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On the matheron theorem for topological spaces

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Abstract. In this paper we study the extending of the Matheron theorem for general topological spaces. We also show some examples about the spaces $\mathcal{F}$ such that the miss-and-hit topology on those spaces are unseparated or non-Hausdorff.

1. Introduction

The Choquet theorem (see [1, 2]) plays very importance role in theory of random sets. The proof of this theorem is based on the Matheron theorem and especially, the locally compact property of the space $\mathcal{F}$, where $\mathcal{F}$ is a space of all close subsets of a given space E and $\mathcal{F}$ is equipped with the miss-and-hit topology (see [1]). The Matheron theorem is stated as follows.

Theorem. *Let E be a complete, separable and locally compact metric space. Then the miss-and-hit topology on $\mathcal{F}$ space of all closed subsets of E is compact, separable and Hausdorff.*

Note that the natural domain of the probability theory is a Polish space, which is, in general, not locally compact. So in [3], the authors extended the Matheron theorem for general metric space. They showed that if E is a separable metric space, then the miss-and-hit topology on space $\mathcal{F}$ is separable and compact. And if E has a non-locally compact point, then the miss-and-hit topology on space $\mathcal{F}$ is not Hausdorff. Now we extend the Matheron theorem for general topological space.

Let E be a topological space. Denote $\mathcal{F}, \mathcal{K}$ and $\mathcal{G}$ the families of all close, compact and open subsets of E respectively.

For every $A \subset E$, we denote

$$\mathcal{F}_A = \{F : F \in \mathcal{F}, F \cap A \neq \emptyset\}; \ \mathcal{F}^A = \{F : F \in \mathcal{F}, F \cap A = \emptyset\}.$$

For every $K \in \mathcal{K}$ and a finite family of sets $G_1, \ldots, G_n \in \mathcal{G},\ n \in \mathbb{N}$, we put

$$\mathcal{F}^K_{G_1,\ldots,G_n} = \mathcal{F}^K \bigcap \mathcal{F}_{G_1} \cdots \bigcap \mathcal{F}_{G_n}.$$

Then

$$\{\mathcal{F}^K_{G_1,\ldots,G_n} : K \in \mathcal{K},\ G_1, \ldots, G_n \in \mathcal{G},\ n \in \mathbb{N}\}$$

is a base of topology on $\mathcal{F}$. Which is called a *miss-and-hit topology* on $\mathcal{F}$.

We have

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Main theorem

i) If E is a separable and Hausdorff topological space, then the miss-and-hit topology on space $\mathcal{F}$ is separable.

ii) Let E be a topological space. Then the miss-and-hit topology on space $\mathcal{F}$ is compact.

iii) Let E be a topological space.

i) Then the space $\mathcal{F}$ with the miss-and-hit topology is a T_1-space.

ii) If E is a T_1-space and has a non-locally compact point, then the miss-and-hit topology on space $\mathcal{F}$ is not Hausdorff.

iv) If E is an uncountable set with Zariski topology, then the miss-and-hit topology on space $\mathcal{F}$ is Hausdorff and unseparated.

v) There exists a topology on the set of all natural numbers $\mathbb{N}$ such that this topology space is a compact and T_1-space. Moerover, space $\mathcal{F}$ with the miss-and-hit topology is non-Hausdorff space.

The paper is organized as follows. In section 2 we will prove some results on the extending of Matheron theorem for topological space. In Section 3 we will show some examples about the spaces $\mathcal{F}$ which are unseparated or non-Hausdorff for the miss-and-hit topology.

2. On the Matheron theorem

Theorem 2.1. *If E is a separable and Hausdorff topological space, then the miss-and-hit topology on space $\mathcal{F}$ is separable.*

Proof. Let A be a countable and dense subset in E. For every $F \in \mathcal{F}$, suppose that $\mathcal{F}^K_{G_1,\ldots,G_n}$ is a neighborhood of F. Then $G_i \backslash K$ are open and non-empty, so we can choose $x_i \in A \cap (G_i \backslash K)$ for $i = 1, \ldots, n$. We obtain

$$\{x_1, \ldots, x_n\} \cap K = \emptyset \text{ and } \{x_1, \ldots, x_n\} \cap G_i \neq \emptyset$$

for all $i = 1, \ldots, n$.

Thus,

$$\{x_1, \ldots, x_n\} \in \mathcal{F}^K_{G_1,\ldots,G_n}.$$

Since the class of finite subsets of A is countable, we conclude that $\mathcal{F}$ is a separable space.

Theorem 2.2. *Let E be a topological space. Then the miss-and-hit topology on space $\mathcal{F}$ is compact.*

Proof. By Alexandroff theorem, in order to prove that the miss-and-hit topology on space $\mathcal{F}$ is compact, it is sufficient to show that if

$$\{\mathcal{F}^{K_i} : K_i \in \mathcal{K},\ i \in I\} \bigcup \{\mathcal{F}_{G_j} : G_j \in \mathcal{G},\ j \in J\}$$

is a cover of $\mathcal{F}$, then it has a finite subcover. Put $\Omega = \bigcup_{j \in J} G_j$, then Ω is an open set. Since

$$\mathcal{F} = (\bigcup_{i \in I} \mathcal{F}^{K_i}) \bigcup (\bigcup_{j \in J} \mathcal{F}_{G_j}),$$

we have

$$\begin{aligned}\emptyset &= \Big(\bigcap_{i\in I}(\mathcal{F}\backslash\mathcal{F}^{K_i})\Big)\bigcap\Big(\bigcap_{j\in J}(\mathcal{F}\backslash\mathcal{F}_{G_j})\Big)\\ &= \Big(\bigcap_{i\in I}\mathcal{F}_{K_i}\Big)\bigcap\Big(\bigcap_{j\in J}\mathcal{F}^{G_j}\Big)\\ &= \Big(\bigcap_{i\in I}\mathcal{F}_{K_i}\Big)\bigcap\mathcal{F}^{\Omega}\\ &= \bigcap_{i\in I}\mathcal{F}^{\Omega}_{K_i}.\end{aligned}$$

From the later there is an index $i_0 \in I$ such that $K_{i_0} \subset \Omega$.

Indeed, assume on the contrary that $K_i \cap (E\backslash\Omega) \neq \emptyset$ for every $i \in I$. Then $\emptyset \neq E\backslash\Omega \in \bigcap_{i\in I}\mathcal{F}^{\Omega}_{K_i}$ is a contradition. Since K_{i_0} is a compact set, there is a set $\{j_1, \ldots, j_n\} \subset J$ such that $\{G_{j_1}, \ldots, G_{j_n}\}$ is a cover of K_{i_0}. Let F be an arbitrary closed subset of E. Then either $F \cap K_{i_0} = \emptyset$ or $F \cap G_{j_k} \neq \emptyset$ for some $k \in \{1, \ldots, n\}$. Therefore

$$F \in \mathcal{F}^{K_{i_0}} \bigcup \mathcal{F}_{G_{j_1}} \bigcup \cdots \bigcup \mathcal{F}_{G_{j_n}}.$$

The theorem is proved.

Remark. The proofs of Theorem 2.1 and 2.2 are analogous as the proof of the Main theorem in [3]. In [3], the authors showed that if E is a separable metric space and has at least a non-locally compact point, then the miss-and-hit topology on space $\mathcal{F}$ is not Hausdorff.

Theorem 2.3. *Let E be a topological space. Then*

i) the miss-and-hit topology on space $\mathcal{F}$ is a T_1-space.

ii) if E is a T_1-space and has a non-locally compact point, then the miss-and-hit topology on space $\mathcal{F}$ is not Hausdorff.

Proof. i) Take $F_1, F_2 \in \mathcal{F}$, $F_2 \neq F_1$. If there is a point $x \in F_2\backslash F_1$, then $F_1 \in \mathcal{F}^{\{x\}}_E$ and $F_2 \notin \mathcal{F}^{\{x\}}_E$. Otherwise, $F_1 \in \mathcal{F}^{\emptyset}_{E\backslash F_2}$ and $F_2 \notin \mathcal{F}^{\emptyset}_{E\backslash F_2}$. It implies that $\mathcal{F}$ is a T_1-space with the miss-and-hit topology.

ii) Let $x_0 \in E$ is a point which has not any compact neighborhood. Take $x_1 \in E\backslash\{x_0\}$ and put $F = \{x_0, x_1\}$, $F' = \{x_1\}$. We will show that $U_F \cap U_{F'} \neq \emptyset$ for any neighborhoods $U_F = \mathcal{F}^{K}_{G_1,\ldots,G_n}$ of F and $U_{F'} = \mathcal{F}^{K'}_{G'_1,\ldots,G'_m}$ of F'.

Put

$$I_0 = \{i : 1 \leq i \leq n, x_0 \in G_i\}.$$

If $I_0 = \emptyset$ then $F' \in U_F \cap U_{F'}$. And if $I_0 \neq \emptyset$, put $G = \bigcap_{i\in I_0} G_i$. Then there exists $x_2 \in G\backslash(K \cup K')$. In fact, if it is not the case, then $G \subset (K \cup K')$. Hence $K \cup K'$ is a compact neighborhood of x_0. It contradicts to x_0 is a non-locally compact point.

Put $F'' = \{x_1, x_2\}$, then $F'' \in \mathcal{F}^{K\cup K'}$ and $F'' \cap G'_i \neq \emptyset$ for all $i = 1, \ldots, m$. Therefore, $F'' \in U_{F'}$. Since $G = \bigcap_{i=1}^{n} G_i$ contains x_1 or x_2, $F'' \cap G_i \neq \emptyset$ for all $i = 1, \ldots, n$. It implies $F'' \in U_F$.

Hence,

$$F'' \in U_F \cap U_{F'}.$$

The proof is completed.

3. Some examples

For a given set E, we say that τ is the Zariski topology on E if τ contains $\emptyset$ and for every $\emptyset \neq U \subset E$, $U \in \tau$ then $E\backslash U$ is a finite set.

Theorem 3.1. *If E is an uncountable set with Zariski topology, then the miss-and-hit topology on $\mathcal{F}$ is Hausdorff and unseparated.*

Proof. Let Δ be an arbitrary countable subset of $\mathcal{F}$. We will show that Δ is not dense in $\mathcal{F}$. In fact, put

$$\mathcal{R} = \bigcup\{F : F \in \Delta, F \neq E\}.$$

For each $F \in \Delta$, $F \neq E$, then F is a finite set. It implies that $\mathcal{R}$ is a countable set. Hence, there exists $x \in E\backslash\mathcal{R}$. It is easy to see that every subset of E is compact. Then $\mathcal{F}_E^{\mathcal{R}}$ is a neighborhood of $\{x\}$ and

$$\Delta \bigcap \mathcal{F}_E^{\mathcal{R}} = \emptyset.$$

Therefore Δ is not dense in $\mathcal{F}$. Thus, $\mathcal{F}$ is unseparated.

Now we show that $\mathcal{F}$ is Hausdorff space. Let $F, F' \in \mathcal{F}$, $F \neq F'$.

If $F \subset F'$, we put

$$K = G' = E\backslash F,\ K' = E\backslash F',\ G = E,$$

and if $F \not\subset F'$ and $F' \not\subset F$, we put

$$K = E\backslash F,\ K' = G = E\backslash F',\ G' = E.$$

Then we have

$$F \in \mathcal{F}_G^K,\ F' \in \mathcal{F}_{G'}^{K'} \text{ and } \mathcal{F}_G^K \bigcap \mathcal{F}_{G'}^{K'} = \emptyset.$$

It implies that $\mathcal{F}$ is Hausdorff space.

Remark. The space E in Theorem 3.1 is separable and non-Hausdorff. But the miss-and-hit topology on $\mathcal{F}$ is Hausdorff and not separable. Hence the assumption that E is Hausdorff in Theorem 2.1 is only a sufficient condition.

Denote $\mathbb{N}$ a set of all natural numbers, put $X = \mathbb{N}$. Let Φ be a family consisting of $\emptyset$, X and all of subsets $A \subset X$ which satisfies the condition: There exists a finite subset α of A such that for every $a \in A$, a can be represented in the form $a = mp$, where $m \in \alpha$, $p \in \mathbb{P} \cup \{1\}$ ($\mathbb{P}$ is the set of all prime numbers). We say that α is a *finite generating set* of A [4, 5].

Theorem 3.2. *Assume that Φ and X are defined as above. Then Φ is the family of close subsets of a topology on X and X with this topology is a compact and T_1-space. Moreover, the miss-and-hit topology on Φ is not Hausdorff.*

Proof. It is easy to see that if A is a finite subset of X then $A \in \Phi$, and if $A, B \in \Phi$ then $A \cup B \in \Phi$. Therefore, to show that Φ is the family of close subsets for a topology in X, it is sufficient to show that for every family of $\{A_i\}_{i\in I} \subset \Phi$, we have $\bigcap_{i\in I} A_i \in \Phi$.

Let α_i be the finite generating set of A_i, $i \in I$. Take an arbitrary α_{i_1}, $i_1 \in I$, choose $i_2 \in I$ such that

$$\emptyset \neq \alpha_{i_1} \cap \alpha_{i_2} \neq \alpha_{i_1}.$$

Next, choose $i_3 \in I$ such that

$$\emptyset \neq \alpha_{i_1} \cap \alpha_{i_2} \cap \alpha_{i_3} \neq \alpha_{i_1} \cap \alpha_{i_2}$$

and go on. Then we have α_{i_1}, $\alpha_{i_1} \cap \alpha_{i_2}$, ... is a decreasing sequence of finite sets. So, after k steps, it will happen one of following two cases.

Case 1. $\alpha_{i_1} \cap \ldots \cap \alpha_{i_k} \neq \emptyset$ and for every $i \notin \{i_1, \ldots, i_k\}$ we have

$$\alpha_{i_1} \cap \ldots \cap \alpha_{i_k} \subset \alpha_i.$$

Case 2. $\alpha_{i_1} \cap \ldots \cap \alpha_{i_k} \neq \emptyset$ and there exists $i \in I$ such that $\alpha_{i_1} \cap \ldots \cap \alpha_{i_k} \cap \alpha_i = \emptyset$.

Suppose that the first case happens. Put $\alpha_0 = \bigcup_{j=1}^{k} \alpha_{i_j}$ and

$$B = \{mp : m \in \alpha_0,\ p \in \{1\} \cup \mathbb{P},\ p|\mathring{a} \text{ for some } a \in \alpha_0\}.$$

Then B is a finite set.

For any $a \in (\bigcap_{i\in I} A_i)\backslash B$ we have

$$a = m_1 p_1 = \ldots = m_k p_k,$$

where $m_j \in \alpha_{i_j}, p_j$ are primer numbers and p_s is not a divisor of m_t if $t \neq s$. Hence $p_1 = p_2 = \ldots = p_k = p$ and $m_1 = m_2 = \ldots = m_k = m \in \bigcap_{j=1}^{k} \alpha_{i_j}$. So $\bigcap_{i\in I} A_i$ has a finite generating set which is

$$(B \bigcap (\bigcap_{i\in I} A_i)) \bigcup (\bigcap_{j=1}^{k} \alpha_{i_j}).$$

Now suppose that the second case happens. Denote B as in the first case. Then for every $a \in (\bigcap_{i\in I} A_i)\backslash B$, we have $a = mp = nq$, where $m \in \bigcap_{j=1}^{k} \alpha_{i_j}$, $n \in \alpha_i$, p, q are prime numbers. Since $p \neq q$, p is divisor of n. On the other hand, α_i and B are finite sets. Hence $(\bigcap_{i\in I} A_i)\backslash B$ is a finite set. So $\bigcap_{i\in I} A_i$ is a finite set. Therefore $\bigcap_{i\in I} A_i \in \Phi$. Thus, every finite set of X is closed, in particular, X is a T_1-space.

Now we will prove that X is a compact space. In fact, suppose that $\{G_i\}_{i\in I}$ is an arbitrary open cover of X. For every $i \in I$, put $A_i = X\backslash G_i$ and α_i is the finite generating set of A_i. Then $\bigcap_{i\in I} A_i \neq \emptyset$.

If $\bigcap_{i\in I} \alpha_i \neq \emptyset$, then we have a contradiction to the fact that $\{G_i\}_{i\in I}$ is an open cover of X. Therefore, $\bigcap_{i\in I} \alpha_i = \emptyset$. Since α_i is a finite set, there exists $\{i_1, \ldots, i_k\} \subset I$ such that $\bigcap_{j=1}^{k} \alpha_{i_j} = \emptyset$.

According to the second case, the set $\bigcap_{j=1}^{k} A_{i_j} = X\setminus \bigcup_{j=1}^{k} G_{i_j}$ is finite. Thus, $\{G_i\}_{i\in I}$ has a finite subcover.

To complete the proof, we will show that Φ is a non-Hausdorff space. First, we invoke two following facts

1. *For every compact set $K \neq X$ and $k \in \mathbb{N}$, there exists $x \notin K$ such that $\tau(x) > k$, where $\tau(x)$ is a number of divisors of x.*

Indeed, choose $x \notin K$ and denote ith prime number by p_i. Put

$$A_i = \{xp : p \geq p_i, p \in \mathbb{P}\}.$$

Then $\{x\} \cup A_i$ is closed in X and x is a finite generating set of it. Therefore $A_i \cap K$ is closed in K. If $A_i \subset K$ for all $i = 1, 2, \ldots$, we receive a contradiction because $\{A_i\}$ has finite intersection property but their intersection is empty. Hence, there exists $q_1 \in \mathbb{P}$ such that $xq_1 \notin K$. Going on this processing, replacing x by xq_1 and considering A_i for $p_i > q_1$, we find out $q_2 \in \mathbb{P}$ such that $xq_1q_2 \notin K$, $q_1 < q_2$. By induction we have $q_1, \ldots, q_k \in \mathbb{P}$, $q_1 < \ldots < q_k$ such that $z = xq_1 \ldots q_k \notin K$. It is clear that $\tau(z) > k$.

2. *For every closed subset $A \neq X$, there exists $k_0 \in \mathbb{N}$ such that $\tau(x) \leq k_0$ for all $x \in A$.*

Indeed, let α be a finite generating set of A. Put

$$k_0 = 2\, max\ \{\tau(x) : x \in \alpha\}.$$

Then k_0 is the needed number.

Now we will prove that space Φ is a non-Hausdorff space.

Let $F = \{1, 2\}$ and $F' = \{1\} \in \Phi$. Assume that

$$\mathcal{F}^{K}_{G_1,\ldots,G_n} \text{ and } \mathcal{F}^{K'}_{G'_1,\ldots,G'_m}$$

are arbitrary neighborhoods of F, F' respectively. We have to show that

$$\mathcal{F}^{K}_{G_1,\ldots,G_n} \bigcap \mathcal{F}^{K'}_{G'_1,\ldots,G'_m} \neq \emptyset.$$

Indeed, it is clear that $X\setminus G_i$ and $X\setminus G'_j$ are closed sets which are different from X. According to 2), there exists k_0 such that $\tau(x) \leq k_0$ for all $x \in X\setminus G_i$, $i = 1, \ldots, n$ and $\tau(y) \leq k_0$ for all $y \in X\setminus G'_j$, $j = 1, \ldots, m$. Since $K \cup K'$ is a compact set which is different from X, according to 1) there exists $x_0 \notin K \cup K'$ such that $\tau(x_0) \geq k_0$. We have $x_0 \notin X\setminus G_i$ for $i = 1, \ldots, n$ and $x_0 \notin X\setminus G'_j$ for $j = 1, \ldots, m$. Consequently, $x_0 \in G_i$, $x_0 \in G'_j$ for all $i = 1, \ldots, n$, $j = 1, \ldots, m$. Hence

$$\{x_0\} \in \mathcal{F}^{K}_{G_1,\ldots,G_n} \bigcap \mathcal{F}^{K'}_{G'_1,\ldots,G'_m}.$$

The proof is completed.

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References

[1] G. Matheron, *Random Set and Integral Geometry*, John Wiley and Sons, New York, 1975.
[2] M. Marinacci, *Choquet Theorem for the Hausdorff Metric*, preprint, 1998.
[3] Nguyen Nhuy, Vu Hong Thanh, On Matheron Theorem for Non-locally Compact metric Spaces. *Vietnam J. Math.* 27 (1999) 115.
[4] N. Bourbaki, *Algebre*, Paris, 1995.
[5] J. L. Kelley, *General Topology*, Van Notrand, Princeton, N. J. 1955.