

Fully parallel methods for a class of linear partial differential-algebraic equations

Vu Tien Dung*

*Department of Mathematics, Mechanics, Informatics, College of Science, VNU
334 Nguyen Trai, Thanh Xuan, Hanoi, Vietnam*

Received 30 November 2007; received in revised form 12 December 2007

Abstract. This note deals with two fully parallel methods for solving linear partial differential-algebraic equations (PDAEs) of the form:

$$Au_t + B\Delta u = f(x, t) \quad (1)$$

where A is a singular, symmetric and nonnegative matrix, while B is a symmetric positive definite matrix. The stability and convergence of proposed methods are discussed. Some numerical experiments on high-performance computers are also reported.

Keywords: Differential-algebraic equation (DAE), partial differential-algebraic equation (PDAE), nonnegative pencil of matrices, parallel method

1. Introduction

Recently there has been a growing interest in the analysis and numerical solution of PDAEs because of their importance in various applications, such as plasma physics, magneto hydro dynamics, electrical, mechanical and chemical engineering, etc...

Although the numerical solution for differential-algebraic equations (DAEs) and (PDAEs) has been studied intensively [1, 2], until now we have not found any results on parallel methods for PDAEs. This problem will be studied here for a special case.

The paper is organized as follows. Section 2 deals with some properties of the so called nonnegative pencils of matrices. In Section 3 we describe two parallel methods for solving linear PDAEs, whose coefficients found a nonnegative pencil of matrices. The solvability and convergence of these methods are studied. Finally in section 4 some numerical examples are discussed.

2. Properties of nonnegative pencils of matrices

In what follows we will consider a pencil of matrices $\{A, B\}$, where $A \in \mathbb{R}^{n \times n}$ is a singular, symmetric and nonnegative matrix with $\text{rank}(A) = r < n$ and $B \in \mathbb{R}^{n \times n}$ is a symmetric positive definite matrix. Such a pencil will be called shortly a nonnegative pencil.

We begin with the following property of nonnegative pencils.

* Tel.: 084-48686532.

E-mail: duzngvt@gmail.com

Proposition 1. Any nonnegative pencil $\{A, B\}$ can be reduced to the Kronecker-Weierstrass form $\{\text{diag}(I_r, O_{n-r}), \text{diag}(D, I_{n-r})\}$ with a symmetric and positive definite matrix $D \in \mathbb{R}^{r \times r}$. Here I_r and O_{n-r} stand for the identity and zero matrices of appropriate dimensions respectively.

Proof. The symmetric and nonnegative matrix A can be diagonalized by an orthogonal matrix U , such as $U^T A U = \text{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_r > 0$ are positive eigenvalues of A . Define two matrices $S := \text{diag}((\lambda_1)^{-\frac{1}{2}}, \dots, (\lambda_r)^{-\frac{1}{2}}, 1, \dots, 1)$ and $\tilde{B} := S U^T B U S$. Clearly, \tilde{B} is also symmetric and positive definite. Moreover, $S U^T A U S = \text{diag}(I_r, O_{n-r})$.

Now let

$$\tilde{B} = \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

It is easy to verify that B_4 and its Schur complement defined by $B_1 - B_2 B_4^{-1} B_3$ are also symmetric and positive definite. Putting

$$P := \begin{pmatrix} I_r & B_2 B_4^{-1} \\ 0 & I_{n-r} \end{pmatrix}; \quad \hat{B} := \begin{pmatrix} B_1 - B_2 B_4^{-1} B_3 & 0 \\ 0 & B_4 \end{pmatrix}$$

and

$$Q = \begin{pmatrix} I_r & 0 \\ B_4^{-1} B_3 & I_{n-r} \end{pmatrix}$$

we get $\tilde{B} = P \hat{B} Q$ and $P \text{diag}(I_r, O_{n-r}) Q = \text{diag}(I_r, O_{n-r})$. From the last relations it follows $P^{-1} \tilde{B} Q^{-1} = \hat{B}$ and $P^{-1} \text{diag}(I_r, O_{n-r}) Q^{-1} = \text{diag}(I_r, O_{n-r})$. Finally, letting $\bar{P} = \text{diag}(I_r, B_4^{-1})$ we find $\bar{P} \text{diag}(I_r, O_{n-r}) = \text{diag}(I_r, O_{n-r})$ and $\bar{P} \hat{B} = \text{diag}(D, I_{n-r})$, where $D := B_1 - B_2 B_4^{-1} B_3$ and $D^T = D > 0$. Thus, there hold decompositions $MAN = \text{diag}(I_r, O_{n-r})$, $MBN = \text{diag}(B, I_{n-r})$ with nonsingular matrices $M := \bar{P} P S U^T$ and $N := U S Q^{-1}$, which was to be proved.

In what follows, we need two Toeplitz tridiagonal matrices P and Q of dimension $k \times k$, where as a rule k is much greater than n .

$$P = \begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 \end{pmatrix}; \quad Q = \begin{pmatrix} 4 & -1 & & & & \\ -1 & 4 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 4 & -1 & \\ & & & -1 & 4 \end{pmatrix} \tag{2}$$

Clearly, if $D \in \mathbb{R}^{r \times r}$ is a symmetric and positive definite matrix, then the Kronecker product $P \otimes D$ and $Q \otimes D$ are again symmetric and positive definite. Let $h > 0$ be a positive parameter.

Proposition 2. Let the pencil $\{A, B\}$ be nonnegative and let M and N be two nonsingular matrices, such as $MAN = \text{diag}(I_r, O_{n-r})$, $MBN = \text{diag}(D, I_{n-r})$, where $D^T = D > 0$. Then one can explicitly define two nonsingular matrices K and H , transforming the pencil $\{I_k \otimes A, \frac{1}{h^2} P \otimes B\}$ to the corresponding Kronecker-Weierstrass form

$$\left\{ \text{diag}(I_k, O_{k(n-r)}), \text{diag}\left(\frac{1}{h^2} \hat{D}, I_{k(n-r)}\right) \right\}, \tag{3}$$

with symmetric and positive definite matrix \hat{D} .

Proof. According to Proposition 1, the nonnegative pencil $\{I_k \otimes A, \frac{1}{h^2} P \otimes B\}$ can be reduced to the canonical form (3).

Further, for the Toeplitz matrix P , there exists an orthogonal matrix U , such that $U^T P U := \Lambda = \text{diag}(\lambda_1, \dots, \lambda_k)$, where $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_k > 0$. Then, the matrix $S := P \otimes I_{n-k}$ is diagonalized, $(U^T \otimes I_{n-r})S(U \otimes I_{n-r}) = \Lambda \otimes I_{n-r} = \text{diag}(\lambda_1 I_{n-r}, \dots, \lambda_k I_{n-r})$. Let $\overline{M} := I_k \otimes M$; $\overline{N} := I_k \otimes N$; $\overline{A} := I_k \otimes \text{diag}(I_r, O_{n-dir})$; $\overline{B} := \frac{1}{h^2} P \otimes \text{diag}(D, I_{n-r})$. Clearly, $\overline{M}(I_k \otimes A)\overline{N} = \overline{A}$ and $\overline{M}(\frac{1}{h^2} P \otimes B)\overline{N} = \overline{B}$.

Now we define two special matrices $E_1 \in \mathbb{R}^{r \times n}$ and $E_2 \in \mathbb{R}^{(n-r) \times r}$ as

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ & & \dots & \\ & & & 1 & 0 & 0 & 0 \\ & & & 0 & 0 & 1 & 0 \end{pmatrix}; E_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ & & \dots & \\ & & & 0 & 1 & 0 & 0 \\ & & & 0 & 0 & 0 & 1 \end{pmatrix} \quad (4)$$

and put $\xi_1 := \begin{pmatrix} E_1 \otimes I_k \\ E_2 \otimes I_k \end{pmatrix}$; $\xi_2 = ((E_1 \otimes I_k)^T, (E_2 \otimes I_k)^T)$. Further, let $\overline{D} := P \otimes D$; $J_1 := \text{diag}(I_{kr}, U^T \otimes I_{n-k})$, $J_2 := \text{diag}(I_{kr}, U \otimes I_{n-r})$; $J_3 := \text{diag}(I_{kr}, \Lambda^{-1} \otimes I_{n-r})$. Finally, set $K := J_1 \Lambda_1 \overline{M}$; $H := \overline{N} \Lambda_2 J_2 J_3$. We will show that K and H transform the pencil $\{I_k \otimes A, \frac{1}{h^2} P \otimes B\}$ to the canonical form (3). Indeed, a simple calculation shows that $\xi_1 \overline{A} \xi_2 = \text{diag}(I_{kr}, O_{k(n-k)})$ and $\xi_1 \overline{B} \xi_2 = \text{diag}(\frac{1}{h^2} \overline{D}, S)$. Further, $K(I_k \otimes A)H = J_1 \xi_1 \overline{A} \xi_2 J_2 J_3 = \text{diag}(I_{kr}, O_{k(n-r)})$. Similarly, $K(\frac{1}{h^2} P \otimes B)H = J_1 \xi_1 \overline{B} \xi_2 J_2 J_3 = \text{diag}(\frac{1}{h^2} \overline{D}, I_{k(n-k)})$. Thus the proposition 2 is complete.

3. Fully parallel methods for linear PDAEs

In this section we study the numerical solution of the following initial boundary value problems (IBVPs) for linear PDAEs:

$$A u_t + B \Delta u = f(x, t), \quad x \in \Omega, t \in (0, 1), \quad (5)$$

$$E u(x, 0) = u_0(x), \quad x \in \overline{\Omega}, \quad (6)$$

$$u(x, t) = 0, \quad x \in \partial\Omega, \quad (7)$$

where $\Delta u := \sum_{i=1}^d \frac{\partial^2 u}{\partial x_i^2}$, $\overline{\Omega} = \{x(x_1, \dots, x_d); 0 \leq x_i \leq 1; i = \overline{1, d}\}$,

A, B, E are given $n \times n$ matrices and the pencil $\{A, B\}$ is nonnegative. Further, u, f are vector functions, $u, f : \overline{\Omega} \times [0, 1] \rightarrow \mathbb{R}^n$ and the given function $f(x, t)$ is assumed to be sufficiently smooth. We propose two parallel methods for solving the IBVP (5)-(7) where the parallelism will be performed across both the problem and the method.

According to Proposition 1, there exist nonsingular matrices M, N such as $MAN = \text{diag}(I_r, O_{n-r})$ and $MBN = \text{diag}(D, I_{n-r})$, where as above, $r = \text{rank}(A)$ and $D = D^T > 0$. We will partition $N^{-1}u, Mf$ and u_0 into two parts, $N^{-1}u := (v^T, w^T)^T$; $Mf := (F_1^T, F_2^T)^T$, $u_0 := (v_0^T, w_0^T)^T$, where $v_0, v, F_1 \in \mathbb{R}^r$ and $w_0, w, F_2 \in \mathbb{R}^{n-r}$. From (5) we get $MAN \frac{\partial}{\partial t}(N^{-1}u) + MBN \Delta(N^{-1}u) = Mf$, or equivalently,

$$v_t + D \Delta v = F_1,$$

and

$$\Delta W = F_2.$$

Further, as in DAE's case, the initial condition (6) cannot be given arbitrarily. It must satisfy some so-called hidden constraints. Indeed, suppose that the matrix EN is partitioned accordingly to the

partition of the vector $N^{-1}u(x, 0)$ such that $EN = \begin{pmatrix} E_1 & E_2 \\ E_3 & E_4 \end{pmatrix}$, where E_1, E_4 are square matrices of dimension $r \times r$ and $(n - r) \times (n - r)$, respectively. For the sake of simplicity, we assume that $E_2 = 0, E_4 = 0$ and E_1 is nonsingular. Then condition (6) can be rewritten as $E_1v(x, 0) = v_0(x)$ and $E_3v(x, 0) = w_0(x)$. From the last relations, it is clear that the value $w(x, 0)$ will not participate in further computations. Besides, the initial condition $u_0(x) = (v_0^T(x), w_0^T(x))^T$ satisfies a hidden constraint

$$E_3E_1^{-1}v_0(x) = w_0(x) \tag{8}$$

Thus, IBVP (5)-(7) is split into an IBVP for the parabolic equation

$$v_t + D\Delta v = F_1, \tag{9}$$

$$v(x, 0) = E_1^{-1}v_0(x), \quad x \in \Omega \tag{10}$$

$$v(x, t) = 0, \quad x \in \partial\Omega, t \in (0, 1), \tag{11}$$

and a BVP for the elliptic equation

$$\Delta W = F_2 \tag{12}$$

$$w(x, t) = 0, \quad x \in \partial\Omega, t \in (0, 1), \tag{13}$$

A parallel fractional step (PFS) method, proposed in [3] and developed in [4], will be exploited for solving the IBVP (9)-(11). For this purpose, we first discretize in the spatial variable $x = (x_1, \dots, x_d)$ by choosing a mesh size $h > 0$ and approximate the problem in the discrete domain Ω_h by using the second order centered difference formula. It leads to the ODE

$$\frac{dv_h}{dt} = Hv_k + F_{1h} \tag{14}$$

$$v_h(0) = v_{0h} \tag{15}$$

Thanks to the symmetry and positive definiteness of D , in many cases, the matrix H is symmetric and positive definite. For example, using the matrices P and Q defined by (2) we get $H = \frac{1}{h^2}P \otimes D$

in 1D case ($d = 1$) and $H = \frac{1}{h^2}L \otimes D$, where $L = \begin{pmatrix} Q & -I & & & \\ -I & Q & -I & & \\ & \ddots & \ddots & \ddots & \\ & & -I & Q & -I \\ & & & -I & Q \end{pmatrix}$ for 2D case

($d = 2$). Further, suppose H can be split into the sum of symmetric, pairwise commutative and positive semidefinite matrices H_k ,

$$H = \sum_{k=1}^d H_k; \quad H_k^T = H_k \geq 0; \quad H_k H_l = H_l H_k; \quad k, l = 1, d; \tag{16}$$

We discretize the time interval $[0,1]$ with step $\tau > 0$ and apply the PFS method [4];

Algorithm PFS

Step 1. Initialize $v^0 := v_{0h}$

Step 2. For given $m \geq 0$ and v^m (an approximation of $v_h(m\tau)$) find $v^{m+1,k}$ by solving (in parallel) systems of linear equations

$$(I + \frac{\tau d}{2} H_k) v^{m+1,k} = (I - \frac{\tau d}{2} \sum_{j=1, j \neq k}^d H_j) v^m + \frac{\tau d}{2} F_{1h}^k \quad (17)$$

where $F_{1h}^k := F_{1h}((k + 1/2)\tau)$

Step 3. Compute

$$v^{m+1} = \frac{2}{d} \sum_{k=1}^d v^{m+1,k} + (1 - \frac{2}{d}) v^m \quad (18)$$

Note that the linear systems (17) can be solved by any parallel iterative methods [5,6,7,8,9]. Now we turn to the BVP (12)-(13). For its solution we implement the parallel splitting up (PSU) method, proposed by T. Lu, P. Neittaanmaki, and X. C. Tai [3].

Discretizing the BVP (12)-(13) one obtains a large-scale system of linear equations

$$Lw = g, \quad (19)$$

where L is a symmetric positive definite matrix of dimension $p \times p$, where $p = p(h)$ depends on the discretization parameter h . Assume that L can be decomposed into the sum m of symmetric and positive definite matrices, which commute with each other

$$L = \sum_{i=1}^m L_i; \quad L_i^T = L_i > 0; \quad L_i L_j = L_j L_i, \quad i, j = 1, m. \quad (20)$$

The PSU method consists of the following steps:

Algorithm PSU

Step 1. Choose an initialization w^0

Step 2. Supposing w^j is known, we compute the fractional step values

$$L_i w^{j+\frac{i}{2m}} = f - \sum_{k=2, k \neq i}^m L_k w^j, \quad i = 1, \dots, m. \quad (21)$$

Step 3. For chosen parameters ω_j , set

$$w^{j+1} = \frac{\omega_j}{m} \sum_{i=1}^m w^{j+\frac{i}{2m}} + (1 - \omega_j) w^j. \quad (22)$$

Note that for different j system (21) can be solved by parallel processors.

Theorem 3. *The PFS-PSU method (17)-(18), (21)-(22) for solving the IBVP (5)-(7) with a nonnegative pencil $\{A, B\}$ and the consistent initial conditions (6), (8), is convergent.*

Proof. Thanks to the nonnegativity of the pencil $\{A, B\}$, we can split the IBVP (5)-(7) into the IBVP (9)-(11) and the BVP (12)-(13). Theorems 4.11 and 5.2 [4] ensure that the PFS method in the symmetric and commutative case is stable provided $\tau \leq 2\{d \max_{1 \leq k \leq d} \|H_k\|\}^{-1}$. Moreover it is convergent with global error $O(h^2 + \tau^2)$. Further, according to [3] the PSU method is convergent. If

all the eigenvalues of the matrix $S := \frac{1}{m} \sum_{i=1}^m L_i^{-1} L$ belong to some segment $[a, b]$, where $a \geq 0$, then the asymptotic rate of the PSU with $\omega_j = \frac{2}{a+b}$ is $\frac{2}{p}$, where $p := \text{cond}(S) \leq (b/a)$.

We end this section by considering the IBVP (5)-(7) in 1D case ($d = 1$). Besides, the matrix E in (6) is supposed to be the identity matrix.

Discretizing the IBVP in the spatial variable we get a system of ODEs

$$A \frac{\partial u}{\partial t}(x_k, t) + \frac{1}{h^2} B[u(x_{k+1}, t) - 2u(x_k, t) + u(x_{k-1}, t)] = f(x_k, t)$$

$k = \overline{1, M-1}, M = M(h)$. Putting $U := (u_1^T, \dots, u_M^T)^T; F := (f_1^T, \dots, f_{M-1}^T)^T$, where $u_k := u(x_k, t); f_k := f(x_k, t)$, we can rewrite the last system of equations as

$$(I_{M-1} \otimes A) \frac{dU}{dt} + \frac{1}{h^2} (P \otimes B) U = F, \tag{23}$$

where the matrix P is determined by (2). By Proposition 2 we can find nonsingular matrices K and H transforming the pencil $\{I_{M-1} \otimes A, \frac{1}{h^2} P \otimes B\}$ to the Kronecker-Weierstrass form (3). Multiplying both sides of (23) by K and putting $H^{-1}U = (V^T, W^T)^T; KF = (\tilde{F}_1^T, \tilde{F}_2^T)^T$, where $V, \tilde{F}_1 \in \mathbb{R}^{(M-1)r}$ and $W, \tilde{F}_2 \in \mathbb{R}^{(M-1)(n-r)}$, we come to the system

$$\frac{dV}{dt} + \frac{1}{h^2} \overline{D} V = \tilde{F}_1, \tag{24}$$

$$W = \tilde{F}_2, \tag{25}$$

where as in Proposition 2, \overline{D} is a symmetric and positive definite matrix. Note that the boundary condition (7) has been included in Equation (23). Now let $H^{-1}(u_0^T(x_1), \dots, u_0^T(x_{M-1}))^T = (V_0^T, W_0^T)^T$, where $V_0 \in \mathbb{R}^{(M-1)r}$ and $W_0 \in \mathbb{R}^{(M-1)(n-r)}$. Then, the initial condition (6), with $E \equiv I$ becomes

$$V(0) = V_0. \tag{26}$$

Moreover, the initial condition (6) must satisfy a hidden constraint $W_0 = \tilde{F}_2(0)$.

For solving the IVP (23-26) in parallel, the PFS method described above for the problem (14)-(15) can be applied. We shall not give the lengthy details.

4. Numerical experiment

Consider the boundary - value problem (5)-(7) with the following data:

$$n = 3; d = 2; A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; B = \begin{pmatrix} 2 & -0.5 & 1 \\ -0.5 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}; E = I. \tag{27}$$

The function $f(x, t)$ is chosen such that the exact solution of the BVP (5)-(7) is $u = 10^3(tx_1(1-x_1)x_2^2(1-x_2), tx_1^2(1-x_1)x_2(1-x_2), tx_1x_2(1-x_1)(1-x_2))^T$

Using nonsingular matrices

$$M = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} N = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \tag{28}$$

we can split the IBVP (5)-(7) into an IBVP for the parabolic equation

$$\begin{cases} v_t - W(v_{x_1x_1} + v_{x_2x_2}) = F_1(x_1, x_2, t) \\ v(x, 0) = 0 \\ v(x, t) = 0 \end{cases} \quad \begin{matrix} x \in \Omega \\ x \in \partial\Omega \text{ and } t \in (0, 1) \end{matrix} \quad (29)$$

and a BVP for the elliptic equation

$$\begin{cases} -(w_{x_1x_1} + w_{x_2x_2}) = F_2(x_1, x_2, t) \\ w(x, t) = 0. \end{cases} \quad (30)$$

The PFS method and PSU method [4,3] are implemented in C and MPI and executed on a Linux Cluster 1350 with eight computing nodes of 51.2GFlops. Each node contains two Intel Xeon dual core 3.2GHz, 2GB Ram.

The following table shows the dependence of the error of the approximate solutions on the number $N = \frac{1}{h}$ while the ratio $\frac{\tau}{h^2}$ remains constant.

N	$\frac{\tau}{h^2}$	16	24	30	40	50	60
Residual	0.5	0.000345	0.00008	0.000038	0.000014	0.0000609	0.0000309
Residual	0.2	0.000064	0.000016	0.000007	0.000002	0.000001	0.0000005

In what follows, we study the relation between the total (CPU) time spent on the performance of a program, the speedup and the efficiency of this performance. The speedup of the performance is defined as $S = T_s/T_p$, where T_s (T_p) is serial execution time (parallel execution time), respectively. The efficiency of the performance is determined as $E = S/P$, where P is the number of processors. The result of an experiment with PFS method for (29) is reported in the following table

Table 1. Speed up and Efficiency on Cluster 1350 with N=120.

Processors	1	2	4	6	8	10
Total times(minutes)	252	126	62	43	37	32
Speedup		2	4	5.8	6.8	7.8
Efficiency		1	1	0.97	0.85	0.78

Using 2 processors of Cluster 1350 and applying the PSU methods to the BVP (30) we observe that the total time increases together with the growth of the number $N = \frac{1}{h}$.

Table 2.

N	24	30	40
Total times(seconds)	120	180	300

For better convergence, we use other methods, such as the parallel Jacobi method [5], the parallel SOR Red/Black [6,7,8,9] method.

The parallel Jacobi method and Parallel SOR Red/Black method are implemented in C and MPI and executed on 1 node of AIX Cluster 1600 of 5 computing nodes, whose total computing power is 240GFlops. Each node contains 8 CPU Power4 64bit RSIC 1.7GHz.

Below are some results for parallel Jacobi method and parallel SOR Red/Black method

Table 3. Speed up and Efficiency on 1 node of Cluster 1600, N=240.

<i>Parallel Jacobi method</i>					
<i>Processors</i>	1	2	4	6	8
<i>Total times(seconds)</i>	937	484	232	191	155
<i>Speedup</i>		1.94	4.0	4.9	6.05
<i>Efficiency</i>		0.97	1	0.81	0.75

Although the parallel Jacobi method converges faster than the PSU methods, it is rarely used as a parallel solver for elliptic problems.

Table 4. Speed up and Efficiency on Cluster 1600 with N=1200.

<i>Parallel Red - Black SOR method</i>					
<i>Processors</i>	1	2	4	6	8
<i>Total times(seconds)</i>	275	154	83	64	54
<i>Speedup</i>		1.79	3.31	4.3	5.1
<i>Efficiency</i>		0.9	0.83	0.72	0.64

The number of iterations needed for convergence and the total time for the serial computation of Red - Black SOR and Jacobi method are given in the following tables.

Table 5. Number of Iterations of sequential Red - Black SOR and Jacobi method.

<i>N</i>	60	120	180	240	300
<i>SOR</i>	284	565	836	1101	1351
<i>Jacobi</i>	10599	39680	86119	149311	

Table 6. Total times of Red - Black SOR method and Jacobi method.

<i>N</i>	60	120	180	240	300
<i>SOR(seconds)</i>	1	2	7	12	19
<i>Jacobi (seconds)</i>	4	45	200	720	

The Red - Black SOR method is clearly the fastest one in terms of serial time and the number of iterations.

Table 1,3,4 show that when the number of processors increases, the speedup increases. The actual speedup is smaller than the ideal speedup because the communication cost is relatively higher when implemented and executed on a Linux Cluster 1350 and AIX Cluster 1600. From Table 1,3,4 it is clear that the more processors are used, the communication cost increases, and the efficiency decreases.

Acknowledgements. The author thanks Prof. Dr. Pham Ky Anh for suggesting the considered topic and for helpful discussions. Partially supported by the VNU's Key Project QGT 05.10.

References

- [1] W. Lucht, K. Strehmel, C. Eichler-Liebenow, Linear Partial Differential Algebraic Equation, *Report No .18* (1997) 430.
- [2] W. Marszalek, Z. Trzaska, A Boundary-value Problem for Linear PDAEs, *Int.J.Appl.Math.Comput.Sci.*, Vol. 12, No. 4 (2002) 487.
- [3] T. Lu, P. Neittaanmaki, X.C. Tai, A Parallel Splitting-up Method for Partial Differential Equations and Its Applications to Navier-Stokes Equations, *Applied Mathematics Letters* Vol. 4, No. 2 (1992) 25.

- [4] J.R. Galo, I. Albarreal, M.C. Calzada, J.L. Cruz, E. Fernández-Cara, M. Mari'n, Stability and Convergence of a Parallel Fractional Step Method for the Solution of Linear Parabolic Problems, *Applied Mathematics Research Express* No. 4 (2005) 117.
- [5] R.D. da Cunha, T.R. Hopkins, Parallel Over Relaxation Algorithms for Systems of Linear Equations, *World Transputer user group conference*, Sunnyvale transputing '91 Amsterdam: IOS Press Vol 1 (1991) 159.
- [6] D.J.Evans, Parallel SOR Iterative Methods, *Parallel Computing* 1 (1984) 3.
- [7] W. Niethammer, The SOR Method on Parallel Computers, *Numer. Math.* 56 (1989) 247.
- [8] D.Xie, L. Adams, New Parallel Method by Domain Partitioning, *SIAM J. Sci. Comput.* Vol. 20, No. 6 (1999) 2261.
- [9] C.Zhang, Hong Lan, Y.E.Yang, B.D. Estrade, Parallel SOR Iterative Algorithms and Performance Evaluation on a Linux Cluster, *Proceedings by the International Conference on Parallel and Distributed Processing Techniques and Applications (PDDTA 2005)*, CSREA Press Vol. 1 (2005) 263.