for all* $(A,B,C)\in\Gamma(\Delta)$

- *Determine all function* $f:(0,\infty)\to(0,\infty)$ *such that* $(f(a),f(b),f(c))\in\Gamma(\Delta)$ *for all* $(a,b,c)\in\Gamma(\Delta)$

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1. On the general solution of function equations induced by triangle angles

In the sequel, Let $\Gamma(\Delta)$ be the set of all triples of positive numbers (A,B,C) such that

$$A+B+C=\pi,$$

i.e. every triple $(A,B,C)\in\Gamma(\Delta)$ forms a triangle ΔABC with 3 angles A,B,C, and denote by $\Gamma_0(\Delta)$ the set of all triples of non-negative numbers (A,B,C) such that $A+B+C=\pi$.

Let $\Gamma(\Delta)$ be the set of all triples of positive numbers (a,b,c) such that

$$|b-c|<a<b+c,$$

i.e. every triple $(A,B,C)\in\Gamma(\Delta)$ forms a triangle ΔABC with 3 side-lengths being a; b; c:

The main purpose of the paper is to find the general solutions of the following functional equations.

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Main problem 1. Determine all functions $f : (0,\pi) \to (0,\pi)$ such that $(f(A), f(B), f(C)) \in \Gamma(\Delta)$ for all $(A, B, C) \in \Gamma(\Delta)$.

Main problem 2. Determine all functions $f : (0,\infty) \to (0,\infty)$ $(f : \mathbb{R}^+ \to \mathbb{R}^+)$ such that $(f(a), f(b), f(c)) \in F(\Delta)$ for all $(a, b, c) \in F(\Delta)$.

Firstly we deal with continuous and differential solutions.

Problem 1.1 Determine the general continuous solution $f(x)$ in $[0,\pi]$ and differentiabe in $(0,\pi)$ with $f(0) = 0$ such that $(f(A),\ f(B),\ f(C)) \in \Gamma(\Delta)$ for all $(A, B, C) \in \Gamma(\Delta)$.

Solution. We determine a differentiable function $f(x)$ such that

$$\begin{cases} f(x) > 0, & \forall x \in (0,\pi) \\ f(0) = 0 \\ f(A) + f(B) + f(C) = \pi. \end{cases}$$

The assumption $f(0) = 0$ follows $f(\pi) = \pi$ and $C = \pi - (A + B)$.

That follows

$$f(A) + f(B) + f(\pi - A - B) = \pi, \quad \forall A, B, A + B \in [0,\pi]$$

hay

$$f(x) + f(y) + f(\pi - x - y) = \pi, \quad \forall x, y, x + y \in [0,\pi]. \tag{1}$$

The derivative in x of the both side of (1) is given by

$$f'(x) - f'(\pi - x - y) = 0, \quad \forall x, y, x + y \in [0,\pi]. \tag{2}$$

Equality (8) follows that $f'(x)$ is constant in $(0,\pi)$ and then $f(x) = px + q$. Since $f(0) = 0$ then $q = 0$ and $f(x) = px$. Since $f(\pi) = \pi$ then $p = 1$ and we find $f(x) = x$.

Hence, only the function $f(x) = x$ is a continuous in $[0,\pi]$ and differentiabe in $(0,\pi)$ with $f(0) = 0$ such that $f(A), f(B), f(C)$ form 3 angles of a triangle for all given ΔABC.

Problem 1.2. Determine all functions $f(x)$ defined in $[0,\pi]$ such that $(f(A),\ f(B),\ f(C)) \in \Gamma(\Delta)$ for all given $(A, B, C) \in \Gamma(\Delta)$ and $f(0) = 0$.

Solution. We formulate Problem 1.2 in the following equivalent form:

Determine the general solution in $[0,\pi]$ of the functional equation

$$f(x) + f(y) + f(\pi - x - y) = \pi, \quad \forall x, y \in (0,\pi), x + y < \pi. \tag{3}$$

$$f(0) = 0, \quad f(x) > 0, \quad \forall x \in (0,\pi).$$

Since $f(0) = 0$, from (3) we get

$$f(x) + f(0) + f(\pi - x) = \pi, \quad \forall x \in [0,\pi].$$

Puting $f(x) = x + g(x)$ then $g(0) = 0$ and

$$(3) \Leftrightarrow x + g(x) + (\pi - x) + g(\pi - x) = \pi$$

$$\Leftrightarrow g(x) + g(\pi - x) = 0, \quad \forall x \in [0,\pi]$$

or

$$g(\pi - x) = -g(x), \quad \forall x \in [0, \pi]. \tag{4}$$

Putting $f(x) = x + g(x)$ to (3) and using (4), we find

$$x + g(x) + y + g(y) + \pi - (x + y) + g(\pi - (x + y)) = \pi, \quad \forall x, y \in [0, \pi], x + y \leqslant \pi$$

or

$$g(x + y) = g(x) + g(y), \quad \forall x, y \in [0, \pi], x + y \leqslant \pi. \tag{5}$$

Hence $g(x)$ is additive in $[0, \pi]$. On the other hand, since $f(x) > 0$ for all $x \in (0, \pi)$, it follows $g(x) > -x > -\pi$, i.e. g is bounded from the lower and then g is linear (cf.[1]-[3]). Hence, $g(x) = ax > -x$ for all $x \in (0, \pi)$. It follows $a > -1$.

Hence, the general solution of the problem 1.2 is $f(x) = (1 + a)x$, $a > -1$. Futhermore, by the assumption, the equality $f(A) + f(B) + f(C) = \pi$ follows $1 + a = 1$, i.e. $a = 0$ and $f(x) \equiv x$.

Theorem 1.1. All functions $f(x)$ defined in $[0, \pi]$ such that $(f(A), f(B), f(C)) \in \Gamma(\Delta)$ for all given $(A, B, C) \in \Gamma(\Delta)$ and $(f(A), f(B), f(C)) \in G_0(\Delta)$ for all given $(A, B, C) \in G_0(\Delta)$ are of the form $f(x) = bx + \frac{\pi}{3}(1 - b)$, where $-\frac{1}{2} \leqslant b \leqslant 1$.

Proof. Note that two functions $f(x) = x$ and $f(x) \equiv \frac{\pi}{3}$ are solutions.

We determine the general solution $f(x)$ in $[0, \pi]$ with

$$f(x) + f(y) + f(\pi - x - y) = \pi, \quad \forall x, y \in [0, \pi], x + y \leqslant \pi. \tag{6}$$
$$f(x) > 0, \quad \forall x \in (0, \pi)]$$

Let $y = 0$, then

$$f(x) + f(0) + f(\pi - x) = \pi, \quad \forall x \in [0, \pi]$$

or

$$f(\pi - x) = \pi - f(0) - f(x), \quad \forall x \in [0, \pi].$$

Putting $f(\pi - x) = \pi - f(0) - f(x)$ into (6), we find

$$x + g(x) + y + g(y) + \pi - (x + y) + g(\pi - (x + y)) = \pi, \quad \forall x, y \in [0, \pi], x + y \leqslant \pi$$

or

$$f(x + y) + f(0) = f(x) + f(y), \quad \forall x, y \in [0, \pi], x + y \leqslant \pi. \tag{7}$$

Putting $f(x) = f(0) + g(x) \geqslant 0$. Then $g(x)$ is additive in $[0, \pi]$ and (7) is of the form

$$g(x + y) = g(x) + g(y), \quad \forall x, y \in [0, \pi], x + y \leqslant \pi. \tag{8}$$

Since $g(x)$ is additive in $[0, \pi]$ and $g(x) \geqslant f(0)$ then (6) has the general solution of the form $f(x) = bx + \beta$, where $bx + \beta \geqslant 0$ for all $x \in [0, \pi]$. That follows $f(x)$ is of the form $f(x) = bx + \frac{\pi}{3}(1 - b)$, where $-\frac{1}{2} \leqslant b \leqslant 1$.

2. On the general solution of functional equations induced by side lengths of triangles

Let $F(\Delta)$ be the set of all triples of positive numbers (a, b, c) such that

$$|b-c| < a < b+c,$$

i.e. every triple $(a, b, c) \in F(\Delta)$ forms a triangle ΔABC with its side lengths being a, b, c.

To determine the general solution $f(x)$ in $[0,1]$ such that $f(a), f(b), f(c)$ form 3 side lengths of a triangle for all given ΔABC we need some additional discussions:

In the plane, consider the cirle O with diameter length 1 (unique circle). Denote by $M(\Delta)$ the set of all triangles inscribed in the cirle O. Note that, if f is a solution of Problem 2 then $F(x) = \lambda f(x)$, with any $\lambda > 0$, also satisfies Problem 2 and conversely. So it enough to examine the Problem 2 in the case when the triples of positive numbers (a, b, c) being the side lengths of triangles in $M(\Delta)$.

The sine theorem follows that a necessary and sufficient condition for three positive numbers α, β, γ to be 3 angles of a triangle in $M(\Delta)$ are $\sin\alpha, \sin\beta, \sin\gamma$ form 3 side lengths of a triangle in $M(\Delta)$.

Indeed, if α, β, γ are 3 angles of a triangle in $M(\Delta)$ then $2R\sin\alpha$, $2R\sin\beta$, $2R\sin\gamma$ or $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are 3 side lengths of a triangle inscribed in the cirle O with diameter length 1.

Conversely, if $\sin\alpha, \sin\beta, \sin\gamma$ are 3 side lengths of a triangle inscribed in the cirle O with diameter length 1 and α, β, γ are positive then α, β, γ form 3 angles of a triangle.

Firstly, we formulate propositions for some simple specialized cases.

Proposition 2.1. The function $f(x) = x + \alpha$ possesses the property that $(f(a), f(b), f(c)) \in F(\Delta)$ for all $(a, b, c) \in F(\Delta)$ iff $\alpha \geqslant 0$.

Proposition 2.2. The function $f(x) = \alpha x$ possesses the property that $f(a), f(b), f(c)$ are side lengths of a triangle for all $(a, b, c) \in F(\Delta)$ iff $\alpha > 0$.

Proposition 2.3. The function $f(x) = \alpha x + \beta$ possesses the property that $f(a), f(b), f(c)$ are side lengths of a triangle for all $(a, b, c) \in F(\Delta)$ iff $\alpha \geqslant 0,\ \ \beta \geqslant 0$ and $\alpha + \beta > 0$.

Proposition 2.4. The function $f(x) = \dfrac{1}{\alpha x + \beta}$ possesses the property that $f(a), f(b), f(c)$ are side lengths of a triangle for all $(a, b, c) \in F(\Delta)$ iff $\alpha = 0,\ \ \beta > 0$.

Now we deal with the set $M(\Delta)$, i.e. the set of all triangles inscribed in the cirle O with diameter length 1.

Theorem 2.1. Any function $f : [0,1] \to [0,1]$ such that $(f(a), f(b), f(c)) \in M(\Delta)$ for all $(a, b, c) \in M(\Delta)$ is of the form

$$f(x) = \sin\Big(\alpha \arcsin x + \frac{(1-\alpha)\pi}{3}\Big),\ \ -\frac{1}{2} \leqslant \alpha \leqslant 1. \tag{9}$$

Proof. Note that, if α, β, γ are 3 angles of a triangle in $M(\Delta)$ then $2R\sin\alpha$, $2R\sin\beta$, $2R\sin\gamma$ or $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are 3 side lengths of a triangle inscribed in the cirle O with diameter length 1.

Conversely, if $\sin\alpha$, $\sin\beta$, $\sin\gamma$ are 3 side lengths of a triangle inscribed in the cirle O with diameter length 1 and α, β, γ are positive then α, β, γ form 3 angles of a triangle.

On the other hand, by theorem thm1, all functions $f(x)$ defined in $[0, \pi]$ such that $(f(A), f(B), f(C)) \in \Gamma(\Delta)$ for all given $(A, B, C) \in \Gamma(\Delta)$ and $(f(A), f(B), f(C)) \in G_0(\Delta)$ for all given $(A, B, C) \in G_0(\Delta)$ are of the form $f(x) = bx + \frac{\pi}{3}(1-b)$, where $-\frac{1}{2} \leqslant b \leqslant 1$.

Hence, the general solution is of the form (10).

Now we formulate the main result.

Theorem 2.2. Any function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $(f(a), f(b), f(c)) \in F(\Delta)$ for all $(a, b, c) \in F(\Delta)$ is of the form

$$f(x) = u\sin\left(\alpha\arcsin\{x\} + \frac{(1-\alpha)\pi}{3}\right), \quad -\frac{1}{2} \leqslant \alpha \leqslant 1. \tag{10}$$

Proof. Applying the above additional discussion and theorem , it is easy to obtain the form (10).

Remark 1. Some other types of functional equations in geometry were considered firstly by S. Galab [4].

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