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On the asymptotic behavior of delay differential equations and its relationship with C_0- semigroup

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Abstract. In this paper, we study the asymptotic behavior of linear differential equations under nonlinear perturbation. Let's consider the delay differential equations:

$$\frac{dx}{dt} = Ax + f(t, x_t),$$

where $t \in R^+, A \in \mathcal{L}(E), f : R^+ \times E \longrightarrow E$ and $(T(t))_{t\geq 0}$ is C_0-semigroup be generated by A. We will give some sufficient conditions for uniformly stable and asymptotic equivalence of above equations.

1. Introduction

Consider the following delay differential equations (Eq.):

$$\frac{dx(t)}{dt} = Ax(t) + \mu f(t, x(t+\theta)), \quad t \geq 0, \quad -h \leqslant \theta \leqslant 0, \tag{1}$$

where $x(.) \in E, A \in \mathcal{L}(E)$, E is a Banach space, the operator $f : R^+ \times E \longrightarrow E$ is continuous in t and satisfies all following conditions:

$$f(t, 0) = 0, \tag{2}$$

$$\|f(t, y(t+\theta)) - f(t, z(t+\theta))\| \leqslant L \sup_{-h\leqslant\theta\leqslant 0} \|y(t+\theta) - z(t+\theta)\|. \tag{3}$$

In [1], K.G.Valeev proved that if Eq.(1) satisfies (2) and (3) with given initial condition

$$x(t) = \varphi(t), \quad -h \leqslant t \leqslant 0, \quad \varphi(.) \in C([-h, 0], E),$$

then Eq.(1) has a unique solution on the half-line.

In recent years, much attentions have been devoted to the qualitative theory of solutions of differential equation with time delay (see [1-5]). In this direction, a particular attentions has been focused on extending the classical results on the asymptotic behavior of solutions of differential equations. In many applied models concerned to mechanics, models of biology and population (see [6-9]).

In this paper, we give some extending results for sufficient conditions of stable and asymptotic equivalence (see [1-5]) of linear delay differential equations under nonlinear perturbation in Banach space. The obtained results thank to use of the theories of general dynamic systems (see [10, 11]).

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2. Main results

2.1. The uniformly stable of null solution of delay differential equations

Let us consider the delay differential equations

$$\frac{dx(t)}{dt} = Ax(t) + \mu f(t, x(t+\theta)) \quad t \geq 0, -h \leqslant \theta \leqslant 0, \tag{4}$$

with given initial condition $x(t) = \varphi(t), -h \leqslant t \leqslant 0$. Where $x(.) \in E; A \in \mathcal{L}(E); f : R^+ \times E \to E$ is continuous in t and satisfies following conditions

$$f(t, 0) = 0, \tag{5}$$

$$\|f(t, y(t+\theta)) - f(t, z(t+\theta))\| \leqslant g(t) \sup_{-h\leqslant\theta\leqslant0} \|y(t+\theta) - z(t+\theta)\|, \tag{6}$$

$$\int_0^\infty g(\tau)d\tau \leqslant m < \infty. \tag{7}$$

Let $(T(t))_{t\geq0}$ be a continuous semigroup of linear operators in the Banach space E and (A, D(A)) is generator of T(t) (see [10]). Throughout this paper, we always assume that $(T(t))_{t\geq0}$ is strongly continuous semigroup (C_0- semigroup). We show that if Eq.(4) satisfies conditions (5), (6), (7) then the solution of Eq.(4), with given initial condition $x(t) = \varphi(t); -h \leqslant t \leqslant 0$, can be written in the form

$$\begin{cases} x(t) = T(t)\varphi(0) + \mu \int_0^t T(t-s)f(s, x(s+\theta))ds, & t \geq 0, \\ x(t) = \varphi(t), & -h \leqslant t \leqslant 0 \end{cases} \tag{8}$$

First of all, we investigate an extention of the conditions for stable (see [12]) of solution of delay linear diffirential equation under nonlinear perturbation. We recall that, the conditions (5), (6), (7) are satisfied. By using the Gronwall-Bellman's lemma(see [12]), we can get the following result:

Theorem 2.1. *Suppose $(T(t))_{t\geq0}$ is C_0- semigroup with the generator (A, D(A)). The following assertions are true:*

(1) If $\|T(t)\| \leqslant M, \forall t \geq 0$, then the null solution $x(t) \equiv 0$ of Eq.(4) is uniformly stable.

(2) If $\lim_{t\to\infty} \|T(t)\| = 0$, then the null solution $x(t) \equiv 0$ of Eq.(4) is uniformly exponential stable.

Proof. Throughout this paper in proof of theorems, we always use the following norm

$$|||x(t)||| = \sup_{t_0\leqslant\tau\leqslant t} \|x(\tau)\|, \quad t \geqslant t_0 \geqslant -h.$$

i) Since $\|T(t)\| \leqslant M, \forall t \geq 0$, the solution of Eq.(4) with intial condition $x(t) = \varphi(t); -h \leqslant t \leqslant 0$ exists solely. It can be written in the form:

$$x(t) = T(t)\varphi(0) + \mu \int_0^t T(t-s)f(s, x(s+\theta))ds, \quad t \geq 0.$$

Thus

$$\|x(t)\| \leqslant \|T(t)\|\|\varphi(0)\| + \mu \int_0^t \|T(t-s)\|\|f(s, x(s+\theta))\|ds, \quad t \geq 0.$$

By using assumptions (5), (6), (7), (i), we have

$$\|x(t)\| \leqslant M\|\varphi(0)\| + \int_0^t M\|f(s, x(s+\theta))\|ds, \quad t \geq 0.$$

and

$$\|x(t)\| \leqslant M\|\varphi(0)\| + \int_0^t Mg(s)\|x(s+\theta)\|ds, \quad t \geq 0.$$

Hence

$$|\|x(t)\| \leqslant M\|\varphi(0)\| + \int_0^t Mg(s).\||x(s+\theta)\||ds, \quad t \geq 0.$$

Using the Gronwall-Bellman's inequality, we obtain:

$$\|x(t)\| \leqslant M\|\varphi(0)\|.e^{M\mu\int_{t_0}^t g(s)ds}.$$

Consequently,

$$\|x(t)\| \leqslant M\|\varphi(0)\|.e^{\mu Mqm}.$$

Put

$$\delta = \frac{\varepsilon}{Me^{\mu Mqm}}.$$

Since definition, we can show that the null solution $x(t) \equiv 0$ of Eq.(4) is uniformly stable.
ii) By assumption (ii) of the theorem there exist the positive constants $C \geqslant 1$ and λ

$$\|T(t)\| \leqslant Ce^{-\lambda t}, \quad \forall t > 0.$$

By the similar argument as (i), we have

$$\|x(t)\| \leqslant Ce^{-\lambda t}\|\varphi(0)\| + \mu\int_0^t Ce^{-\lambda(t-s)}\|f(s, x(s+\theta))\|ds, \quad t \geq 0.$$

By (7), we have

$$\|x(t)\| \leqslant Ce^{-\lambda t}\|\varphi(0)\| + \int_0^t Ce^{-\lambda(t-s)}g(s)\|x(s+\theta))\|ds, \quad t \geq 0.$$

and,

$$\|x(t)\|e^{\lambda t} \leqslant C\|\varphi(0)\| + \mu\int_0^t Ce^{\lambda(s)}g(s).\||x(s+\theta))\||ds, \quad t \geq 0.$$

hence

$$\|x(t)\|e^{\lambda t} \leqslant C\|\varphi(0)\|.e^{C\mu\int_{t_0}^t g(s)ds}.$$

Consequently,

$$\|x(t)\| \leqslant C\|\varphi(0)\|e^{\mu Cqm}e^{-\lambda t}.$$

It proves that the null solution $x(t) \equiv 0$ of Eq.(4) is uniformly exponential stable.

2.2. The asymptotic equivalence of linear delay differential equations under nonlinear perturbation in Banach space

In this section, we are interested in finding conditions such that the solution of Eq.(4) in the case $\mu = 0$ will be asymptotic equivalence to the solution of Eq.(4) in the case $\mu \neq 0$(in the following we will give $\mu = 1$).The obtained result of this part is an extention of Levinson's theorem to the case of linear delay differential equations under nonlinear perturbation (see [1, 13, 14]). Let's consider the two following differential equations:

$$\frac{dx(t)}{dt} = Ax(t), t \geq 0, \tag{9}$$

$$\frac{dy(t)}{dt} = Ay(t) + f(t, y(t+\theta)), t \geq 0, \tag{10}$$

where $x(.) \in E, A \in \mathcal{L}(E)$, $f : R^+ \times E \longrightarrow E$ is satisfied (5), (6), (7).

Definition 2.2. Eq.(9) and Eq.(10) are said to be asymptotic equivalence if for every solution $x(t)$ of Eq.(9), there exists a solution $y(t)$ of Eq.(10) such that

$$\lim_{t \to +\infty} \|y(t) - x(t)\| = 0,$$

and conversely.

Next, we prove the following theorem :

Theorem 2.3. *Suppose that there exist positive constants M, C, ω and a projector $P : E \to E$ such that:*

(1) $\|T(t)P\| \leqslant Me^{-\omega t}$, for all $t \in R^+$,

(2) $\|T(t)(I - P)\| \leqslant C$, for all $t \in R$.

Then Eq.(9) and Eq.(10) are asymptotic equivalence.

Proof. In order to prove the theorem, we recall that assumptions (5),(6) and (7) hold for (10). Put

$$U(t) = T(t)P, \quad V(t) = T(t)(I - P).$$

We get

$$T(t) = U(t) + V(t).$$

Hence

$$\begin{aligned} T(t-s)V(s-\tau) &= T(t-s)T(s-\tau)(I-P) \\ &= V(t-\tau). \end{aligned}$$

Next, The proof of the theorem falls into two steps.

Step 1: Assume that $y(t)$ is the solution of Eq.(9), for each sufficiently large $s \geq 0$, $y(s) \in E$ we set

$$x(s) = y(s) + \int_s^\infty V(s-\tau) f(\tau, y(\tau+\theta)) d\tau.$$

Therefore, the solution of Eq.(10) and Eq.(9) can be written in the form

$$\begin{aligned} x(t) &= T(t-s)x(s) \\ &= T(t-s)y(s) + \int_s^\infty V(t-\tau) f(\tau, y(\tau+\theta)) d\tau. \end{aligned}$$

$$y(t) = T(t-s)y(s) + \int_0^t T(t-\tau) f(\tau, y(\tau+\theta)) d\tau; t \geq s,$$

Consequently

$$\|y(t) - x(t)\| = \left\| -\int_s^\infty V(t-\tau) f(\tau, y(\tau+\theta)) d\tau + \int_s^t T(t-\tau) f(\tau, y(\tau+\theta)) d\tau \right\|.$$

By the suppositions (i), (ii) we have

$$\|y(t)-x(t)\| \leqslant M.\int_s^t e^{-\omega(t-\tau)}g(\tau)\|y(\tau+\theta)\|d\tau + C\int_t^\infty g(\tau)\|y(\tau+\theta)\|d\tau$$

$$\leqslant M.\int_s^t e^{-\omega(t-\tau)}g(\tau)|||y(\tau+\theta)|||d\tau + C\int_t^\infty g(\tau)|||y(\tau+\theta)|||d\tau$$

$$\leqslant MM_0\int_s^t e^{-\omega(t-\tau)}g(\tau)d\tau + CM_0\int_t^\infty g(\tau)dt, \quad \forall t \geq s,$$

where M_0 is constant such that $\|y(s)\| \leqslant M_0, \forall s \geq 0$ (see theorem 2.1). Hence

$$\|y(t)-x(t)\| \leqslant M_1\int_s^t e^{-\omega(t-\tau)}g(\tau)d\tau + M_2\int_t^\infty g(\tau)dt, \quad \forall t \geq s,$$

where $M_1 = MM_0$, $M_2 = CM_0$. By the similar arguments as in [1], we have

$$\|y(t)-x(t)\| < \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3} = \varepsilon.$$

This means that

$$\lim_{t\to\infty}\|y(t)-x(t)\| = 0.$$

Step 2: Let $x(t)$ is the solution of Eq.(10). By successive approximations method, we can show that for each $x(s) \in E$ (with sufficiently large $s \geq \Delta > 0$), there are a solution $y(t)$ of Eq.(9) satisfies the following condition

$$x(s) = y(s) + \int_s^\infty V(s-\tau)f(\tau, y(\tau+\theta))d\tau.$$

Put

$$y(t) = T(t-s)y(s) + \int_0^t T(t-s)f(\tau, y(\tau+\theta))d\tau, \quad t \geq s.$$

Continuing the above process by the same arguments as step 1, we obtain

$$\|y(t)-x(t)\| < \varepsilon$$

Consequently

$$\lim_{t\to\infty}\|y(t)-x(t)\| = 0.$$

2.3. Application

In recent years, many new dynamic systems of population have been formulated and studied. In this direction, G.F.Webb has established the theory of nonlinear age-dependent population dynamics in 1985 (see [9]). After, G.F.Webb and H.Inaba have studied the asymptotic properties of the population dynamics in the following model (see [6, 7, 9]):

$$\left(\frac{\partial}{\partial a}+\frac{\partial}{\partial t}\right)p(a,t) = Q(a)p(a,t) + \mu f(a,t), \tag{11}$$

$$p(0,t) = \int_0^{\omega} M(a)p(a,t)da. \quad t > 0,$$
$$p(a,0) = \phi(a).$$

This inhomogeneous model (11) is rewritten as an astract Cauchy problem (see [6, 7]):

$$\frac{d}{dt}p(t) = Ap(t) + \mu f(t, p(t)), \quad t > 0.$$
$$p(0) = \varphi$$

In next, we assume that $E = L^1(0, \omega; C^n)$ (see [9], H.Inaba), $A : D(A) \subset E \to E$ and operator $f : R^+ \times E \to E$ is continuous in t and satisfies all conditions (5), (6), (7). And now, we introduce the delay differential equation:

$$\frac{d}{dt}p(t) = Ap(t) + \mu f(t, p(t+\theta)), \quad t > 0, \quad -h \leqslant \theta \leqslant 0, \tag{12}$$
$$p(t) = \varphi(t), \quad -h \leqslant t \leqslant 0.$$

In the following, we recall that operator A will be unbounded (the hypothesis $A \in \mathcal{L}(E)$ is not right). However, if we further assume that $(T(t))_{t \geq 0}$ is bounded C_0- semigroup with populations generator (A, D(A)), then we can investigate the assertion (2.5) for Eq.(13) (see [7, 11, 12]). Applying the process of argument of parts 2.1 and 2.2 to the uniformlly stable and asymptotic equivalence of above model of population, we will give the following results :

a. If the C_0- semigroup operator $(T(t))_{t \geq 0}$ is uniformly bounded then the null solution of Eq.(12) is uniformly stable.

b. If the C_0- semigroup operator $(T(t))_{t \geq 0}$ satisfies the hypothesis of theorem (2.3) then the solution of Eq.(12) in the case $\mu = 0$ is asymptotic equivalence to the solution of Eq.(12) in the case $\mu \neq 0$.

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