VNU Journal of Science, Mathematics - Physics 23 (2007) 221-224

Some problem on the shadow of segments infinite boolean rings

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Received 18 September 2007; received in revised form 8 October 2007

Abstract. In this paper , we consider finite Boolean rings in which were defined two orders: natural order and antilexicographic order. The main result is concerned to the notion of shadow of a segment. We shall prove some necessary and sufficient conditions for the shadow of a segment to be a segment.

1. Introduction

Consider a finite Boolean ring: $B(n) = \{x = x_1x_2...x_n : x_i \in \{0,1\}\}$ with natural order $\leq_N$ defined by $x \leq_N y \Leftrightarrow xy = x$. For each element x ∈ B(n), weight of x is defined to be: w(x) = $x_1 + x_2 + ... + x_n$ i.e the number of members $x_i \neq 0$.In the ring B(n), let B(n,k) be the subset of all the elements x∈ B(n) such that w(x)= k.

We define a linear order $\leq_L$ on B(n,k) by following relation. For each pair of elements x, y ∈ B(n,k), where x = $x_1...x_n$, y = $y_1...y_n$, $x \leq_L y$ if and only if there exists an index t such that $x_t < y_t$ and $x_i = y_i$ whenever $i > t$. That linear order is also called antilexicographical order. Note that each element $x = x_1...x_n \in$ B(n,k) can be represented by sequence of all indices $n_1 < ... < n_k$ such that x_{n_i} = 1. Thus we can identify the element x with its corresponding sequence and write x =($n_1..., n_k$). Using this identification, we have: $x = (n_1, ..., n_k) \leq_L (m_1, ..., m_k) = y$ whenever there is an index t such that $n_t < m_t$ and $n_i = m_i$ if $i > t$.

It has been shown by Kruskal (1963), see [1], [2] that the place of element x=($n_1..., n_k$) ∈B(n,k) in the antilexicographic ordering is:

$$\varphi(x) = 1 + \binom{n_1 - 1}{1} + ... + \binom{n_k - 1}{k} \qquad 1$$

(Note that $\binom{n}{r}$ is a binomial coefficient (n-choose-r) and $\binom{m}{t} = 0$ whenever $m < t$) . We remark that φ is the one-one correspondence.Therefore $\varphi(A) = \varphi(B)$ is equivalent to A = B, for every subsets A, B in B(n,k).

Now, suppose a∈ B(n,k) with $k > 1$, *the shadow of element a* is defined to be $\Delta a = \{x \in B(n, k-1) : x \leq_N a\}$. If A ⊂ B(n,k), *the shadow of A* is the union of all Δa, a ∈A i.e $\Delta A =$

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$\bigcup_{a\in A} \Delta a = \{x \in B(n, k-1) : x \leq_N a \; for \; some \; a \in A\}$.Thus the shadow of A contain all the elements x ∈B(n,k-1) which can be obtained by removing an index from the element in A.The conception about the shadow of a set was used efficiently by many mathematicians as: Sperner, Kruskal, Katona, Clement,....?

We shall study here the shadow of segments in B(n,k) and make some conditions for that the shadow of a segment is a segment. As in any linearly ordered set, for every pair of elements a,b ∈ B(n,k), the segment [a,b] is defined to be: [a,b]=$\{x \in B(n,k) : a \leq_L x \leq_L b\}$. However, if a=(1,2,...?,k)∈ B(n,k) is the first element in the antilexicographic ordering, the segment [a,b] is called an *initial segment* and denoted by IS(b) so IS(b)=$\{x \in B(n,k) : x \leq_L b\}$. We remind here a very useful result, proof of which had been given by Kruskal earlier (1963), see [4], [2]. We state this as a lemma

Lemma 1.1. *Given* $b = (m_1, m_2, ..., m_k)$∈*B(n,k) with* $k > 1$ *then* $\Delta IS(b) = IS(b')$, *where* $b' = (m_2, ..., m_k) \in B(n, k-1)$?

This result is a special case of more general results and our aim in the next section will state and prove those. Let a =$(n_1, n_2, ..., n_k)$ and b =$(m_1, m_2, ..., m_k)$ be elements in B(n,k). Comparing two indices n_k *and* m_k , it is possible to arise three following cases:

(a) $m_k = n_k = M$

(b) $m_k = n_k + 1 = M + 1$

(c) $m_k > n_k + 1$

In each case ,we shall study necessary and sufficient conditions for the shadow of a segment to be a segment.

2. Main result

Before stating the main result of this section, we need some following technical lemmas. First of all, we establish a following lemma as an application of the formula (1):

Lemma 2.1. *Let a* =$(n_1, n_2, ..., n_k)$ *and b* =$(m_1, m_2, ..., m_k)$ *be elements in B(n,k) such that* $n_k \leq m_k < n$. *and let M be a number such that* $m_k < M \leq n$. *Define x* =$(n_1, n_2, ..., n_k, M)$, *y*=$(m_1, m_2, ..., m_k, M)$ ∈*B(n,k+1). Then we have:[x,y]*=$\{$*c +M : c* ∈ *[a,b]*$\}$ *and [a,b]*=$\{$*z -M : z* ∈ *[x,y]*$\}$.

(Note that here we denote x = a+M and a = x-M)

Proof. It follows from the formula (1) that, for any c∈ [a,b],

$\varphi(c+M) = \varphi(c) + \binom{M-1}{k+1}$, therefore $\varphi(\{c+M : c \in [a,b]\}) = [\varphi(a) + \binom{M-1}{k+1}; \varphi(b) + \binom{M-1}{k+1}] = [\varphi(x); \varphi(y)] = \varphi([x;y])$ So [x; y]= {c + M : c [a,b]}. By using similar argument for the remaining equality, we finish the prove of the lemma.

As an immediate consequence,we get the following

muc Lemma 2.2. *Let a,b* ∈ *B(n,k) be elements such that a =(1,...,k-1, M) and b =(M-k+1,...,M-1,M) then the shadow* $\Delta[a, b]$ = *IS(c) with c =(M-k+2,...,M-1,M)*∈ *B(n,k-1).*

Proof Choose g =(1,...,k-1); d=(M-k+1,...,M-1) in B(n,k-1).Then it follows from lemma 2.1 that A ={x-M : x ∈[a,b]}= [g; d]=IS(d). However, we also have from the lemma 1.1 that $\Delta A = \Delta IS(d) = IS(c - M)$. Repeating to apply the lemma 2.1 to the set B =$\{z + M : z \in \Delta A\}$. We have obtained

B=[h;c] where h=(1,...?,k-2, M). Note that $\varphi(d)+1=\varphi(h)$ so A and B are two consecutive segments. Therefore their union: $\Delta[a;b]=A\cup B=IS(d)\cup[h;c]=IS(c)$ is an initial segment. The proof is completed. We now get some useful consequences of this lemma as follows:

Corollary 2.1. *Let a=$(n_1,...,n_{k-1},M)$ and b=$(M-k+2,...,M,M+1)$ be elements in B(n,k) then $\Delta[a,b]$ =IS(c) with c =$(M-k+3,...,M,M+1)\in$ B(n,k-1).*
Proof. Choose d=(1,...?,k-1, M+1)∈ B(n,k) then [d; b] ⊂ [a;b].By the lemma 2.2, we have $\Delta[d,b]$ =IS(c) with c =(M-k+3,...?, M, M+1)∈ B(n,k-1). However, we also have: [a,b]⊂IS(b) so $\Delta[a,b]\subset$ $\Delta IS(b)=IS(c)$. Thus $\Delta[a,b]\subset\Delta(b)$ =IS(c) as required.

Corollary 2.2. *Let a =(1,...?,k-1,M); b =$(m_1,...,m_{k-1},M+1)$ be elements in B(n,k) then $\Delta[a,b]$ =IS(c) where c =$(m_2,...,m_{k-1},M+1)\in$ B(n,k-1).*
Proof. In the proof of this result, we denote: h =(M-k+1,...?,M)∈ B(n,k), d =(M-k+2,...?,M), g =(1,...?,k-2, M+1), c =$(m_2,...,m_{k-1},M+1)$ in B(n,k-1). Then ,again by the lemma 2.2, we have: $\Delta[a,h]=IS(d)\subset\Delta[a,b]$. Obviously, we also have [g;c]⊂ $\Delta[a,b]$. Therefore, $\Delta[a;b]\supset(IS(d)\cup[g;c])=IS(c)$ and as in above proof it follows that$\Delta[a,b]$ =IS(c).

Corollary 2.3.*Let a =$(n_1,n_2,...,n_k)$ and b =$(m_1,m_2,...,m_k)\in$ B(n,k) be given such that $m_k>n_k+1$ then $\Delta[a,b]$ =IS(c) where c =$(m_2,...,m_k)\in$ B(n,k-1).*
Proof. Since $m_k>n_k+1$, there must be a number M such that $n_k+1\le m_k-1=M$. Choose d=(1,...?,k-1, M)∈ B(n,k), we therefore have [d;b]⊂ [a;b]. Note that the segment [d;b] satisfys conditions of corollary 2.2, we now imitate the above proof to finish the corollary. Certaintly, the last corollary is a solution for our key questions, in the case (b). What about the remaining case ? First of all, we turn our attention to the case (a) and have that:

Theorem 2.1. *Let a,b ∈ B(n,k) be elements such that a =$(n_1,...,n_{k-1},M)$ and b =$(m_1,...,m_{k-1},M)$ then $\Delta[a,b]$ is a segment if and only if $m_1=M-k+1$and either $n_{k-1}<M-1$ or $n_{k-2}=k-2$*
Proof. Take c =a -M; d=b - M ∈B(n,k-1) then $\Delta[a;b]=[c;d]\cup\{x+M:x\in\Delta[c,d]\}$. Suppose that $\Delta[a,b]$ is a segment then there must have g =(1,...?,k-1)∈$\Delta[c;d]$ and $\varphi(d)+1=\varphi(g+M)$. Therefore we have that d =(M-k+1,...?,M-1) i.e $m_1=M-k+1$. In the case $n_{k-1}=M-1$, since g+M ∈ $\Delta[a,b]$ so h=(1,...?,k-2, M-1, M)∈[a,b]. Therefore, $a\le h$. However, $n_{k-1}=M-1$ follows that $h=(1,?,k-2,M-1,M)\le(n_1,...,n_{k-2},M-1,M)=a$. Thus a =h , i.e, $n_{k-2}=k-2$. Conversely, suppose that a =(1,...?,k-2,M-1, M) and b =(M-k+1,...?, M-1, M). We shall prove that $\Delta[a,b]$ is a segment. Apply the lemma 2.2 to segment [a-M; b-M], we obtain $\Delta[a-M,b-M]=IS(c)$ where c =(M-k+2,...?, M-1). We now have $\Delta[a;b]=[a-M;b-M]\cup\{x+M:x\in IS(c)\}$ to be the union of two consecutive segments. Therefore,it is a segment. In the case $n_{k-1}<M-1$, apply the corollary 2.1 (if $n_{k-1}=M-2$) or the corollary 2.3 (if $n_{k-1}<M-2$) to the segment [a-M; b-M] we obtain $\Delta[a-M,b-M]=IS(c)$ for some c ∈ B(n, k-2). Thus $\Delta[a;b]=[a-M;b-M]\cup\{x+M:x\in IS(c)\}$ as above is the union of two consecutive segments, therefore is a segment.

Finally, we return attention to the case (b)with $m_k=n_k+1$. There are two ablities for index m_1 : $m_1=M-k+2$ and $m_1<M-k+2$. The former is easily answered by the corollary 2.1 so here we only give the proof for the latter.In fact, We define the number s as follows

$$s=min\{t:m_{k-t}\le M-t\} \qquad 2$$

We close this section with the following theorem:

Theorem 2.2. *If* $a = (n_1, ..., n_{k-1}, M)$ *and* $b = (m_1, ..., m_{k-1}, M+1) \in$ *B(n,k) satisfying* $m_1 \leq M - k + 1$. *then we have that:*

(a) In the case $n_{k-s+1} < M - s + 1$, $\Delta[a, b]$ *is a segment.*

(b) In the case $n_{k-s+1} = M - s + 1$, $\Delta[a, b]$ *is a segment if and only if* $\varphi(a') \leq \varphi(b') + 1$ *and either* $n_{k-s} < M - s$ *or* $n_{k-s-1} = k - s - 1$ *where a'=*$(n_1, ..., n_{k-s})$ *and b'=*$(m_1, ..., m_{k-s}) \in$ *B(n,k-s).*

Proof. Choose h =(M-k+1,..., M-1); c =a-M; d = b-(M+1)$\in$ B(n, k-1) and define set X=$\{y + (M + 1) : y \in \Delta IS(d)\}$. Since [a;b]= [a; h+M]$\cup\{x + (M + 1) : x \in IS(d)\}$. We have $\Delta[a; b] = IS(d) \cup \Delta[a; h + M] \cup X$. Note that two members IS(d) and X of this union are segments and $\varphi(\max \Delta[a, h+M])+1 = \varphi(\min X)$ so $\Delta[a, b]$ is a segment if and only if the union IS(d) $\cup\Delta[a; h+M]$ is a segment. In the case that $n_{k-s+1} < M - s + 1$, there must be g=(1,...,k-s, M-s+1,...,M)$\in$ B(n,k) such that g $\in$[a; h+M]. Denote g'=(1,...?,k-s, M-s+1) and h'=(M-k+1,...,M-s,M-s+1) $\in$B(n, k-s+1). By lemma 2.2, we obtain an initial segment. Therefore the set Y defined by Y=$\{z+(M-s+2, ..., M) : z \in \Delta[g', h']\}$ is a segment in B(n, k-1). It is easy to see that d=$(m_1, ..., m_{k-s}, M - s + 2, ..., M) \in$ Y and this follows that IS(d) $\cup$ Y is also a segment. Thus, It is clear that IS(d) $\cup$ X =IS(d) $\cup$ Y is a segment as required. In the case $n_{k-s+1} = M - s + 1$, we consider first s =1. Since $m_{k-1} \leq M - 1$, $d = b - (M + 1) \leq h$ in B(n, k-1). Note that $\Delta[a; h + M] = [c; h] \cup \{z + M : z \in \Delta[c; h]\}$, therefore IS(d)$\cup\Delta[a; h + M]$ is a segment if and only if $\varphi(c) \leq \varphi(d) + 1$ and $\Delta[a, h + M]\}$ is a segment. According to the theorem 2.1, last condition is equavalent to that $n_{k-1} < M - 1$ or $n_{k-2} = k - 2$ is required. Next, suppose that $s > 1$ with $n_{k-s+1} = M - s + 1$ then a=$(n_1, ..., n_{k-s}, M - s + 1, ..., M)$ and d=$(m_1, ..., m_{k-s}, M - s + 2, ..., M)$. Take

A=$\{x+(M-s+2, ..., M) : x \in \Delta[a'+(M-s+1); h'+(M-s+1)]\}$, where a'=$(n_1, ..., n_{k-s})$ and h'=(M-k+1,...,M-s)$\in$ B(n,k-s). It is clear that the union IS(d) $\cup\Delta[a; h + M]$ is a segment if and only if the union IS(d) $\cup A$ is a segment. Note that $m_{k-s} \leq M - s$, therefore $b' = (m_1, ..., m_{k-s}) \leq h'$. Hence, the last requirement is equivalent to the requirement that $\varphi(a') \leq \varphi(b')+1$ and $\Delta[a'+(M-s+1); h'+(M-s+1)] = [a'; h'] \cup \{y+(M-s+1) : y \in \Delta[a'; h']\}$ is a segment. By the theorem 2.1, the latter is equivalent to the requirements that $n_{k-s} < M - s$ or $n_{k-s-1} = k - s - 1$. The proof is completed.

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