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# Vibration of corrugated cross-ply laminated composite plates

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**Abstract.** In the present paper the governing equations for dynamical analysis of corrugated cross-ply laminated composite plates in the form of a sin wave are developed based on the Kirchoff-Love's theory and the extension of Seydel's technique. The problems of natural vibration and forced vibration of a plate with various boundary conditions are studied. Effects of factors as geometry dimensions, order of laminate as well as waved-amplitude on frequency of natural vibration, amplitude of forced vibration of the corrugated cross-ply laminated composite plates are also analysed.

## 1. Introduction

Laminated structures like corrugated cross-ply laminated composite plates in the form of sine wave or fiber-reinforced composite plates are used widely in practice. Results of research about statical and dynamical problems of laminated composite flat plates were presented mainly in Chia's book [1]. A series of general articles about studying vibration of plates were reviewed by Sathyamoonthy [2]. However, the analysis of corrugated laminated composite plates in the form of sine wave has received comparitively little attention.

Corrugated plates of wave form made of isotropic elastic material were considered as flat orthotropic plates with corresponding orthotropic constants determined by the Seydel's technique. In this paper, the governing equations for dynamical analysis of corrugated cross-ply laminated composite plates in the form of sine wave are established based on the Kirchoff-Love's theory and the extension of Seydel's technique.

## 2. Constitutive equations of corrugated laminated composite plates

Consider a rectangular symmetrically laminated composite corugated plate in the form of sine wave (see Fig. 1), each layer of which is an unidirectional composite material. Suppose the cross-section line of corrugated plate in the plane $(x, z)$ has the form of sine wave

$$z = H \sin \frac{\pi x}{l}.$$

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Linear displacement – strain relationships in the middle surface for a such corrugated plate are [3]:

$$\varepsilon_x = \frac{\partial u}{\partial x} - kw, \qquad \chi_x = -\frac{\partial^2 w}{\partial x^2},$$
$$\varepsilon_y = \frac{\partial v}{\partial y}, \qquad \chi_y = -\frac{\partial^2 w}{\partial y^2}, \tag{1}$$
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \qquad \chi_{xy} = -2\frac{\partial^2 w}{\partial x \partial y},$$

where $u$, $v$ and $w$ denote displacements of the middle surface point along $x$, $y$ and $z$ directions respectively, $\varepsilon_i$ (i = 1, 2 and 6) are strains in the middle surface; $k$ is the curvature of the cross-section line in ($x$, $z$) plane, which is defined as:

$$k = \frac{z''}{(1 + z'^2)^{3/2}} \approx z'' = -H.\frac{\pi^2}{l^2}.\sin\frac{\pi x}{l},$$

From the stress - strain relation, after intergrating through the thickness of the plate we obtain the expressions for stress resultants:

$$N_x = A_{11}.\varepsilon_x + A_{12}\varepsilon_y, \qquad M_x = D^*_{11}.\chi_x + D^*_{12}.\chi_y,$$
$$N_y = A_{12}.\varepsilon_x + A_{22}.\varepsilon_y, \qquad M_y = D^*_{12}.\chi_x + D^*_{22}.\chi_y, \tag{2}$$
$$N_{xy} = A_{66}.\gamma_{xy}, \qquad M_{xy} = D^*_{66}.\chi_{xy},$$

where: $A_{ij}$ (i, j =1, 2 and 6) are extensional stiffnesses of the plate and $D_{ij}$ ($i$, $j$ = 1, 2 and 6) are bending stiffnesses of the corresponding flat plate.

Coefficients of the bending stiffness $D^*_{ij}$ of a corrugated plate are determined by the extension of Seydel's technique [4] as follows:

$$D^*_{11} = \frac{l}{s}D_{11}; \qquad D^*_{22} = E_2 I; \qquad D^*_3 = (D^*_{12} + 2D^*_{66}) = \frac{l}{s}(1 - v_1).D_3$$

where: $D_3 = D_{12} + 2D_{66}$

$$I = \frac{h.H^2}{2}\left[1 - \frac{0,81}{1 + 2,5.(\frac{H}{2l})^2}\right],$$

$$s = \int_0^l \sqrt{1 + \frac{\pi^2 H^2}{l^2}.cos^2\frac{\pi x}{l}}dx \approx l.\sqrt{1 + \frac{\pi^2 H^2}{2l^2}} \approx l(1 + \frac{\pi^2 H^2}{4l^2})$$

Substituting (1) into (2) we obtain:

$$N_x = A_{11}\left(\frac{\partial u}{\partial x} - k\,w\right) + A_{12}\frac{\partial v}{\partial y}, \qquad M_x = -D^*_{11}.\frac{\partial^2 w}{\partial x^2} - D^*_{12}.\frac{\partial^2 w}{\partial y^2},$$
$$N_y = A_{12}\left(\frac{\partial u}{\partial x} - k\,w\right) + A_{22}\frac{\partial v}{\partial y}, \qquad M_y = -D^*_{12}.\frac{\partial^2 w}{\partial x^2} - D^*_{22}.\frac{\partial^2 w}{\partial y^2}, \tag{3}$$
$$N_{xy} = A_{66}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right), \qquad M_{xy} = -2D^*_{66}.\frac{\partial^2 w}{\partial x\,\partial y}.$$

## 3. Motional equations of waved plate

According to [5], the motional equations of a plate are of the form:

$$\frac{\partial N_x}{\partial x}+\frac{\partial N_{xy}}{\partial y}=J_0\frac{\partial^2 u}{\partial t^2}$$
$$\frac{\partial N_{xy}}{\partial x}+\frac{\partial N_y}{\partial y}=J_0\frac{\partial^2 v}{\partial t^2}$$
$$\frac{\partial^2 M_x}{\partial x^2}+2\frac{\partial^2 M_{xy}}{\partial x\,\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=J_0\frac{\partial^2 w}{\partial t^2}-J_2\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right) \tag{4}$$

where: $J_i=\sum\limits_{k=1}^{n}\int\limits_{h_{k-1}}^{h_k}\rho^{(k)}z^{(i)}dz \qquad i=0,1,2.$

Substituting (3) into (4), we obtain a set of motional equations of a corrugated plate in terms of displacements:

$$A_{11}\frac{\partial^2 u}{\partial x^2}+A_{66}\frac{\partial^2 u}{\partial y^2}+(A_{12}+A_{66})\frac{\partial^2 v}{\partial x\,\partial y}+A_{11}\frac{H\pi^2}{l^2}\sin\frac{\pi x}{l}\frac{\partial w}{\partial x}+A_{11}\frac{H\pi^3}{l}.\cos\frac{\pi x}{l}.\mathrm{w}=J_0\frac{\partial^2 u}{\partial t^2},$$
$$A_{22}\frac{\partial^2 v}{\partial y^2}+A_{66}\frac{\partial^2 v}{\partial x^2}+(A_{12}+A_{66})\frac{\partial^2 u}{\partial x\,\partial y}+A_{12}.\frac{H\pi^2}{l^2}.\sin\frac{\pi x}{l}.\frac{\partial w}{\partial y}=J_0\frac{\partial^2 v}{\partial t^2}, \tag{5}$$
$$D_{11}^*\frac{\partial^4 w}{\partial x^4}+2\left(D_{12}^*+2D_{66}^*\right)\frac{\partial^4 w}{\partial x^2\,\partial y^2}+D_{22}^*\frac{\partial^4 w}{\partial y^4}-q=-J_0\frac{\partial^2 w}{\partial t^2}+J_2\left(\frac{\partial^4 w}{\partial x^2\partial t^2}+\frac{\partial^4 w}{\partial y^2\partial t^2}\right)$$

These equations are used to study static and dynamic states of laminated composite corugated plate in the form of sine wave.

## 4. Solution method

Consider a simply supported rectangular laminated composite corugated plate in the form of sine wave. The displacement field satisfying boundary conditions can be chosen as follows:

$$u=U(t)\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b},$$
$$v=V(t)\sin\frac{m\pi x}{a}\cos\frac{n\pi y}{b}, \tag{6}$$
$$w=W(t)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}$$

where $m$, $n$ are natural numbers representing the number of half waves in the $x$ and $y$ directions respectively.

Substituting (3) into (4) and applying the Bubnov-Galerkin procedure, we obtain a set of algebraic equations in matrix form as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{Bmatrix}\ddot U\\ \ddot V\\ \ddot W\end{Bmatrix}+\begin{bmatrix} b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{bmatrix}\begin{Bmatrix}U\\ V\\ W\end{Bmatrix}=\begin{Bmatrix}0\\ 0\\ \dfrac{4ab}{mn\pi^2}q\end{Bmatrix}, \tag{7}$$

where:

$$a_{11} = \frac{1}{4}J_0ab, \qquad a_{12} = a_{21} = 0, \qquad a_{13} = a_{31} = 0, \quad a_{23} = a_{32} = 0,$$

$$a_{22} = \frac{1}{4}J_0ab, \qquad a_{33} = \frac{1}{4}\left[J_0 + J_2\left(\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}\right)\right]ab,$$

$$b_{11} = \frac{1}{4}\left(A_{11}\frac{m^2\pi^2 b}{a} + A_{66}\frac{n^2\pi^2 a}{b}\right),$$

$$b_{12} = b_{21} = \frac{1}{4}\left(A_{12} + A_{66}\right)mn\pi^2,$$

$$b_{13} = -\frac{m^3 lbH\pi^2 A_{11}\left(\cos\frac{a\pi}{l} - 1\right)}{(a + 2ml)(a - 2ml)},$$

$$b_{22} = \frac{1}{4}\left(A_{22}\frac{n^2\pi^2 a}{b} + A_{66}\frac{m^2\pi^2 b}{a}\right),$$

$$b_{23} = \frac{m^2 nH\pi^2 lA_{12}\left(\cos\frac{a\pi}{l} - 1\right)}{(a + 2ml)(a - 2ml)},$$

$$b_{31} = b_{32} = 0,$$

$$b_{33} = \frac{1}{4}\left[D^*_{11}\frac{m^4\pi^4 b}{a^3} + 2\left(D^*_{12} + 2D^*_{66}\right)\frac{m^2n^2\pi^4}{ab} + D^*_{22}\frac{n^4\pi^4 a}{b^3}\right].$$

Similar, for a plate hinged at $x = 0; x = a$ and clamped at $y = 0; y = b$, the displacements $u, v$ can be chosen such as (6), but the deflection has of the form:

$$w = W(t)\sin\frac{m\pi x}{a}\left(1 - \cos\frac{2n\pi y}{b}\right).$$

### *4.1. Natural vibration problem*

For natural vibration, then $q(t) = 0$, functions $U(t), V(t), W(t)$ in (7) are taken as follows:

$$\begin{aligned} U(t) &= U_{mn}.e^{i\omega t}, \\ V(t) &= V_{mn}.e^{i\omega t}, \\ W(t) &= W_{mn}.e^{i\omega t}, \end{aligned} \tag{8}$$

equation (7) becomes:

$$\begin{bmatrix} b_{11} - a_{11}\omega^2 & b_{12} & b_{13} \\ b_{21} & b_{22} - a_{22}\omega^2 & b_{23} \\ b_{31} & b_{32} & b_{33} - a_{33}\omega^2 \end{bmatrix} \begin{Bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{Bmatrix} = 0 \tag{9}$$

Because of $U_{mn}$,$V_{mn}$,$W_{mn}$ being not equal to zero simultanously, the determinant of the homogeneous algebraic equation (9) must be to zero:

$$Det\begin{bmatrix} b_{11} - a_{11}\omega^2 & b_{12} & b_{13} \\ b_{21} & b_{22} - a_{22}\omega^2 & b_{23} \\ b_{31} & b_{32} & b_{33} - a_{33}\omega^2 \end{bmatrix} = 0 \tag{10}$$

This is an equation for $\omega^2$ to obtain fundamental frequencies of the natural vibration.

### *4.2. Forced vibration problem*

For forced vibration problem, when the plate is subjected to uniformly distributed excited force in the form $q(t) = q_0\sin\Omega t$, we can select functions $U(t), V(t), W(t)$ as follows:

$$\begin{aligned} U(t) &= u_0 \sin\Omega t, \\ V(t) &= v_0 \sin\Omega t, \\ W(t) &= w_0 \sin\Omega t, \end{aligned} \tag{11}$$

Substituting $q(t)$ and (11) into equation (7), we obtain a set of algebraic equations for $u_0, v_0, w_0$:

$$\begin{bmatrix} b_{11} - a_{11}\Omega^2 & b_{12} & b_{13} \\ b_{21} & b_{22} - a_{22}\Omega^2 & b_{23} \\ b_{31} & b_{32} & b_{33} - a_{33}\Omega^2 \end{bmatrix} \begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ \dfrac{4ab}{mn\pi^2}q_0 \end{Bmatrix} \tag{12}$$

When $\Omega \neq \omega$ determinant of equation (12) is not vanished, $Det\left|b_{ij} - a_{ij}\Omega^2\right| \neq 0$, from (12) the amplitudes of forced vibration of the corrugated plate can be determined:

$$\begin{Bmatrix} u_0 \\ v_0 \\ w_0 \end{Bmatrix} = \begin{bmatrix} b_{11} - a_{11}\Omega^2 & b_{12} & b_{13} \\ b_{21} & b_{22} - a_{22}\Omega^2 & b_{23} \\ b_{31} & b_{32} & b_{33} - a_{33}\Omega^2 \end{bmatrix}^{-1} \begin{Bmatrix} 0 \\ 0 \\ \dfrac{4ab}{mn\pi^2}q_0 \end{Bmatrix} \tag{13}$$

where $[...]^{-1}$ denotes an inverse matrix.

## 5. Numerical solution

Consider a rectangular symmetrically laminated composite corugated plate in the form of sine wave. The plate has geometry dimensions and structure as follows:
a = 0,9 m; b = 1,5m; and laminate: $\left[45^o/-45^0/-45^0/45^0\right]$

Thickness of a lamina t = 1mm.

Elastic coefficients of material AS4/3501 graphite/epoxy:

$$E_1 = 144.8GPa, \quad E_2 = 9.67GPa, \quad G_{12} = G_{13} = 4.14GPa, \quad \nu_{12} = 0.3, \quad \rho = 1389.23kg/m^3$$

We have studied the effects of dimensions, boundary conditions and order of lamina on the natural vibration frequency. The results are compared to flat plate with equivalent loads.

Table 1 shows the results of three first fundamential frequencies of waved plate hinged at all edges with two way of laminate order and comparing to flat plate.

Table 1.

| Laminate | $45^0/-45^0/-45^0/45^0$ | | $0^0/90^0/90^0/0^0$ | |
|---|---|---|---|---|
| Plate Mode | Waved plate | Flat plate | Waved plate | Flat plate |
| 1 | 278 (1,1) | 227,7 (1,1) | 201,7 (1,1) | 151,1 (1,1) |
| 2 | 610 (2,1) | 434 (1,2) | 479,6 (1,2) | 191,6 (1,2) |
| 3 | 651 (1,2) | 532,4 (2,1) | 639,3 (2,1) | 558,3 (2,1) |

Effect of laminate order on natural vibration frequency is shown in the fig 1.

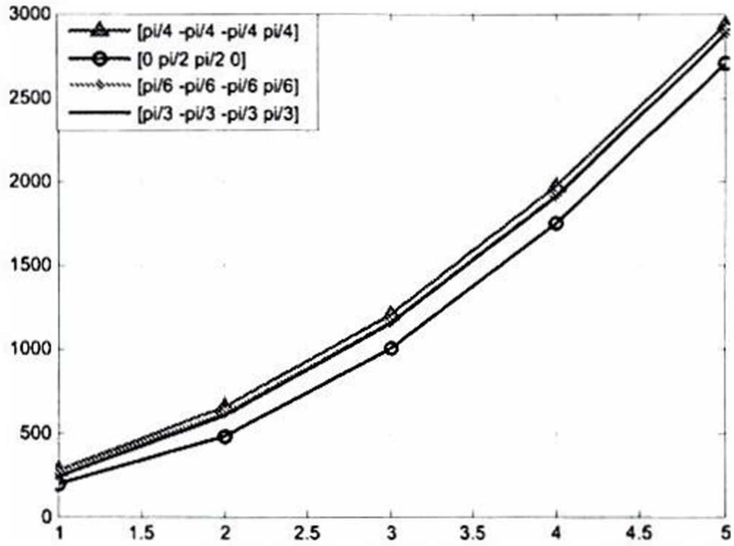


Fig. 1. Effect of laminate order on natural vibration frequency.

Table 2 shows the results of effect of boundary conditions and order of laminate on natural vibration frequency.

Table 2.

| Laminate | 45/-45/-45/45 | | 0/90/90/0 | |
|---|---|---|---|---|
| boundary Mode | Clamped- hinged | Hinged | Clamped- hinged | Hinged |
| 1 | 297 (1,1) | 278 (1,1) | 294 (1,1) | 201,7 (1,1) |
| 2 | 517 (2,1) | 610 (2,1) | 665 (2,1) | 479,6 (1,2) |
| 3 | 943 (3,1) | 651 (1,2) | 1004 (1,2) | 639,3 (2,1) |

Effect of dimensions and boundary conditions of laminate plate $[45^0/-45^0/-45^0/45^0]$ on fundamential vibration frequency shows on table 3 and fig 2.

Table 3.

| b(m) Boundary | 0,9 | 1,5 | 2,1 | 2,7 | 3,3 |
|---|---|---|---|---|---|
| B4 | 508 (1,1) | 278 (1,1) | 201,6 (1,1) | 164,5 (1,1) | 143,3 (1,1) |
| N4 | 730 (1,1) | 297 (1,1) | 183,9 (1,1) | 140,9 (1,1) | 121,3 (1,1) |
| Flat plate B4 | 334,4 (1,1) | 227,7 (1,1) | 171,9 (1,1) | 142,8 (1,1) | 125,5 (1,1) |

Effects of the height H on natural vibration frequency and buckling amplitude are shown on the fig 3 and fig 4, respectively.

## 6. Discussion

- Tables of data and graphs above show that a waved composite plate has natural vibration frequency much more greater than that of a flat plate. It shows that stiffness of a waved composite plate is much more greater than stiffness of a flat plate.

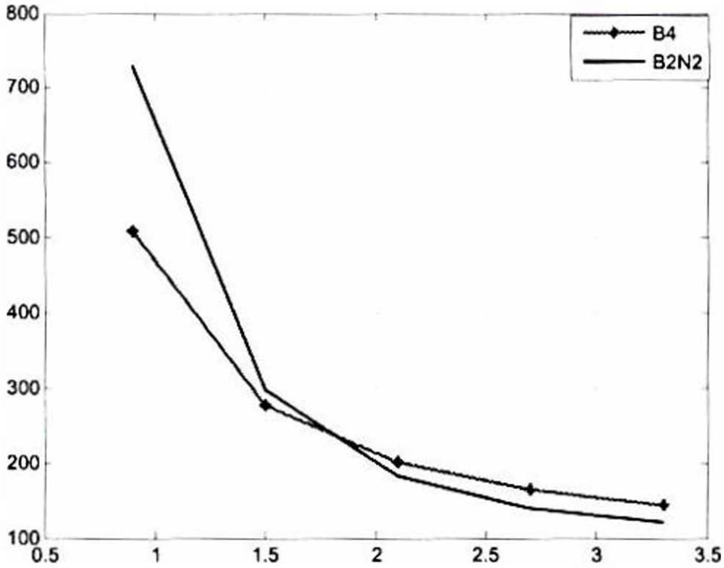


Fig. 2. Effect of boundary condition on fundamential vibration frequency.

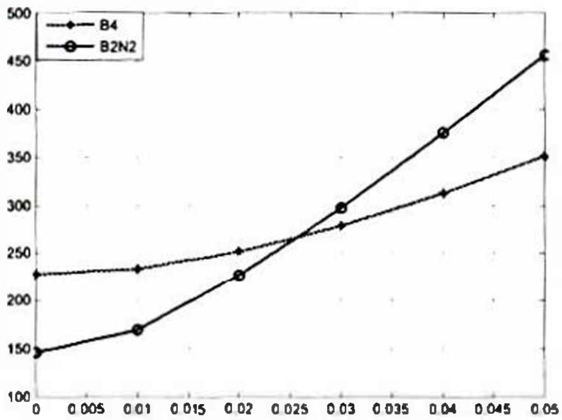


Fig. 3. Effect of the height H on natural vibration frequency.

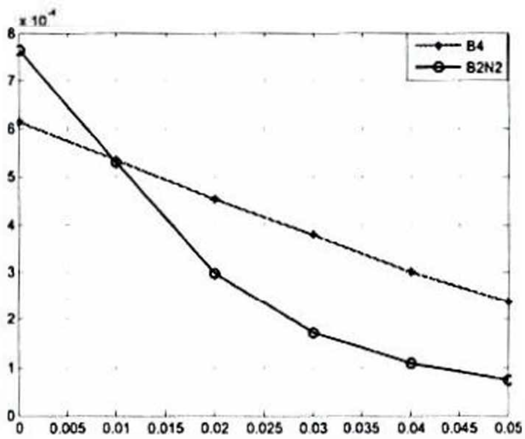


Fig. 4. Effect of the height H on vibration amplitude.

- When the height H increases, the vibration frequency also increases (see Fig 3), but the vibration amplitude reduces, it means that stiffness ofa plate increases when increas H.
- When the length of a plate increases, the amplitude also increases (see Fig 5), it means that stiffness of a plate reduces. Therefore, when manufacturing a plate, we have to design dimensions of length and width so that it is the most sensible plate.

## 7. Conclusion

- Based on the proposed strain expression and Seldel's technique, the governing equations for dynamical analysis of corrugated cross-ply laminated composite plates in the form of sine wave are formulated.

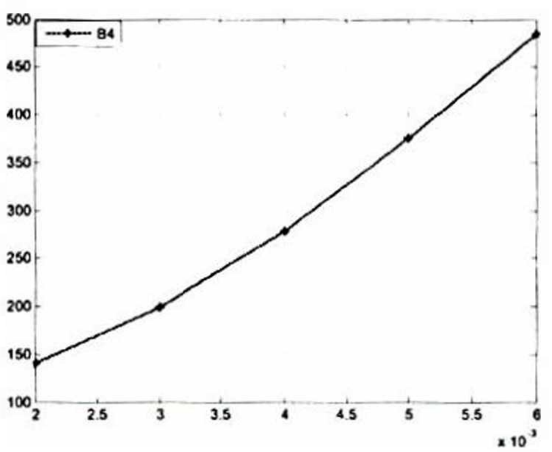


Fig. 5. Effect of thickness h on natural vibration frequency.

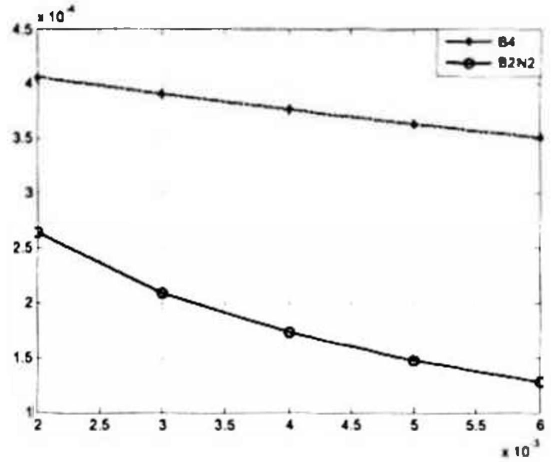


Fig. 6. Effect of thickness h on vibration amplitude.

- The natural vibration and forced vibration of waved composite plate and analysis of some effects on the vibration are studied from that some discussion are given for this kind of plates, which can be used in practice.

- Obtained results can be extended to the other form of corrugated plates which satisfy proposed requirements

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