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On the stability of the distribution function of the composed random variables by their index random variable

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Abstract. Let us consider the composed random variable $\eta = \sum_{k=1}^{\nu} \xi_k$, where $\xi_1, \xi_2, ...$ are independent identically distributed random variables and ν is a positive value random, independent of all ξ_k.
In [1] and [2], we gave some the stabilities of the distribution function of η in the following sense: the small changes in the distribution function of ξ_k only lead to the small changes in the distribution function of η.
In the paper, we investigate the distribution function of η when we have the small changes of the distribution of ν.

1. Introduction

Let us consider the random variable (r.v):

$$\eta = \sum_{k=1}^{\nu} \xi_k \tag{1}$$

where $\xi_1, \xi_2, ...$ are independent identically distributed random variables with the distribution function $F(x)$, ν is a positive value r.v independent of all ξ_k and ν has the distribution function $A(x)$.

In [1] and [2], η is called to be the composed r.v and ν is called to be its index r.v. If $\Psi(x)$ is the distribution function of η with the characteristic function $\psi(x)$ respecrively then (see [1] or [2])

$$\psi(x) = a[\varphi(t)] \tag{2}$$

where $a(z)$ is the generating function of ν and $\varphi(t)$ is the characteristic function of ξ_k.

In [1] and [2], we gave some the stabilities of $\Psi(x)$ in the following sence: the small changes in the distribution function $F(x)$ only lead to the small changes in the distribution function $\Psi(x)$.

In this paper, we shall investigate the stability of η's distribution function when we have the small change of the distribution of the index r.v ν.

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2. Stability theorem

Let us consider the r.v now:

$$\eta_1 = \sum_{k=1}^{\nu_1} \xi_k \tag{3}$$

where ν_1 has the distribution function $A_1(x)$ with the generating function $a_1(z)$. Suppose ξ_k have the stable law with the characteristic function

$$\varphi(t) = \exp\{i\mu t - c|t|^\alpha[1 - e\beta\frac{t}{|t|}\omega(t;\alpha)]\} \tag{4}$$

where c, μ, α, β are real number, $c \geq 0$; $|\beta| \leqslant 1$,

$$2 \geq \alpha \geq \alpha_1 > 1; \quad \omega(t;\alpha) = tg\frac{\alpha t}{2}. \tag{5}$$

For every $\varepsilon > 0$ is given, such that

$$\varepsilon < (\frac{\pi}{3c_2})^3 \tag{6}$$

where $c_2 = (c + c|\beta||tg\frac{\alpha_1\pi}{2} + |\mu|)$.
We have the following theorem:
Theorem 2.1 (Stability Theorem). *Assume that*

$$\rho(A; A_1) = \sup_{x\in R^1} |A(x) - A_1(x)| \leqslant \varepsilon$$

$$\mu_A^\alpha = \int_0^{+\infty} z^\alpha dA(z) < +\infty; \quad \mu_{A_1}^\alpha = \int_0^{+\infty} z^\alpha dA_1(z) < +\infty, \ \forall\alpha > 0. \tag{7}$$

Then we have

$$\rho(\Psi, \Psi_1) \leqslant K_1\varepsilon^{1/6}$$

where K_1 is a constant independent of ε, $\Psi(x)$ and $\Psi_1(x)$ are the distribution function of η and η_1 respectively.
Lemma 2.1. *Let a is a complex number, $a = \rho e^{i\theta}$, such that $|\theta| \leqslant \frac{\pi}{3}; 0 \leqslant \rho \leqslant 1$. Then we have the following estimation:*

$$|a^t - 1| \leqslant \frac{\sqrt{14}|a-1|}{(1 - |a-1|)} \quad (forevery\ t > 0) \tag{8}$$

Proof. Since $a = \rho(\cos\theta + i\sin\theta)$, it follows that $a^t = \rho^t(\cos t\theta + i\sin t\theta)$.
Hence

$$|a^t - 1|^2 = (\rho^t\cos t\theta - 1)^2 + (\rho^t\sin t\theta)^2, \tag{9}$$

we also have

$$(\rho^t\cos t\theta - 1) = (\rho^t - 1)\cos t\theta + (\cos t\theta - 1),$$

Notice that $|1 - \cos x| \leqslant |x|$ for all x, thus

$$|\rho^t\cos t\theta - 1| \leqslant |\rho^t - 1| + |t\theta|.$$

On the other hand, since $|\sin u| \leqslant |u|$ for all u,

$$|a^t - 1|^2 \leqslant 2|\rho^t - 1|^2 + 2t^2\theta^2 + \rho^{2t}t^2\theta^2, \tag{10}$$

we can see

$$|a-1|^2 = (\rho\cos\theta - 1)^2 + (\rho^2\sin^2\theta).$$

It follows that

$$|\rho\sin\theta| \leqslant |a-1|. \tag{11}$$

Furthermore,

$$||a|-1| \leqslant |a-1| \Rightarrow |\rho - 1| \leqslant |a-1| \Rightarrow \rho \geq 1 - |a-1|.$$

From (11) we obtain

$$|\sin\theta| \leqslant \frac{|a-1|}{\rho} \leqslant \frac{|a-1|}{1-|a-1|}. \tag{12}$$

Since $|\theta| \leqslant \frac{\pi}{3} \Rightarrow |\sin\theta| \geq \frac{|\theta|}{2}$, so that

$$|\theta| \leqslant \frac{2|a-1|}{(1-|a-1|)}. \tag{13}$$

From (10) and (13), we have

$$|a^t - 1|^2 \leqslant 2|\rho^t - 1|^2 + \frac{8t^2|a-1|^2}{(1-|a-1|)^2} + 4\frac{\rho^{2t}t^2|a-1|^2}{(1-|a-1|)^2}. \tag{14}$$

For all $t \geq 0$, the following inequality holds:

$$1 - \rho^t \leqslant \frac{t(1-\rho)}{\rho}. \tag{15}$$

Using (11) and notice that $|1-\rho| = |1-|a|| \leqslant |a-1|$, we shall have

$$1 - \rho^t \leqslant \frac{t|a-1|}{\rho}. \tag{16}$$

Hence by (14) we get

$$|a^t - 1|^2 \leqslant \frac{14t^2|a-1|^2}{(1-|a-1|)^2}.$$

Lemma 2.2. *Under the notation in* (2), *let* $\delta(\varepsilon)$ *be sufficiently small postive number such that* $\delta(\varepsilon) \to 0$ *when* $\varepsilon \to 0$ *and*

$$|arg\varphi(t)| \leqslant \frac{\pi}{3} \quad \forall t, \quad |t| \leqslant \delta(\varepsilon).$$

Then

$$|\psi(t) - \psi_1(t)| \leqslant C|t| \quad \forall t, \quad |t| \leqslant \delta(\varepsilon)$$

where C *is a constant independent of* ε *and* $\psi_1(t)$ *is the characteristic function with the distribution function* $\Psi_1(t)$ *respectively.*

Proof. We have

$$|\psi(t) - \psi_1(t)| = |\int_0^{+\infty} |\varphi(t)|^z d[A(z) - A_1(z)]| \leqslant \int_0^{+\infty} |\varphi^z(t) - 1| d[A(z) + A_1(z)]. \tag{17}$$

Notice that, for all $t \in R^1$

$$|e^{itx} - 1| \leqslant 3|\sin(\frac{tx}{2})| \leqslant \frac{3}{2}|tx| < 2|tx|.$$

Hence, if we put

$$\mu_F = \int_{-\infty}^{+\infty} |x| dF(x) < +\infty; \quad \varphi(t) = \int_{-\infty}^{+\infty} e^{itx} dF(x),$$

then

$$|\varphi(t) - 1| \leqslant \int |e^{itx} - 1| dF < 2|t|\mu_F.$$

From lemma 2.1, (with $a = \varphi(t); |t| \leqslant \delta(\varepsilon)$)

$$|\varphi(t) - 1| \leqslant \frac{\sqrt{14}z|\varphi(t) - 1|}{(1 - |\varphi(t) - 1|)}. \tag{18}$$

Because there exits moments (from (7)) and with t, $|t| \leqslant \delta(\varepsilon)$ we can see $|1 - \varphi(t)| \leqslant \frac{1}{2}$, therefore

$$|\psi(t) - \psi_1(t)| \leqslant \int_0^{+\infty} \frac{\sqrt{14}z|\varphi(t) - 1|}{(1 - |\varphi(t) - 1|)} d[A(z) + A_1(z)] \leqslant 4\sqrt{14}\mu_F(\mu_A + \mu_{A_1})|t| = C|t|$$

(do $|\varphi(t) - 1| \leqslant \mu_F|t| \quad \forall t$)
where C is a constant independent of ε and $\mu_F = \int_{-\infty}^{+\infty} |x| dF(x) < \infty$.
Proof of Theorem 2.1.

For every $N > 0$ and $t \in R^1$, we have

$$\begin{aligned}
|\psi(t) - \psi_1(t)| &= |\int_0^{+\infty} \varphi^z(t) d[A(z) - A_1(z)]| \\
&\leqslant |\int_0^{N} \varphi^z(t) d[A(z) - A_1(z)]| + |\int_N^{+\infty} \varphi^z(t) d[A(z) - A_1(z)]| \\
&\leqslant |[A(z) - A_1(z)]|_0^N| + \int_0^N |A(z) - A_1(z)||\varphi^z(t)||\ln\varphi(t)|dz + \int_N^{+\infty} d[A(z) + A_1(z)] \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{19}$$

First, it casy to see that

$$I_1 \leqslant 2\varepsilon. \tag{20}$$

In order to estimate I_2, notice that $\varphi(t)$ has form (4) with the condition (5) so we have

$$|\ln\varphi(t)| \leqslant |\mu||t| + |t|^\alpha(c + c|\beta||tg\frac{\alpha\pi}{2}|) \leqslant |\mu||t| + C_1|t|^\alpha \tag{21}$$

where $C_1 = c + c|\beta||tg\frac{\alpha\pi}{2}| \leqslant c + c|\beta||tg\frac{\alpha_1\pi}{2}|$.
If $T = T(\varepsilon)$ is a positive number which will be chosen later ($T(\varepsilon) \to \infty$ when $\varepsilon \to 0$), we can see that

$$|\ln\varphi(t)| \leqslant |\mu|T + C_1T^\alpha \leqslant (C_1 + |\mu|)T^\alpha \leqslant C_2T^\alpha \quad \forall t, |t| \leqslant T(\varepsilon)$$

where $C_2 = c + c|\beta||tg\frac{\alpha_1\pi}{2}| + |\mu|; \quad (\alpha \geq \alpha_1 > 1)$.
Then

$$I_2 \leqslant \varepsilon \int_0^N C_2T^\alpha dz \leqslant C_2\varepsilon T^\alpha N. \tag{22}$$

Finally, with α from condition (5), we have

$$I_3 \leqslant \frac{\mu_A^\alpha + \mu_{A_1}^\alpha}{N^\alpha}. \tag{23}$$

By using (19), (20), (21), (23), we conclude that

$$|\psi(t) - \psi_1(t)| \leqslant 2\varepsilon + C_2\varepsilon T^\alpha N + \frac{\mu_A^\alpha + \mu_{A_1}^\alpha}{N^\alpha}. \tag{24}$$

Choosing $T = \varepsilon^{-\frac{1}{3^\alpha}}$ and $N = T = \varepsilon^{-\frac{1}{3^\alpha}}$, we can see that

$$C_2\varepsilon T^\alpha N \leqslant C_2\varepsilon^{1-\frac{1}{3}-\frac{1}{3}} = C_2\varepsilon^{\frac{1}{3}},$$

$$(\mu_A^\alpha + \mu_{A_1}^\alpha)N^{-\alpha} = (\mu_A^\alpha + \mu_{A_1}^\alpha)\varepsilon^{\frac{1}{3}}.$$

Thus

$$|\psi(t) - \psi_1(t)| \leqslant 2\varepsilon + C_2\varepsilon^{\frac{1}{3}} + (\mu_a^\alpha + \mu_{A_1}^\alpha)\varepsilon^{\frac{1}{3}} = C_3\varepsilon^{\frac{1}{3}} \tag{25}$$

for every t with $|t| \leqslant T = \varepsilon^{-\frac{1}{3^\alpha}}$ and C_3 is a constant independent of ε.

For all $\delta(\varepsilon) > 0$, we consider now

$$\int_{-T}^{T} |\frac{\varphi(t) - \varphi_1(t)}{t}|dt = \int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\varphi(t) - \varphi_1(t)}{t}|dt + \int_{\delta(\varepsilon)\leqslant|t|\leqslant T} |\frac{\varphi(t) - \varphi_1(t)}{t}|dt.$$

Since

$$\ln z = \ln|z| + i arg(z) \quad (0 \leqslant arg z \leqslant 2\pi),$$

for all complex number z, letting $z = \varphi(t)$, $(|t| \leqslant \delta(\varepsilon))$

$$|arg\varphi(t)| \leqslant |\ln\varphi(t)| \leqslant C_2\delta(\varepsilon)$$

with $\delta(\varepsilon) = \varepsilon^{\frac{1}{3}}$, we shall get $|arg\varphi(t)| \leqslant C_2\varepsilon^{\frac{1}{3}}$ and from (6)

$$C_2\varepsilon^{\frac{1}{3}} \leqslant \frac{\pi}{3} \Rightarrow |arg\varphi(t)| \leqslant \frac{\pi}{3} \text{ for every } t, \ |t| \leqslant \delta\varepsilon.$$

Hence, using lemma 2.2, we obtain:

$$\int_{-\delta(\varepsilon)}^{\delta(\varepsilon)} |\frac{\varphi(t) - \varphi_1(t)}{t}|dt \leqslant 2C\delta(\varepsilon) = 2C\varepsilon^{\frac{1}{3}}. \tag{26}$$

On the other hand, using (25), we get

$$\int_{\delta(\varepsilon)\leqslant|t|\leqslant T} |\frac{\varphi(t) - \varphi_1(t)}{t}|dt \leqslant C_3\varepsilon^{\frac{1}{3}} \int_{\delta(\varepsilon)}^{T} \frac{dt}{t} = C_3\varepsilon^{\frac{1}{3}} \ln\frac{T}{\delta(\varepsilon)} = C_3\varepsilon^{\frac{1}{3}} \ln(\frac{1}{\varepsilon^{\frac{1+\alpha}{3\alpha}}}) \leqslant C_4\varepsilon^{\frac{1}{6}}. \tag{27}$$

From (26) and (27)

$$\int_{-T}^{T} |\frac{\varphi(t) - \varphi_1(t)}{t}|dt \leqslant 2C\varepsilon^{\frac{1}{3}} + C_4\varepsilon^{\frac{1}{6}} \leqslant C_5\varepsilon^{\frac{1}{6}}$$

where C_5 is constant independent of ε.

Indeed, by using Essen's inequality (see [3]) we have

$$\rho(\Psi; \Psi_1) \leqslant C_5\varepsilon^{\frac{1}{6}} + C_6\varepsilon^{\frac{1}{4}} \leqslant K_1\varepsilon^{\frac{1}{6}}$$

where K_1 is a constant independent of ε.

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