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The total specialization of modules over a local ring

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Abstract. In this paper we introduce the total specialization of an finitely generated module over local ring. This total specialization preserves the Cohen-Macaulayness, the Gorensteiness and Buchsbaumness of a module. The length and multiplicity of a module are studied.

1. Introduction

Given an object defined for a family of parameters $u = (u_1, \ldots, u_m)$ we can often substitute u by a family $\alpha = (\alpha_1, \ldots, \alpha_m)$ of elements of an infinite field K to obtain a similar object which is called a specialization. The new object usually behaves like the given object for almost all α, that is, for all α except perhaps those lying on a proper algebraic subvariety of K^m. Though specialization is a classical method in Algebraic Geometry, there is no systematic theory for what can be "specialized".

The first step toward an algebraic theory of specialization was the introduction of the specialization of an ideal by W. Krull in [1]. Given an ideal I in a polynomial ring $R = k(u)[x]$, where k is a subfield of K, he defined the specialization of I as the ideal

$$I_\alpha = \{f(\alpha, x)|\ f(u, X) \in I \cap k[u, x]\}$$

of the polynomial ring $R_\alpha = k(\alpha)[x]$. For almost all $\alpha \in K^m$, I_α inherits most of the basic properties of I. Let $\mathfrak{p}_u$ be a separable prime ideal of R. In [2], we introduced and studied the specializations of finitely generated modules over a local ring $R_{\mathfrak{p}_u}$ at an arbitrary associated prime ideal of $\mathfrak{p}_\alpha$ (For specialization of modules, see [3]). Now, we will introduce the notation about the total specializations of modules. We showed that the Cohen-Macaulayness, the Gorensteiness and Buchsbaumness of a module are preserved by the total specializations.

2. Specializations of prime separable ideals

Let $\mathfrak{p}_u$ be an arbitrary prime ideal of R. The first obstacle in defining the specialization of $R_{\mathfrak{p}_u}$ is that the specialization $\mathfrak{p}_\alpha$ of $\mathfrak{p}_u$ need not to be a prime ideal. By [1], $\mathfrak{p}_\alpha = \bigcap_{i=1}^{s} \mathfrak{p}_i$ is an unmixed ideal of R_α.

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Assume that $\dim \mathfrak{p}_u = d$ and (ξ) is a generic point of $\mathfrak{p}_u$ over k. Without loss of generality, we may suppose that this is normalised so that $\xi_0 = 1$. Denote by $(v) = (v_{ij})$ with $i = 0, 1, \ldots, d$, $j = 1, \ldots, n$, a system of $(d+1)n$ new indeterminates v_{ij}, which are algebraically independent over $k(u, \xi_1, \ldots, \xi_n)$. We enlarge $k(u)$ by adjoining (v). We form $d+1$ linear forms

$$y_i = -\sum_{j=1}^{n} v_{ij} x_j, i = 0, 1, \ldots, d.$$

Then $\mathfrak{p}k(u,v)[x] \cap k(u,v)[y] = (f(u, v; y_0, \ldots, y_d))$ is a principal ideal. We put $\lambda_i = \sum_{j=0}^{n} v_{ij}\xi_j$ with $i = 0, 1, \ldots, d$. Then $\lambda_0, \ldots, \lambda_d$ satisfies $f(u, v; \lambda_0, \ldots, \lambda_d) = 0$ and is called the *ground-form* of $\mathfrak{p}_u$. The prime ideal $\mathfrak{p}_u$ is called a *separable* prime ideal if it's ground-form is a separable polynomial. We have the following lemma:

Lemma 2.1.[1, Satz 14] *A specialization of a prime separable ideal is an intersection of a finite prime ideals for almost all α.*

Let the prime ideal $\mathfrak{p}_u$ be separable. Assume that $\mathfrak{p}_\alpha = \bigcap_{i=1}^{s} \mathfrak{p}_i$ and set $T = \bigcap_{i=1}^{s} (R_\alpha \setminus \mathfrak{p}_i)$.

Lemma 2.2. *For almost all α, we have $(R_\alpha)_T$ is a semi-local ring.*

Proof. Note that T is a multiplicative subset of R_α. We show that $(R_\alpha)_T$ is a semi-local ring. Indeed, let $\mathfrak{m}$ be a maximal ideal of $(R_\alpha)_T$. Then, there is a prime ideal $\mathfrak{q}$ of R_α such that $\mathfrak{m} = \mathfrak{q}(R_\alpha)_T$. Suppose that $\mathfrak{m} \supset \mathfrak{p}_1(R_\alpha)_T, \mathfrak{m} \neq \mathfrak{p}_1(R_\alpha)_T$. We have $\mathfrak{q} \supset \mathfrak{p}_1, \mathfrak{q} \neq \mathfrak{p}_1$. Since $\mathfrak{m} = \mathfrak{q}(R_\alpha)_T$ is a maximal ideal, $\mathfrak{q} \cap T = \emptyset$. Hence $\mathfrak{q} \subset \bigcup_{i=1}^{s} \mathfrak{p}_i$. Therefore, it exists j such that $\mathfrak{q} \subseteq \mathfrak{p}_j$. Then $\mathfrak{p}_1 \subset \mathfrak{p}_j$, contradiction. Hence $\mathfrak{m} = \mathfrak{p}_1(R_\alpha)_T$.

The natural candidate for the total specialization of $R_{\mathfrak{p}_u}$ is the semi-local ring $(R_\alpha)_T$.

Definition We call $(R_\alpha)_T$ a *total specialization* of $R_{\mathfrak{p}_u}$ with respect to α. For short we will put $S = R_{\mathfrak{p}_u}, S_\alpha = (R_\alpha)_\mathfrak{p}$ and $S_T = (R_\alpha)_T$, where $\mathfrak{p}$ is one of the $\mathfrak{p}_i$. Then there is $(S_T)_{\mathfrak{p}_T} = S_\alpha$.

3. The total specialization of $R_{\mathfrak{p}_u}$-modules

Let f be an arbitrary element of R. We may write $f = p(u,x)/q(u)$, $p(u,x) \in k[u,x]$, $q(u) \in k[u] \setminus \{0\}$. For any α such that $q(\alpha) \neq 0$ we define $f_\alpha := p(\alpha, x)/q(\alpha)$. It is easy to check that this element does not depend on the choice of $p(u,x)$ and $q(u)$ for almost all α. Now, for every fraction $a = f/g$, $f, g \in R$, $g \neq 0$, we define $a_\alpha := f_\alpha/g_\alpha$ if $g_\alpha \neq 0$. Then a_α is uniquely determined for almost all α.

The following lemma shows that the above definition of S_T reflects the intrinsic substitution $u \to \alpha$ of elements of R.

Lemma 3.1. *Let a be an arbitrary element of S. Then $a_\alpha \in S_T$ for almost all α.*

Proof. Since $\mathfrak{p}_u$ is a separable prime ideal of R, $\mathfrak{p}_\alpha \neq R_\alpha$ for almost all α. Let $a = f/g$ with $f, g \in R$, $g \notin \mathfrak{p}_u$. Since $\mathfrak{p}$ is prime, $\mathfrak{p}_u : g = \mathfrak{p}_u$. By [1, Satz 3], $\mathfrak{p}_\alpha = (\mathfrak{p}_u : g)_\alpha = \mathfrak{p}_\alpha : g_\alpha$. Hence $g_\alpha \in T$. Then $a_\alpha \in S_\alpha$ for almost all α.

First we want to recall the definition of specialization of finitely generated S-module by [2]. Let F, G be finitely generated free S-modules. Let $\phi : F \to G$ be an arbitrary homomorphism of free S-modules of finite ranks. With fixed bases of F and G, ϕ is given by a matrix $A = (a_{ij})$, $a_{ij} \in S$.

By Lemma 3.1, the matrix $A_\alpha := ((a_{ij})_\alpha)$ has all its entries in $(R_\alpha)_\mathfrak{p}$ for almost all α. Let F_α and G_α be free $(R_\alpha)_\mathfrak{p}$-modules of the same rank as F and G, respectively.

Definition. [2] For fixed bases of F_α and G_α, the homomorphism $\phi_\alpha : F_\alpha \to G_\alpha$ given by the matrix A_α is called the *specialization* of ϕ with respect to α.

The definition of ϕ_α does not depend on the choice of the bases of F, G in the sense that if B is the matrix of ϕ with respect to other bases of F, G, then there are bases of F_α, G_α such that B_α is the matrix of ϕ_α with respect to these bases.

Definition. [2] Let L be a finitely generated S-module and $F_1 \xrightarrow{\phi} F_0 \to L \to 0$ a finite free presentation of L. The $(R_\alpha)_\mathfrak{p}$-module $L_\alpha :=$Cokerϕ_α is called a *specialization* of L (with respect to ϕ).

Then, we have the following results.

Lemma 3.2. [2, Theorem 2.2] *Let* $0 \to L \to M \to N \to 0$ *be an exact sequence of finitely generated S-modules. Then* $0 \to L_\alpha \to M_\alpha \to N_\alpha \to 0$ *is exact for almost all* α.

Lemma 3.3. [2, Theorem 2.6] *Let L be a finitely generated S-module. Then, for almost all α, we have*

(i) *(Ann $L)_\alpha$ =Ann (L_α).*

(ii) $\dim L = \dim L_\alpha$.

Lemma 3.4. [2, Theorem 3.1] *Let L be a finitely generated S-module. Then, for almost all α, we have*

(i) $\mathrm{proj}L_\alpha = \mathrm{proj}L$.

(ii) $\mathrm{depth}L_\alpha = \mathrm{depth}L$.

Now we will define the total specialization of an arbitrary finitely generated S-module as follows. As above, the matrix $A_\alpha := ((a_{ij})_\alpha)$ has all its entries in S_T for almost all α. Let F_T and G_T be free S_T-modules of the same rank as F and G, respectively, and B_α is the matrix of ϕ_T with respect to these bases.

Definition. Let L be a finitely generated S-module and $F_1 \xrightarrow{\phi} F_0 \to L \to 0$ a finite free presentation of L. The S_T-module $L_T :=$ Cokerϕ_T is called a *total specialization* of L (with respect to ϕ). The module L_T depends on the chosen presentation of L, but L_T is uniquely determined up to isomorphisms. Hence the finite free presentation of L will be chosen in the form $S^s \xrightarrow{\phi} S^r \to L \to 0$.

Lemma 3.5. *Let L be a finitely generated S-module. Suppose that $\mathfrak{p} = \mathfrak{p}_1$. Then $(L_T)_{\mathfrak{p}_{1T}} \cong L_\alpha$ for almost all α.*

Proof. Let $S^s \xrightarrow{\phi} S^r \to L \to 0$ be a finite free presentation of L. There exists an exact sequence $(R_\alpha)_T^s \xrightarrow{\phi_T} (R_\alpha)_T^r \to L_T \to 0$. This will induces also an exact sequence $|(R_\alpha)_T|_{\mathfrak{p}_T}^s \xrightarrow{|\phi_T|_{\mathfrak{p}_T}} |(R_\alpha)_T|_{\mathfrak{p}_T}^r \to |L_T|_{\mathfrak{p}_T} \to 0$. By an easy computation $A_\alpha = ((a_{ij})_\alpha) = (\frac{(f_{ij})_\alpha/1}{(g_{ij})_\alpha/1})$, it follows that $(\phi_T)_{\mathfrak{p}_T} = \phi_\alpha$. Since $|(R_\alpha)_T|_{\mathfrak{p}_T} \cong (R_\alpha)_\mathfrak{p} = S_\alpha$, we have a commutative diagram

$$\begin{array}{ccccccc}
|(R_\alpha)_T|_{\mathfrak{p}_T}^s & \xrightarrow{\phi_\alpha} & |(R_\alpha)_T|_{\mathfrak{p}_T}^r & \longrightarrow & |L_T|_{\mathfrak{p}_T} & \longrightarrow & 0 \\
\downarrow\cong & & \downarrow\cong & & \downarrow & & \\
S_\alpha^s & \xrightarrow{\psi_\alpha} & S_\alpha^r & \longrightarrow & L_\alpha & \longrightarrow & 0,
\end{array}$$

where to rows are finite free presentations of $|L_T|_{\mathfrak{p}_T}$ and L_α, and an isomorphism $(L_T)_{\mathfrak{p}_T} \to L_\alpha$. Hence $(L_T)_{\mathfrak{p}_T} \cong L_\alpha$ for almost all α.

Proposition 3.6. *Let L be a finitely generated S-module. For almost all α, we have*

(i) $(\mathrm{Ann}L)_\alpha = \mathrm{Ann}(L_T)_{\mathfrak{p}_T}$.
(ii) $\dim L = \dim L_T$.

Proof. (i) Since $(L_T)_{\mathfrak{p}_T} \cong L_\alpha$ by Lemma 3.5, there is $\mathrm{Ann}(L_T)_{\mathfrak{p}_T} = \mathrm{Ann}((L_T)_{\mathfrak{p}_T}) = \mathrm{Ann}(L_\alpha)$. Since $\mathrm{Ann}(L)_\alpha = \mathrm{Ann}(L_\alpha)$ by Lemma 3.3, therefore $\mathrm{Ann}(L)_\alpha = \mathrm{Ann}(L_T)_{\mathfrak{p}_T}$ for almost all α.
(ii) We have $\dim L = \dim L_\alpha$ by Lemma 3.3. Then $\dim L = \dim(L_T)_{\mathfrak{p}_T}$. Semilarly, $\dim L = \dim(L_T)_{\mathfrak{p}_{iT}}$ for $i = 1, \ldots, s$. Hence $\dim L = \dim L_T$ for almost all α.

Theorem 3.7. *Let $0 \to L \to M \to N \to 0$ be an exact sequence of finitely generated S-modules. Then $0 \to L_T \to M_T \to N_T \to 0$ is exact for almost all α.*

Proof. Since $0 \to L \to M \to N \to 0$ is an exact sequence, the sequence $0 \to L_\alpha \to M\alpha \to N_\alpha \to 0$ is also exact by Lemma 3.2, or the sequence $0 \to (L_T)_{\mathfrak{p}_T} \to (M_T)_{\mathfrak{p}_T} \to (N_T)_{\mathfrak{p}_T} \to 0$ is exact for every maximal ideal $\mathfrak{p}_T$. Hence $0 \to L_T \to M_T \to N_T \to 0$ is exact for almost all α.

Proposition 3.8. *Let L be a finitely generated S-module. For almost all α, we have*

(i) $\mathrm{proj}L = \mathrm{proj}L_T$,
(ii) $\mathrm{depth}L = \mathrm{depth}L_T$.

Proof. (i) Since $\mathrm{proj}L = \mathrm{proj}L_\alpha$ for almost all α by Lemma 3.4, there is $\mathrm{proj}L_T = \sup\limits_{\mathfrak{m} \in \sup(S_T)} \{\mathrm{proj}(L_\alpha)_\mathfrak{m}\} = \mathrm{proj}L_\alpha = \mathrm{proj}L$.
(ii) By [4, Lemma 18.1], there is a maximal ideal $\mathfrak{m}$ of S_T such that $\mathrm{depth}L_T = \mathrm{depth}(L_T)_\mathfrak{m} = \dim(L_T)_{\mathfrak{p}_T}$. Then $\mathrm{depth}L_T = \mathrm{depth}L_\alpha = \mathrm{depth}L$ by Lemma 3.4.

Proposition 3.9. *Let L be a S-module of finite length. Then L_T is a S_T-module of finite length for almost all α. Moreover, $\ell(L_T) = s\ell(L)$.*

Proof. Since $\ell(L_\alpha) = \ell(L)$ by [2, Proposition 2.8] and $\ell(L_T) = \sum\limits_{\mathfrak{m} \in \sum(R_T)} \ell((L_T)_\mathfrak{m})$ by [5, 3. Theorem 12], there is $\ell(L_T) = s\ell(L)$.

Proposition 3.10. *Let L be a finitely generated S-module of dimension d and $\mathfrak{q} = (a_1, \ldots, a_d)S$ a parameter ideal on L. Then, we have $e(\mathfrak{q}_T, L_T) = se(\mathfrak{q}, L)$ for almost all α, where $e(\mathfrak{q}_T, L_T)$ and $e(\mathfrak{q}, L)$ are the multiplicities of L_T and L with respect to $\mathfrak{q}_T$ and $\mathfrak{q}$, respectively.*

Proof. First, we will to show that $e(\mathfrak{q}_\alpha, L_\alpha) = e(\mathfrak{q}, L)$. Indeed, Since $a_1, \ldots, a_d \in \mathfrak{p}S$, for almost all α there are $(a_1)_\alpha, \ldots, (a_d)_\alpha \in \mathfrak{p}_\alpha S_\alpha$. By Lemma 3.2 and by Lemma 3.3, $\dim L_\alpha/((a_1)_\alpha, \ldots, (a_d)_\alpha) L_\alpha = \dim L/(a_1, \ldots, a_d)L = 0$. Then $(a_1)_\alpha, \ldots, (a_d)_\alpha$ is a system of parameters on L_α. The multiplicity symbol of $a_1, \ldots, a_d$ with respect to L will be denoted by $e(a_1, \ldots, a_d|L)$, and the multiplicity symbol of $(a_1)_\alpha, \ldots, (a_d)_\alpha$ with respect to $L\alpha$ by $e((a_1)_\alpha, \ldots, (a_d)_\alpha|L_\alpha)$. Then we have

$$\begin{aligned} e(\mathfrak{q}_\alpha; L_\alpha) &= e((a_1)_\alpha, \ldots, (a_d)_\alpha|L_\alpha) \\ e(\mathfrak{q}; L) &= e(a_1, \ldots, a_d|L). \end{aligned}$$

We need only show that $e(a_1, \ldots, a_d|L) = e((a_1)_\alpha, \ldots, (a_d)_\alpha|L_\alpha)$. This claim will be proved by induction on d. For $d = 0$, by applying [2, Proposition 2.8], there is

$$e(\emptyset|L_\alpha) = \ell(L_\alpha) = \ell(L) = e(\emptyset|L).$$

Now we assume that $d \geq 1$, and the claim is true for all S-modules with the dimension $\leq d-1$. By [2, Lemma 2.3 and Lemma 2.5], there are

$$L_\alpha/(a_1)_\alpha L_\alpha \cong (L/a_1L)_\alpha \text{ and } 0_{L_\alpha} : (a_1)_\alpha \cong (0_L : a_1)_\alpha.$$

Since the dimensions of these modules $\leq d-1$, therefore

$$\begin{aligned} e((a_2)_\alpha, \ldots, (a_d)_\alpha | L_\alpha/(a_1)_\alpha L_\alpha) &= e(a_2, \ldots, a_d | L/a_1L) \\ e((a_2)_\alpha, \ldots, (a_d)_\alpha | 0_{L_\alpha} : (a_1)_\alpha) &= e(a_2, \ldots, a_d | 0_L : a_1). \end{aligned}$$

The statment follows from the definition of the multiplicity.
Now we prove the result $e(\mathfrak{q}_T, L_T) = se(\mathfrak{q}, L)$ Since

$$e(\mathfrak{q}_T, L_T) = e((a_1)_T, \ldots, (a_d)_T | L_T) = \sum_{\mathfrak{m} \in \sum(R_T)} e_{(R_T)_\mathfrak{m}}(\Phi_\mathfrak{m}(a_1)_T, \ldots, \Phi_\mathfrak{m}(a_d)_T | (L_T)_\mathfrak{m})$$

by [5, 7.8. Theorem 15], there is $e(\mathfrak{q}_T, L_T) = se(\mathfrak{q}, L)$ for almost all α.

4. Preservation of some properties of modules

By virtue of Proposition 3.10 one can show that preservartion of Cohen-Macaulayness by total specializations.

Theorem 4.1. *Let L be a finitely generated S-module. For almost all α, we have*

(i) *L_T is a Cohen-Macaulay S_T-module if L is a Cohen-Macaulay S-module.*
(ii) *L_T is a maximal Cohen-Macaulay S_T-module if L is a maximal Cohen-Macaulay S-module.*

Proof. We need only show that $(L_T)_{\mathfrak{p}_T}$ is a (maximal) Cohen-Macaulay $(S_T)_{\mathfrak{p}_T}$-module if L is a (maximal) Cohen-Macaulay S-module.
(i) Assume that L is a Cohen-Macaulay S-module. Therefore $\dim L = \operatorname{depth} L$. Since $\dim L = \dim L_\alpha$ by Lemma 3.3 and $\operatorname{depth} L = \operatorname{depth} L_\alpha$ by Lemma 3.4, we get $\dim L_\alpha = \operatorname{depth} L_\alpha$. Hence L_α is also a Cohen-Macaulay S_α-module for almost all α. Since $L_\alpha = (L_T)_{\mathfrak{p}_T}$, it follows that L_T is a Cohen-Macaulay S_T-module for almost all α.
(ii) Assume that L is a maximal Cohen-Macaulay S-module. Therefore $\dim L = \dim S$. Since $\dim L_\alpha = \dim L$ and $\dim S_\alpha = \dim S$, it follows that $\dim L_\alpha = \dim S_\alpha$. Hence L_T is a maximal Cohen-Macaulay S_T-module.

The ith *Bass* and ith *Betti numbers* of L, which are denoted by $\mu^i_S(L)$ and $\beta_i(L)$ respectively, are defined as follows:

$$\mu^i_S(L) = \dim_{S/\mathfrak{m}} \operatorname{Ext}^i_S(S/\mathfrak{m}, L), \beta_i(L) = \dim_{S/\mathfrak{m}} \operatorname{Tor}^S_i(S/\mathfrak{m}, L), \forall\, i \geq 0.$$

Lemma 4.2. *Let L be finitely generated S-modules. Then, for almost all α, we have*

$$\mu^i_{S_\alpha}(L_\alpha) = \mu^i_S(L), \beta_i(L_\alpha) = \beta_i(L), \forall\, i \geq 0.$$

Proof. Since L and L_α are the finitely generated modules, all integers $\mu^i_S(L)$ and $\mu^i_{S_\alpha}(L_\alpha)$ are finite. We have

$$\mu^i_S(L) = \ell(\operatorname{Ext}^i_S(S/\mathfrak{m}, L)), \mu^i_{S_\alpha}(L_\alpha) = \ell(\operatorname{Ext}^i_{S_\alpha}(S_\alpha/\mathfrak{m}_\alpha, L_\alpha)).$$

By [2, Proposition 3.3], there is $\mathrm{Ext}^i_{S_\alpha}(S_\alpha/\mathfrak{m}_\alpha, L_\alpha) \cong \mathrm{Ext}^i_S(S/\mathfrak{m}, L)_\alpha$. Since $\mathfrak{p}_\alpha$ is a radical ideal, from [2, Proposition 2.8] it follows that

$$\ell(\mathrm{Ext}^i_{S_\alpha}(S_\alpha/\mathfrak{m}_\alpha, L_\alpha)) = \ell(\mathrm{Ext}^i_S(S/\mathfrak{m}, L)_\alpha) = \ell(\mathrm{Ext}^i_S(S/\mathfrak{m}, L)).$$

Hence $\mu^i_S(L) = \mu^i_{S_\alpha}(L_\alpha)$. Similar, we obtain $\beta_i(L) = \beta_i(L_\alpha)$.

Before invoking Lemma 4.2 to reprove Corollary 3.8 in [2], we will define a quasi-Buchsbaum module. A finitely generated module over a Noetherian commutative ring is said to be a *quasi-Buchsbaum* module if its localization at every maximal ideal is a surjective Buchsbaum.

Corollary 4.3. *Let L be finitely generated S-modules. Then, for allmost all α, we have*

(i) *If L is a surjective Buchsbaum S-module, then L_α is also a surjective Buchsbaum S_α-module.*

(ii) *If L is a quasi-Buchsbaum S-module, then L_T is also a quasi-Buchsbaum S_T-module.*

Proof. (i) Put $d = \dim L$. By Lemma 3.3, $\dim L_\alpha = d$. Since S is a regular ring, by [6, Chapter 2. Theorem 4.2] we known that L is a surjective S-module if and only if

$$\mu^i_S(L) = \sum_{j=0}^{i} \beta_{i-j}(S/\mathfrak{m})\ell(H^j_{\mathfrak{m}}(L)), i = 0, \ldots, d-1.$$

Since $\ell(H^j_{\mathfrak{m}}(L)) < \infty$, therefore $\ell(H^j_{\mathfrak{m}_\alpha}(L_\alpha)) = \ell(H^j_{\mathfrak{m}}(L))$ by [2, Theorem 3.6]. Now the proof is immedialtely from Lemma 4.2.

(ii) It is easily seen that the localization of L_T at every maximal ideal is a surjective Buchsbaum, Hence L_T is also a quasi-Buchsbaum S_T-module.

We will now recall the definition of the Gorenstein module. A non-zero and finitely generated L is said to be a *Gorenstein* module if and only if the cousin complex for L provides a injective resolution for L, see [7]. Before proving the preservation of Gorensteiness of module, we will show that the injective dimension of module L is not change by specialization.

Lemma 4.4. *Let L be finitely generated S-modules. Then, for almost all α, we have*

$$\mathrm{inj.dim}(L_\alpha) = \mathrm{inj.dim}\ (L).$$

In particular, if L is an injective module, then L_α is also an injective module.

Proof. Since S and S_α have finite global dimensions, therefore $\mathrm{inj.dim}L$ and $\mathrm{inj.dim}L_\alpha$ are finite. From [8, Theorem 3.1.17] we obtain the following relations

$$\mathrm{inj.dim}L_\alpha = \mathrm{depth}S_\alpha = \mathrm{depth}S = \mathrm{inj.dim}L.$$

If L is an injective module, then $\mathrm{inj.dim}L = 0$. Hence $\mathrm{inj.dim}L_\alpha = 0$, and therefore L_α is also an injective module.

Theorem 4.5. *Let L be finitely generated S-modules. If L is a Gorenstein S-module, then $(L_T)_{\mathfrak{p}_T}$ is again a Gorenstein $(S_T)_{\mathfrak{p}_T}$-module for almost all α.*

Proof. Assume that L is a Gorenstein S-module of dimension d. Then L is a Cohen-Macaulay S-module and $\dim S = \mathrm{inj.dim}L = d$ by [7, Theorem 3.11]. Since $\dim L_\alpha = \dim L = d$ by Lemma 3.3 and $\mathrm{inj.dim}(L_\alpha) = \mathrm{inj.dim}(L)$ by Lemma 4.2, therefore $\dim S_\alpha = \mathrm{inj.dim}L_\alpha = \dim L_\alpha$. Hence $(L_T)_{\mathfrak{p}_T}$ is again a Gorenstein $(S_T)_{\mathfrak{p}_T}$-module for almost all α.

Corollary 4.6. *Let I be an ideal of S. If S/I is a Gorenstein ring, then S_T/I_T is again a Gorenstein ring for almost all α.*

Proof. We first will recall the definition about the Gorenstein ring. A Noetherian ring is a Gorenstein ring if its localization at every maximal ideal is a Gorenstein local ring. Since the localization of

S_T/I_T at every maximal ideal is also a Gorenstein ring by Theorem 4.5, therefore S_T/I_T is again a Gorenstein ring for almost all α.

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