VNU Journal of Science, Mathematics - Physics 23 (2007) 92-98

Asymptotic equilibrium of the delay differential equation

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Received 16 April 2007; received in revised form 15 August 2007

Abstract. In this paper, we show that if the operator $A(\cdot)$ is strongly continuous on Hilbert space $\mathbb{H}$, $A(t) = A^*(t)$, $\sup_{\|h\|\leq 1} \int_T^{+\infty} \|A(t)h\|dt \leq q < 1$ then the equation

$$\frac{d}{dt}x(t) = A(t)x(t-r), \forall t \geq 0, \quad r \text{ is a given positive constant},$$

is asymptotic equilibrium.

1. Introduction

Consider the delay differential equation

$$\frac{d}{dt}x(t) = A(t)x(t-r) \quad (t \geq 0), \tag{1}$$

where r is a given positive constant, $A(\cdot) \in C(\mathbb{R}^+, \mathbb{L}(H))$, $\mathbb{H}$ is a Hilbert space. We will show a condition for the asymptotic equilibrium of Eq (1)by extending some results obtained from the equation

$$\frac{d}{dt}x(t) = A(t)x(t) \quad (t \geq 0), \tag{2}$$

Finding conditions for the asymptotic equilibrium of a differential equation is considered by many mathematicians. Some of the mathmaticians are L. Cezari, A. Winter, A. Ju. Levin, Nguyen The Hoan,.etc.

In a paper, L. Cezari asserted that

If $\|A(t)\| \in L_1(\mathbb{R}^+, \mathbb{R}^n)$ *then Eq (2) is asymptotic equilibrium*

The result was devoloped by A. Winter (1954) (see [2]), and A. Ju. Levin (1967) (see [3]). However, those results were restricted to finite dimentional spaces. Nguyen The Hoan extended them into any Hilbert spaces.

From this result, we extend on Eq (1) and obtain a similar result (theorem 3.3)

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2. Preliminaries

The section will be devoted entirely to the notation and concept of asymptotic equilibrium of differential equations. Almost all results of this section are more or less known. However, for the reader's convenience we will quote them here and even verify several results which seem to be obviuous but not available in the mathematical literature.

Thoughout this paper we will use the following notations: $\mathbb{H}$ is a given hilbert space. r is a given positive constant. $C([a;b],H)$ stands for the space of all continuous functions from the interval $[a;b]$ into $\mathbb{H}$. $L(H)$ is the set of all continous operators from $\mathbb{H}$ into itself.

The purpose to introduce Pro.The Hoan's theorem 1, we consider the following equation:

$$\frac{d}{dt}x(t) = A(t)x(t), \quad \forall t \in \mathbb{R}^+, \tag{3}$$

where $A(\cdot) : \mathbb{R}^+ \mapsto L(\mathbb{H}), A(t) = A^*(t) \quad (\forall t \in \mathbb{R})$, $A(\cdot)$ is strongly continuous.

Definition 2.1. $x(\cdot)$ is called a solution of Eq (3) if there is such $t_0 \in \mathbb{R}^+, x_0 \in \mathbb{H}$ that $x(\cdot)$ is a solution of the following Cauchy problem:

$$\begin{cases} \frac{d}{dt}x(t) = A(t)x(t) \quad (t \geq t_0), \\ x(t_0) = x_0. \end{cases}$$

Definition 2.2. [Asymptotic equilibrium] The equation (3) is an asymptotic equilibrium equation if all solutions of the equation satisfy:

i) if $x(\cdot)$ is an solution of Eq (3) then $x(t)$ tends to the finite limit, as $t \longrightarrow +\infty$.

ii) For a given c belonging to $\mathbb{H}$, there is such a solution $x(\cdot)$ of Eq (3) that $\lim\limits_{t\to+\infty} x(t) = c$.

Theorem 2.3 *If $A(\cdot)$ satisfies $\sup\limits_{\|h\|\leq 1} \int\limits_T^{+\infty} \|A(t)h\|dt < q < 1$, where T, q are given, then Eq (3) is asymptotic equilibrium.*

To prove this theorem, we must solve the following lemma.

Note: We usually assume that $A(\cdot)$ satifies all the conditions of theorem (2.3).

Lemma 2.4. *If $x(\cdot)$ is a solution of Eq (3) then $x(\cdot)$ satisfies the condition (i) of definition (2.2).*

Proof. Firstly, we see, for $t \geq T, h \in \mathbb{H} : \|h\| \leq 1$,

$$\begin{aligned} < x(t), h > &=< x(T), h > + \int\limits_T^t < A(\tau)x(\tau), h > d\tau, \\ &=< x(T), h > + \int\limits_T^t < x(\tau), A(\tau)h > d\tau. \end{aligned}$$

Hence,

$$\|x(t)\| = \sup_{\|h\|\leq 1} \| < x(t), h > \| \leq \|x(T)\| + \int\limits_T^t \|A(t)h\|\|x(\tau)\|d\tau$$

By the Gronwall-Bellman inequality, we have

$$\|x(t)\| \le \|x(T)\| e^{\int_T^t \|A(\tau)h\| d\tau} \le \|x(T)\| e^q < +\infty.$$

Let $M := \sup_{t\in\mathbb{R}} \|x(t)\|$. From

$$\begin{aligned}\|x(t) - x(s)\| &= \sup_{\|h\|\le 1} \| < x(t), h > - < x(s), h > \| \\ &= \sup_{\|h\|\le 1} \| \int_s^t < x(\tau), A(\tau)h > d\tau \| \\ &\le M \sup_{\|h\|\le 1} \int_s^t \|A(\tau)h\| d\tau \to 0,\end{aligned}$$

as $t \ge s \to +\infty$. This lemma is proved. □

The proof of theorem [2.3]

Let a fixed $h_0 \in \mathbb{H}$. Consider the function:

$$\xi_1(t,h) =< h_0, h > - \int_t^{+\infty} < A(\tau)h_0, h > d\tau, \quad t \ge T, h \in \mathbb{H}.$$

It is easy to show that

i) $\|\xi_1(t,h)\| \le (1+q)\|h_0\|, \ \|h\| \le 1.$

ii) $\|\xi_1(t,h)\| \le (\|h_0\| + q)\|h\|\|h_0\|, \ \forall h \in \mathbb{H}.$

Hence, $\xi_1(t,\cdot) \in H^* = L(\mathbb{H}, \mathbb{R})$. By theorem Riezs, there is a $x_1(t) \in \mathbb{H}$:

$$\xi_1(t,h) =< x_1(t), h > \ \ and \ \ \|x_1(t)\| \le (1+q)\|h_0\|.$$

Let $x_0(\cdot) \equiv h_0$. Obviously,

$$\frac{d}{dt}x_1(t) = A(t)x_0(t), \ \forall t \ge T.$$

By the same way, we establish two sequences $\{\xi_n(\cdot,\cdot)\}$,$\{x_n(\cdot)\}$:

$$\begin{cases} \xi_n(t,h) =< h_0, h > - \int_t^{+\infty} < A(\tau)x_{n-1}(\tau), h > d\tau \ (t \ge T, n \in \mathbb{N}), \\ \xi_n(t,h) =< x_n(t), h >, \\ \|x_n(t)\| \le (1 + q + \cdots + q^n)\|h_0\| \le \frac{1}{1-q}\|h_0\|, \\ \frac{d}{dt}x_n(t) = A(t)x_{n-1}(t) \ \ (t \ge T). \end{cases} \tag{4}$$

Moreover,

$$\begin{aligned}\|x_{n+1}(t) - x_n(t)\| &= \sup_{\|h\|\le 1} \| < x_{n+1}(t) - x_n(t), h > \| \\ &\le \sup_{\|h\|\le 1} \int_T^{+\infty} \|x_n(\tau) - x_{n-1}(\tau)\|\|A(\tau)h\| d\tau \\ &\le q^{n+1}\|h_0\| \ (\forall n \in \mathbb{N}).\end{aligned}$$

Consequently, $\{x_n(\cdot)\}$ converges uniformly on $[T, +\infty)$. Let the limit of $\{x_n(\cdot)\}$ be $x(\cdot)$ belonging to $C([T, +\infty), \mathbb{H})$. For a given $T_1 > T$, By the strongly continuous property of $A(\cdot)$, $\sup_{t\in[T,T_1]} \|A(t)h\| = M_h < +\infty$. It follows from the uniformly bounded priciple, there is such a positive M_1 that $\sup_{t\in[T,T_1]} \|A(t)\| = M_1 < +\infty$.
From

$$\begin{aligned}\|\frac{d}{dt}x_{n+1}(t) - \frac{d}{dt}x_n(t)\| = & \sup_{\|h\|\le 1} \| < A(t)x_n(t) - A(t)x_{n-1}(t), h > \| \\ = & \sup_{\|h\|\le 1} \| < x_n(t) - x_{n-1}(t), A(t)h > \| \\ \le & M_1\|x_n(t) - x_{n-1}(t)\| \le M_1 q^n \|h_0\|, \quad \forall t \in [T, T_1],\end{aligned}$$

the sequence $\{\frac{d}{dt}x_n(\cdot)\}$ converges uniformly to $\frac{d}{dt}x(\cdot)$ on (T, T_1).
On the other hand,

$$< x_n(t), h >=< h_0, h > - \int_t^{+\infty} < x_{n-1}(t), A(t)h > dt \quad (t \in [T, T_1]).$$

Letting $n \to +\infty$, we have

$$< x(t), h >=< h_0, h > - \int_t^{+\infty} < x(t), A(t)h > dt \quad (t \in [T, T_1]).$$

It leads to $< \frac{d}{dt}x(t), h >=< A(t)x(t), h >, \quad \forall h \in \mathbb{H}, t \in (T, T_1)$. By a any given $T_1 > T$,

$$\frac{d}{dt}x(t) = A(t)x(t), \quad \forall t > T.$$

Observe that

$$\begin{aligned}\|x(t) - h_0\| = \sup_{\|h\|\le 1} \| < x(t) - h_0, h > \| = & \sup_{\|h\|\le 1} \| \int_t^{+\infty} < x(\tau), A(\tau)h > d\tau \| \\ \le & \sup_{\|h\|\le 1} \int_t^{+\infty} \|A(\tau)h\| d\tau \frac{1}{1-q} \|h_0\| \to 0,\end{aligned}$$

as $t \to +\infty$. This is proved the theorem.

3. The main result

In the section, we will extend Pro.The Hoan's result to the following delay differential equation:

$$\frac{d}{dt}x(t) = A(t)x(t - r) \quad (t \in \mathbb{R}^+), \tag{5}$$

where r is a given positive constant, $A(\cdot)$ satisfies all the conditions in the section 2.

Definition 3.1. $x(\cdot)$ is called a solution of Eq (5) if there is such $t_0 \in \mathbb{R}^+, \varphi \in C([t_0 - r, t_0], \mathbb{H})$ that $x(\cdot)$ is a solution of the following Cauchy problem:

$$\begin{cases} \frac{d}{dt}x(t) = A(t)x(t-r) \quad (t \geq t_0), \\ x_{t_0} = \varphi. \end{cases}$$

Lemma 3.2. If $x(\cdot)$ is a solution of Eq (5), then $x(\cdot)$ satifies the condition (i) of definition [2.2].

Proof. From

$$< x(t), h > = < x(s), h > + \int_s^t < A(\tau)x(\tau - r), h > d\tau \quad (t \geq s \geq T),$$

we have

$$\|x(t)\| = \sup_{\|h\|\leq 1} \| < x(t), h > \| \leq \|x(s)\| + \sup_{\|h\|\leq 1} \int_s^t \|A(\tau)h\|\|x(\tau - r)\|d\tau, \ t \geq s > T + t_0.$$

Hence,

$$|||x(t)||| \leq \|x(s)\| + q|||x(t)||| \quad or \quad |||x(t)||| \leq \frac{\|x(s)\|}{1-q}, \qquad t \geq s > T + t_0,$$

where $|||x(t)||| = \sup_{t_0 - r \leq \xi \leq t} \|x(\xi)\|$. Let $M := \sup_{t \geq t_0 - r} \|x(t)\|$.
On the other hand,

$$\|x(t) - x(s)\| = \sup_{\|h\|\leq 1} \| < x(t) - x(s), h > \| = \sup_{\|h\|\leq 1} \| \int_s^t < x(\tau - r), A(\tau) > d\tau\|$$

$$\leq \qquad M \sup_{\|h\|\leq 1} \int_s^t \|A(\tau)h\|d\tau \to 0,$$

as $t \geq s \to +\infty$. The lemma is proved. □

Theorem 3.3. The Eq (5) is asymptotic equilibrium.

Proof. Let a fixed $h_0 \in \mathbb{H}$. We consider the following function:

$$\xi_1(t, h) = < h_0, h > - \int_t^{+\infty} < A(\tau)h_0, h > d\tau \quad (t \geq T).$$

Let $x_0(t) \equiv 0$. Use exactly the argument of the proof of theorem 2.3, we establish the functions $x_1(\cdot)$,$\overline{x_1}(\cdot)$ which satisfy :
For $t \geq T$,

i) $\xi_1(t, h) = < \overline{x_1}(t), h >$,
ii) $\frac{d}{dt}\overline{x_1}(t) = A(t)x_0(t)$,
iii) $\overline{x_1}(t) = x_1(t)$,
iv) $\|x_1(t)\| \leq (1+q)\|h_0\|$.

By the same way, we have three sequences $\{\xi_n(t, h)\}, \{x_n(t)\}, \{\overline{x_n}(t)\}$ which satisfy :

i) $\xi_n(t,h) =< \overline{x_n}(t), h >=< h_0, h > - \int\limits_t^{+\infty} < A(\tau)x_{n-1}(\tau - r), h > d\tau \quad (t \geq T + r).$

ii) $\frac{d}{dt}\overline{x_n}(t) = A(t)x_{n-1}(t-r), \quad \forall t > T + r.$

iii)

$$x_n(t) = \begin{cases} \overline{x_n}(t), & t \geq T + r, \\ \overline{x_n}(T + r), & T + r \geq t \geq T. \end{cases}$$

iv) $\|x_n(t)\| \leq (1 + q + \cdots + q^n)\|h_0\| \leq \frac{1}{1-q}\|h_0\|, \quad t \geq T.$

We see that

$$\begin{aligned} \|x_{n+1}(t) - x_n(t)\| = & \sup_{\|h\|\leq 1} \| < x_{n+1}(t) - x_n(t), h > \| \\ \leq & \sup_{\|h\|\leq 1} \int\limits_T^{+\infty} \|x_n(\tau - r) - x_{n-1}(\tau - r)\| \|A(\tau)h\| d\tau \\ \leq & q^{n+1}\|h_0\| \quad (\forall n \in \mathbb{N}, t \geq T + r). \end{aligned}$$

Moreover, $\|x_{n+1}(t) - x_n(t)\| \leq q^{n+1}\|h_0\|, \quad \forall t \geq T, n \in \mathbb{N}.$

Consequently, $\{x_n(\cdot)\}$ converges uniformly on $[T, +\infty)$. Let the limit of $\{x_n(\cdot)\}$ be $x(\cdot)$ belonging to $C([T, +\infty), \mathbb{H})$. From

$$< x_{n+1}(t), h >=< h_0, h > - \int\limits_t^{+\infty} < x_n(\tau - r), A(\tau)h > d\tau \quad (t \geq T + r),$$

letting $n \to +\infty$, we have

$$< x(t), h >=< h_0, h > - \int\limits_t^{+\infty} < x(\tau - r), A(\tau)h > d\tau \quad (t \geq T + r).$$

On the other hand, for a given any $T_1 > T + r$, the sequence $\{\frac{d}{dt}x_n(t)\}$ converges uniformly on $(T + r, T_1]$(see proof of theorem 2.3). This leads to

$$\frac{d}{dt}x(t) = \lim_{n\to+\infty} \frac{d}{dt}x_n(t) \quad \forall t \geq T + r.$$

Hence,

$$\frac{d}{dt}x(t) = A(t)x(t - r) \quad \forall t > T + r.$$

By $\|x(t)| \leq \frac{1}{1-q}\|h_0\|, \forall t \geq T,$

$$\begin{aligned} \|x(t) - h_0\| = \sup_{\|h\|\leq 1} \| < x(t) - h_0, h > \| = & \sup_{\|h\|\leq 1} \| \int\limits_t^{+\infty} < x(\tau - r), A(\tau)h > d\tau\| \\ \leq & \sup_{\|h\|\leq 1} \int\limits_t^{+\infty} \|A(\tau)h\| d\tau \frac{1}{1-q}\|h_0\| \to 0, \end{aligned}$$

as $t \to +\infty$. This is proved the theorem. □

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