

The parameter-dependent cyclic inequality

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Abstract. In this paper we will construct a parameter-dependent cyclic inequality that can be used to prove a lot of hard and interesting inequalities.

1. Introduction

The cyclic inequality is a type of inequality that may be right in just some particular cases but not in general. In this paper, we propose one type of parameter-dependent cyclic inequality from a special inequality. Thanks to this inequality, we can obtain many inequalities by choosing α and n . Note that it can be proved by some ways in particular case. However in order to prove it in general case, we have to use the method that is mentioned in the paper.

2. The general case

Denote $R^+ = \{x \in R \mid x > 0\}$.

Lemma 1.1. Assume that $x_i \in R$, ($i = \overline{1, n}$) we have

$$\sum_{1 \leq i < j \leq n} x_i x_j \leq \frac{n-1}{2n} \left(\sum_{i=1}^n x_i \right)^2.$$

Proof. We have

$$\sum_{1 \leq i < j \leq n} x_i x_j \leq \sum_{1 \leq i < j \leq n} \frac{x_i^2 + x_j^2}{2}.$$

Since $1 + 2 + \dots + (n-1) = \frac{n(n-1)}{2}$, hence the number of terms of $\sum_{1 \leq i < j \leq n} x_i x_j$ is $\frac{n(n-1)}{2}$. It follows

$$2 \sum_{1 \leq i < j \leq n} x_i x_j \leq \sum_{1 \leq i < j \leq n} (x_i^2 + x_j^2) = (n-1) \left(\sum_{i=1}^n x_i^2 \right).$$

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Adding both sides of the above inequality by $2(n - 1) \sum_{1 \leq i < j \leq n} x_i x_j$, we obtain the inequality as was to be proved.

The proof of Lemma 1.1 is complete.

Theorem 1.1. Assume that x_i ($i = \overline{1, n}$), $n \geq 3$ are positive number. Then there holds the following inequality

$$\frac{x_1}{x_1 + \alpha(x_2 + \dots + c_n x_{k+1})} + \frac{x_2}{x_2 + \alpha(x_3 + \dots + c_n x_{k+2})} + \dots + \dots + \frac{x_n}{x_n + \alpha(x_1 + \dots + c_n x_k)} \geq \frac{2n}{2 + \alpha(n - 1)} \quad (1.1)$$

Where $c_n = \frac{(n \bmod 2 + 1)}{2}$, $k = \lfloor \frac{n}{2} \rfloor$ and α is an arbitrary real number satisfies $\alpha \geq 2$.

Proof. First, for the sake convinience, we set

$$P = \frac{x_1}{x_1 + \alpha(x_2 + \dots + c_n x_{k+1})} + \frac{x_2}{x_2 + \alpha(x_3 + \dots + c_n x_{k+2})} + \dots + \dots + \frac{x_n}{x_n + \alpha(x_1 + \dots + c_n x_k)} \geq \frac{2n}{2 + \alpha(n - 1)}.$$

Now let's consider the case $n = 2k + 1$ it gives

$$P = \frac{x_1^2}{x_1^2 + \alpha(x_1 x_2 + \dots + x_1 x_{k+1})} + \frac{x_2^2}{x_2^2 + \alpha(x_2 x_3 + \dots + x_2 x_{k+2})} + \dots + \frac{x_n^2}{x_n^2 + \alpha(x_n x_1 + \dots + x_n x_k)}.$$

Using the fact that

$$\sum_{i=1}^n \frac{x_i^2}{a_i} \geq \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n a_i} \quad (1.2)$$

with $a_i \in R^+$ ($i = \overline{1, n}$), it implies

$$P \geq \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n x_i^2 + \alpha \sum_{1 \leq i < j \leq n} x_i x_j}.$$

Since $\alpha \geq 2$, it can be rewritten as $\alpha = 2 + \beta$ with $\beta \geq 0$. This leads to

$$P \geq \frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2 + \beta \sum_{1 \leq i < j \leq n} x_i x_j}.$$

Applying Lemma (1.1) we obtain

$$P \geq \frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2 + \frac{\beta(n - 1)}{2n} (\sum_{i=1}^n x_i)^2}$$

or $P \geq \frac{2n}{2 + \alpha(n - 1)}.$

Next, for $n = 2k$, we get

$$P = \frac{x_1^2}{x_1^2 + \alpha(x_1x_2 + \dots + x_1x_k + \frac{1}{2}x_1x_{k+1})} + \frac{x_2^2}{x_2^2 + \alpha(x_2x_3 + \dots + x_2x_{k+1} + \frac{1}{2}x_2x_{k+2})} + \dots + \frac{x_n^2}{x_n^2 + \alpha(x_nx_1 + \dots + x_nx_{k-1} + \frac{1}{2}x_nx_k)}$$

Applying the inequality (1.2), we get

$$P \geq \frac{(\sum_{i=1}^n x_i)^2}{\sum_{i=1}^n x_i^2 + \alpha \sum_{1 \leq i < j \leq n} x_i x_j} = \frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2 + \beta \sum_{1 \leq i < j \leq n} x_i x_j}$$

Using the Lemma 1.1 once more, we come to the following inequality

$$P \geq \frac{(\sum_{i=1}^n x_i)^2}{(\sum_{i=1}^n x_i)^2 + \frac{\beta(n-1)}{2n} (\sum_{i=1}^n x_i)^2} = \frac{2n}{2 + \alpha(n-1)}$$

Thus Theorem 1.1 is proved.

3. The special cases

For $n = 3$, we obtain the following inequalities.

Example 1.1. Let a, b, c be positive numbers, $\alpha \geq 2$, prove that

$$\frac{a}{a + \alpha b} + \frac{b}{b + \alpha c} + \frac{c}{c + \alpha a} \geq \frac{3}{1 + \alpha}$$

Take $\alpha = 2$ we obtain

Example 1.2. Let a, b, c be positive numbers, prove that

$$\frac{a}{a + 2b} + \frac{b}{b + 2c} + \frac{c}{c + 2a} \geq 1$$

Take $\alpha = \frac{1}{abc} \geq 2 \Leftrightarrow abc \leq \frac{1}{2}$, we yield

Example 1.3. Let a, b, c be positive numbers satisfy $abc \leq \frac{1}{2}$, prove that

$$\frac{a^2c}{1 + a^2c} + \frac{b^2a}{1 + b^2a} + \frac{c^2b}{1 + c^2b} \geq \frac{3abc}{1 + abc}$$

For $n = 4$ we yield the inequality

Example 1.4. Assume that $a, b, c, d \in R^+, \alpha \geq 2$, prove that

$$\frac{a}{2a + \alpha(2b + c)} + \frac{b}{2b + \alpha(2c + d)} + \frac{c}{2c + \alpha(2d + a)} + \frac{d}{2d + \alpha(2a + b)} \geq \frac{4}{2 + 3\alpha}$$

Take $\alpha = 2$ we obtain

Example 1.5. Assume that $a, b, c, d \in R^+$, prove that

$$\frac{a}{2a + 4b + 2c} + \frac{b}{2b + 4c + 2d} + \frac{c}{2c + 4d + 2a} + \frac{d}{2d + 4a + 2b} \geq \frac{1}{2}$$

Take $a = b, b = c$ we get

Example 1.5. Assume that $a, b \in R^+, \alpha \geq 2$, prove that

$$\frac{a((\alpha + 2)a + 2\alpha b)}{[2a + \alpha(2a + b)][2a + 3\alpha b]} + \frac{b((\alpha + 2)b + 2\alpha a)}{[2b + \alpha(2b + a)][2b + 3\alpha a]} \geq \frac{2}{2 + 3\alpha}$$

For $n = 5$ we yield the inequality

Example 1.7. Give $a, b, c, d, e \in R^+, \alpha \geq 2$, prove that

$$P = \frac{a}{a + \alpha(b + c)} + \frac{b}{b + \alpha(c + d)} + \frac{c}{c + \alpha(d + e)} + \frac{d}{d + \alpha(e + a)} + \frac{e}{e + \alpha(a + b)} \geq \frac{5}{1 + 2\alpha}$$

Take $c = d = e, \alpha = 2$ we yield the inequality

Example 1.8. Given $a, b, c \in R^+$, prove that

$$\frac{a}{a + 2b + 2c} + \frac{b}{b + 4c} + 2c \left(\frac{2a + 2c + b}{[c + 2(c + a)][c + 2(a + b)]} \right) \geq \frac{4}{5}$$

For $n = 6$ we yield

Example 1.9. Given $a_i \in R^+ (i = \overline{1, 6}), \alpha \geq 2$, prove that

$$\begin{aligned} & \frac{a_1}{a_1 + \alpha(a_2 + a_3 + \frac{1}{2}a_4)} + \frac{a_2}{a_2 + \alpha(a_3 + a_4 + \frac{1}{2}a_5)} + \frac{a_3}{a_3 + \alpha(a_4 + a_5 + \frac{1}{2}a_6)} + \\ & + \frac{a_4}{a_4 + \alpha(a_5 + a_6 + \frac{1}{2}a_1)} + \frac{a_5}{a_5 + \alpha(a_6 + a_1 + \frac{1}{2}a_2)} + \frac{a_6}{a_6 + \alpha(a_1 + a_2 + \frac{1}{2}a_3)} \geq \frac{12}{2 + 5\alpha} \end{aligned}$$

Finally, take $a_1 = a_2 = a, a_3 = a_4 = b, a_5 = a_6 = c$ and $\alpha = 2$ we get

Example 1.10. Assume that $a, b, c \in R^+$, prove that

$$a \left(\frac{1}{3a + 3b} + \frac{1}{a + 4b + c} \right) + b \left(\frac{1}{3b + 3c} + \frac{1}{b + 4c + a} \right) + c \left(\frac{1}{3c + 3a} + \frac{1}{c + 4a + b} \right) \geq 1.$$

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References

[1] B.A Troesch, The cyclic inequality for a large number of terms, *Notices Amer. Math Soc.* 25 (1978).
 [2] E.S Freidkin, S.A Freidkin, On a problem by Shapiro, *Elem. Math.* 45 (1990) 137.