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On the set of periods for periodic solusions of some linear differential equations on the multidimensional sphere S^n

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Abstract. The problem about periodic solutions for the family of linear differential equation

$$Lu \equiv \left(\frac{\partial}{i\partial t} - a\Delta\right) u(x,t) = \nu G(u-f)$$

is considered on the multidimensional sphere $x \in S^n$ under the periodicity condition $u|_{t=0} = u|_{t=b}$ and $\int_{S^n} u(x,t)dx = 0$.

Here a is given real, ν is a fixed complex number, $Gu(x,t)$ is a linear integral operator, and Δ is the Laplace operator on S^n. It is shown that the set of parameters (ν, b) for which the above problem admits a unique solution is a measurable set of full measure in $\mathbb{C} \times \mathbb{R}^+$.

This work further develops part of the authors' result in [1, 2], on the problem on the periodic solution to the equation $(L-\lambda)u = \nu G(u-f)$. Here L is Schrödinger operator on sphere S^n and λ belongs to the spectrum of L. Particularly, the authors consider the case that λ is an eigenvalue of L (the case which can be always converted to the case $\lambda = 0$). It is shown that the main results are all right (but) on the complement of eigenspace of λ in the domain of L.

1. We consider the problem on periodic solutions for the nonlocal Schrödinger type equation

$$\left(\frac{1}{i}\frac{\partial}{\partial t} - a\Delta\right) u(x,t) = \nu G(u-f), \tag{1}$$

with these conditions :

$$u|_{t=0} = u|_{t=b}; \quad \int_{S^n} u(x,t)dx = 0. \tag{2}$$

Here $u(x,t)$ - is a complex function on $S^n \times [0,b]$, S^n - is the multidimensional sphere, $n \geq 2$; $a \neq 0$, ν - are given complex numbers, $f(x,t)$ - is a given function. The change of variables $t = b\tau$ reduces our problem to a problem with a fixed period, but with a new equation in which the coefficient of the $\tau-$ derivative is equal to $\frac{1}{b}$:

$$\left(\frac{1}{i}\frac{\partial}{b\partial\tau} - a\Delta\right) u(x,b\tau) = \nu G(u(x,b\tau) - f(x,b\tau)).$$

2. Thus, problem (1), (2) turns into the problem on periodic solution of the equation

$$Lu \equiv \left(\frac{1}{i}\frac{\partial}{b\partial t} - a\Delta\right) u(x,t) = \nu G(u(x,t) - f(x,t)), \tag{3}$$

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with the following conditions:

$$u|_{t=0} = u|_{t=1}, \quad \int_{S^n} u(x,t)dx = 0. \tag{4}$$

Here $Gu(x,t) = \int_{S^n} g(x,y)u(y,t)dy$ (dy is the Lebesgue-Hausdorff measure on the sphere S^n) is an integral operator on the space $L_2(S^n \times [0,1])$ with smooth kernel $g(x,y)$ defined on $S^n \times S^n$ such that

$$\int_{S^n} g(x,y)dx = 0 \tag{5}$$

for all y in S^n.

The differential operation $\frac{1}{i}\frac{\partial}{b\partial t} - a\Delta$ is assumed to be defined for the functions $u(x,t) \in C^\infty(S^n \times [0,1])$ such that $u|_{t=0} = u|_{t=1}$ and with the condition $\int_{S^n} u(x,t)dx = 0$. Let L - denote the closure of this operation $\frac{1}{i}\frac{\partial}{b\partial t} - a\Delta$ in $\mathcal{H} = L_2(S^n \times [0,1])$. So an element $u \in \mathcal{H}$ belongs to the domain $\mathcal{D}(L)$ of $L = \frac{1}{i}\frac{\partial}{b\partial t} - a\Delta$, if and only if there is a sequence $\{u_j\} \subset C^\infty(S^n \times [0,1])$ $u_j|_{t=0} = u_j|_{t=1}$ and $\int_{S^n} u_j dx = 0$, such that $\lim u_j = u$, $\lim Lu_j = Lu$ in $\mathcal{H}$. Let $\mathcal{H}_0$ is a subspace of space $L_2(S^n \times [0,1])$,

$$\mathcal{H}_0 = \{u(x,t) \in L_2(S^n \times [0,1]) \mid \int_{S^n} u(x,t)dx = 0\}.$$

It is well known that the eigenvalues of the Laplace operator Δ on the sphere S^n are of the form $-k(k+n-1)$, $k \in \mathbb{Z}$, $k \geq 0$ and that Δ admits the corresponding orthonormal basis of eigenfunctio $w_k(x) \in C^\infty(S^n)$(see, e.g [3]).

Lemma 1. *The functions $e_{km}(x,t) = e^{i2\pi mt}w_k(x)$, $k, m \in \mathbb{Z}$, $k > 0$ are eigenfunctions of the operator L in the space $\mathcal{H}_0$ that correspond to the eigenvalues*

$$\lambda_{km} = \frac{2m\pi}{b} + ak(k+n-1) = \frac{2m\pi}{b} + \lambda_k \tag{6}$$

These functions form an orthonormal basis in $\mathcal{H}_0$. The domain of L is given by the formula

$$\mathcal{D}(L) = \{u = \sum_{m,k\in\ ,k>0} u_{km}e_{km} \mid \sum |\lambda_{km}u_{km}|^2 < \infty, \sum |u_{km}|^2 < \infty\}$$

The spectrum $\sigma(L)$ in the closure of the set $\{\lambda_{km}\}$.

Lemma 2.

$$||G||^2 \leq M_0^2 = \int_{S^n}\int_{S^n} |g(x,y)|^2 dx\, dy.$$

Proof. We have

$$|Gu(x,t)|^2 = |\int_{S^n} g(x,y)u(y,t)\,dy|^2 \leq \int_{S^n} |g(x,y)|^2 dy \int_{S^n} |u(y,t)|^2 dy$$

$$||Gu(x,t)||^2 = \int_0^1 \int_{S^n} |Gu(x,t)|^2 dx dt \leq$$

$$\int_0^1 \int_{S^n} \left(\int_{S^n} |g(x,y)|^2 dy \int_{S^n} |u(y,t)|^2 dy \right) dx\, dt$$

$$||Gu(x,t)||^2 \leq \int_{S^n}\int_{S^n}|g(x,y)|^2dxdy\int_{S^n}\int_0^1|u(y,t)|^2dydt = M_0^2||u||^2$$
$$||G|| \leq M_0.$$

The lemma is proved.

We note that the Laplace operator is formally selfadjoint relative to the scalar product $(u,v) = \int_{S^n} u(x)\overline{v(x)}dx$ on the space $C^\infty(S^n)$. The product $\Delta_x \circ G = \Delta_x G$ coincides with the integral operator with the kernel $\Delta_x g(x,y)$. Let the function $\Delta_x g(x,y)$ be continuous on $S^n \times S^n$. We put $M = \max\{||\Delta_x G||, ||G||\}$.

Lemma 3. *Let* $v = Gu = \sum_{m,k\in\ ,k>0} v_{km}e_{km}$, *then*

$$|v_{km}|^2 = \frac{|\alpha_{km}|^2}{(k(k+n-1))^2} \leq \frac{M^2||u||^2}{(k(k+n-1))^2}, \tag{7}$$

where $\alpha_{km} = \langle\Delta_x Gu, e_{km}\rangle$, *and* $\sum|\alpha_{km}|^2 \leq M^2||u||^2$.

Proof. Since the Laplace operator is selfadjoint, for $k > 0$ we have

$$\alpha_{km} = \langle\Delta_x Gu, e_{km}\rangle = \langle Gu, \Delta_x e_{km}(x,t)\rangle = \langle Gu, -k(k+n-1)e_{km}(x,t)\rangle$$
$$\alpha_{km} = -k(k+n-1)\langle Gu, e_{km}(x,t)\rangle = -k(k+n-1)v_{km}.$$

It follows that

$$|v_{km}|^2 = \frac{|\alpha_{km}|^2}{(k(k+n-1))^2}.$$

By the Parseval identity, we have $\sum|\alpha_{km}|^2 = ||\Delta_x Gu||^2 \leq M^2||u||^2$, whence

$$|v_{km}|^2 \leq \frac{M^2||u||^2}{(k(k+n-1))^2}.$$

The lemma is proved.

We assume that a is a real number. Then by Lemma 1, the spectrum $\sigma(L)$ lies on the real axis. Most typical and interesting is the case where the number $ab/(2\pi)$ is irrational. In this case, $0 \neq \lambda_{km}\forall k,m \in \mathbb{Z}, k > 0$ and the H.Weyl theorem (see, e.g., [4]), says that, in this case, the set of the numbers λ_{km} is everywhere dense on R and $\sigma(L) = \mathbb{R}$. Then in the subspace $\mathcal{H}_0$ the inverse operator L^{-1} is well defined , but unbounded. The expression for this inverse operator involves small denominators.

$$L^{-1}v(x,t) = \sum\frac{v_{km}}{\lambda_{km}}e_{km}, \tag{8}$$

where v_{km} is the Fourier coefficient of the series

$$v(x,t) = \sum_{k,m\in\ ,k>0} v_{km}e_{km}.$$

For positive numbers σ and C, let $A_\sigma(C)$ denote the set of all positive b such that

$$|\lambda_{km}| \geq \frac{C}{k^{1+\sigma}}. \tag{9}$$

for all $m,k \in \mathbb{Z}, k > 0$.

This definition shows that the set $A_\sigma(C)$ extends as C reduces and as σ grows. Therefore, in what follows, to prove that such a set or its part is nonempty, we require that $C > 0$ be sufficiently small and σ sufficiently large. Let A_σ denote the union of the sets $A_\sigma(C)$ over all $C > 0$. If inequality (9) is fulfilled for some b and all m, k, then it is fulfilled for $m = 0$; this provides a condition necessary for the nonemptiness of $A_\sigma(C)$:

$$C \leq k^{1+\sigma}|ak(k+n-1)| \ \forall\ k > 0. \tag{10}$$

We put $d = min_{k\in Z, k>0} k^{1+\sigma}|ak(k+n-1)| > 0$.

Theorem 1. *The sets $A_\sigma(C)$, A_σ are Borel. The set A_σ has full measure, i.e., its complement to the half-line $\mathbb{R}^+$ is of zero measure.*

Proof. Obviously, the sets $A_\sigma(C)$ are closed in $\mathbb{R}^+$. The set $A_\sigma = \bigcup_{r=1}^{\infty} A_\sigma(1/r)$ - is Borel, being a countable union of closed sets. We show that A_σ has full measure in $\mathbb{R}^+$. Suppose $b,\ l > 0,\quad C \leq \frac{d}{2}$; we consider the complement $(0, l)\backslash A_\sigma(C)$. This set consists of all positive numbers b, for which there exist m, k, such that

$$|\lambda_{km}| < \frac{C}{k^{1+\sigma}}. \tag{11}$$

Solving this inequality for b, we see that, for m, k fixed, the number b form an interval $I_{k,m} = (m\alpha_k, m\beta_k)$, where $m = 1, 2, 3, \ldots$,

$$\alpha_k = \frac{2\pi}{|ak(k+n-1)| + \frac{C}{k^{1+\sigma}}},\ \beta_k = \frac{2\pi}{|ak(k+n-1)| - \frac{C}{k^{1+\sigma}}}.$$

The length of $I_{k,m}$ is $m\delta_k$, with

$$\delta_k = \frac{4\pi C k^{-1-\sigma}}{|ak(k+n-1)|^2 - C^2 k^{-2-2\sigma}}.$$

Since $C \leq \frac{d}{2}$ by assumption, we have

$$\delta_k \leq \frac{16\pi C}{3k^{1+\sigma}|ak(k+n-1)|^2}. \tag{12}$$

For k fixed and m varying, there are only finitely many of intervals I_{km} that intersect the given segment $(0, l)$. Such intervals arise for the values of $m = 1, 2 \ldots$, satisfying $m\alpha_k < l$, i.e.,

$$0 < m < \frac{l}{2\pi}(|ak(k+n-1)| + Ck^{-1-\sigma}).$$

Since $Ck^{-1-\sigma} \leq \frac{1}{2}|ak(k+n-1)|$, we can write simpler restrictions on m :

$$0 < m < \frac{l}{2\pi}\frac{3}{2}|ak(k+n-1)| < \frac{l}{\pi}|ak(k+n-1)|. \tag{13}$$

The measure of the intervals indicated (for k fixed) is dominated by $\delta_k \bar{S}_k$, where $\bar{S}_k = \bar{S}_k(l)$ is the sum of all integers m satisfying (13). Summing an arithmetic progression, we obtain

$$\bar{S}_k \leq \frac{l}{2\pi^2}|ak(k+n-1)|\{l|ak(k+n-1)| + \pi\}. \tag{14}$$

Passing to the union of the intervals in question over k and m, and using (12), we see that

$$\mu((0,l)\backslash A_\sigma(C)) \le \sum_{k=0}^{\infty} \delta_k \tilde{S}_k \le CS(l),$$

where

$$S = S(l) = \sum_{k=0}^{\infty} \frac{8l\{l|ak(k+n-1)| + \pi\}}{3\pi k^{1+\sigma}|ak(k+n-1)|}.$$

Observe that the quantity

$$\frac{l|ak(k+n-1)| + \pi}{\pi|ak(k+n-1)|}$$

is dominated by a constant D; therefore,

$$S(l) \le \frac{8}{3} lD \sum_{k=1}^{\infty} \frac{1}{k^{1+\sigma}} < \infty.$$

We have

$$\mu((0,l) \setminus A_\sigma) \le \mu((0,l) \setminus A_\sigma(C)) \le CS(l) \quad \forall C > 0.$$

It follows that $\mu((0,l) \setminus A_\sigma) = 0 \quad \forall l > 0$. Thus,$\mu((0,\infty) \setminus A_\sigma) = 0$ and A_σ- has full measure. The theorem is proved.

Theorem 2. *Suppose $g(x,y)$ is a function defined on $S^n \times S^n$ such that the function $\Delta_x g(x,y)$ is continuous on $S^n \times S^n$ and $\int_{S^n} g(x,y)dx = 0\ \forall y \in S^n$. Let $0 < \sigma < 1$, and let $b \in A_\sigma(C)$. Then in the space $\mathcal{H}_0$ the inverse operator L^{-1} is well defined, and the operator $L^{-1} \circ G$ is compact.*

Proof. Since $b \in A_\sigma(C)$, we have $\lambda_{km} \neq 0 \quad \forall\, k,m \in \mathbb{Z},\ k > 0$ so that L^{-1} is well defined and looks like the expression in (8). Observe that $\lim \dfrac{k^{2+2\sigma}}{(k(k+n-1))^2} = 0$ as $k \to \infty$. Therefore, given $\varepsilon > 0$, we can find an integer $k_0 > 0$, such that $\dfrac{k^{2+2\sigma}}{(k(k+n-1))^2} < \dfrac{(\varepsilon C)^2}{M^2}$ for all $k > k_0$. We write

$$L^{-1}v(x,t) = Q_{k_{01}}v + Q_{k_{02}}v, \quad v = Gu,$$

where

$$Q_{k_{01}}v = \sum_{0<k\le k_0} \frac{v_{km}}{\lambda_{km}} e_{km}, \quad Q_{k_0 2}v = \sum_{k>k_0} \frac{v_{km}}{\lambda_{km}} e_{km}.$$

For the operator $Q_{k_{01}}$ we have

$$||Q_{k_{01}}v||^2 = \sum_{0<k\le k_0} \frac{|v_{km}|^2}{|\lambda_{km}|^2}$$

Observe that if $0 < k \le k_0$, then

$$\lim \frac{1}{|\frac{2m\pi}{b} + ak(k+n-1)|^2} = 0$$

as $|m| \to \infty$. Therefore, the quantity $\dfrac{1}{|\frac{2m\pi}{b} + ak(k+n-1)|^2}$ is dominated by a constant $C(k_0)$.

Then

$$||Q_{k_0 1}v||^2 \le \sum |v_{km}|^2 C(k_0) \le C(k_0)||v||^2,$$

which means that $Q_{k_{01}}$ is a bounded operator.

Consider the operator $Q_{k_{02}} \circ G$. By Lemma 3 and (9), we have

$$||Q_{k_{02}}v||^2 = ||Q_{k_{02}} \circ Gu||^2 = \sum_{k>k_0} \frac{|v_{km}|^2}{|\lambda_{km}|^2} \leq$$

$$\sum_{k>k_0} \frac{\alpha_{km}^2}{(k(k+n-1))^2}(\frac{1}{C})^2 k^{2+2\sigma} \leq (\frac{1}{C})^2(\frac{\varepsilon C}{M})^2 \sum_{k>k_0} |\alpha_{km}|^2 \leq \varepsilon^2||u||^2.$$

Consequently, $||Q_{k_{02}} \circ G|| \leq \varepsilon$.

Since G is compact and $Q_{k_{01}}$ is bounded, $Q_{k_{01}} \circ G$ is compact. Next, we have

$$||L^{-1} \circ G - Q_{k_{01}} \circ G|| = ||Q_{k_{02}} \circ G|| < \varepsilon.$$

Thus, we see that the operator $L^{-1} \circ G$ is the limit of sequence of compact operators. Therefore, it is compact itself. The theorem is proved.

We denote $K = K_b = L^{-1} \circ G$.

Theorem 3. *Suppose $b \in A_\sigma(C)$. Then problem (1),(2) admits a unique periodic solution with period b for all $\nu \in \mathbb{C}$, except, possibly, an at most countable discrete set of values of ν.*

Proof. Equation (1) reduces to

$$(L^{-1} \circ G - \frac{1}{\nu})u = L^{-1} \circ G(f).$$

We write $L^{-1} \circ G - \frac{1}{\nu} = K - \frac{1}{\nu}$.

Since $K = L^{-1} \circ G$ is a compact operator, its spectrum $\sigma(K)$ is at most countable, and the limit point of $\sigma(K)$ (if any) can only be zero. Therefore, the set $S = \{\nu \neq 0 \mid \frac{1}{\nu} \in \sigma(K)\}$ is at most countable and discrete, and for all $\nu \neq 0$, $\nu \notin S$ the operator $(K - \frac{1}{\nu})$ is invertible, i.e., equation (1) is uniquely solvable. The theorem is proved.

We pass to the question about the solvability of problem (1), (2) for fixed ν. We need to study the structure of the set $E \subset \mathbb{C} \times \mathbb{R}^+$, that consists of all pairs (ν, b), such that $\nu \neq 0$ and $\frac{1}{\nu} \notin \sigma(K_b)$, where $K_b = L^{-1} \circ G$.

Theorem 4. *E is a measurable set of full measure in $\mathbb{C} \times \mathbb{R}^+$.*

For the proof, we need several auxiliary statements.

Lemma 4. *For any $\varepsilon > 0$ there exists an integer k_0 such that $||K_b - \widetilde{K_b}|| < \varepsilon$ for all $b \in A_\sigma(\frac{1}{r}), 0 < \sigma < 1$, where $r = 1, 2, \ldots$,*

$$K_b u = L_b{}^{-1} v = \sum \frac{v_{km}}{\lambda_{km}(b)} e_{km}, \quad \widetilde{K_b} u = \sum_{0<k\leq k_0} \frac{v_{km}}{\lambda_{km}(b)} e_{km}.$$

Proof. Observe that for any $\varepsilon > 0$ there is an integer k_0 such that $\frac{k^{2+2\sigma}}{(k(k+n-1))^2} \leq (\frac{\varepsilon}{rM})^2$ for all $k > k_0$, $0 < \sigma < 1$. We have

$$(K_b - \widetilde{K_b})u = K_{k_0 b} u = \sum_{k>k_0} \frac{v_{km}}{\lambda_{km}(b)} e_{km}$$

$$||(K_b - \widetilde{K_b})u||^2 = ||K_{k_0b}u||^2 = \sum_{k>k_0} |\frac{v_{km}}{\lambda_{km}(b)}|^2 \leq \sum_{k>k_0} \frac{r^2\alpha_{km}^2 k^{2+2\sigma}}{(k(k+n-1))^2} \leq$$

$$r^2(\frac{\varepsilon}{rM})^2 \sum_{k>k_0} |\alpha_{km}|^2 \leq r^2(\frac{\varepsilon}{rM})^2 M^2||u||^2 = \varepsilon^2||u||^2.$$

Thus $||K_b - \widetilde{K_b}|| = ||K_{k_0b}|| < \varepsilon$ as required.

Lemma 5. *The operator-valued function $b \to K_b$ is continuous for $b \in A_\sigma(\frac{1}{r})$.*

Proof. Suppose $b, b+\Delta b \in A_\sigma(\frac{1}{r})$ and $\varepsilon > 0$. By Lemma 4 there exists an integer k_0 (independent of $b, b+\Delta b$) such that $||K_b - \widetilde{K_b}|| = ||K_{k_0b}|| < \varepsilon$ and $||K_{b+\Delta b} - \widetilde{K_{b+\Delta b}}|| = ||K_{k_0(b+\Delta b)}|| < \varepsilon$. Next,

$$K_{b+\Delta b} - K_b = (\widetilde{K_{b+\Delta b}} + K_{k_0(b+\Delta b)}) - (\widetilde{K_b} + K_{k_0b}),$$

whence we obtain

$$||K_{b+\Delta b} - K_b|| \leq ||\widetilde{K_{b+\Delta b}} - \widetilde{K_b}|| + ||K_{k_0(b+\Delta b)}|| + ||K_{k_0b}||.$$

Consider the operators $\widetilde{K_{b+\Delta b}}, \widetilde{K_b}$. We have

$$(\widetilde{K_{b+\Delta b}} - \widetilde{K_b})u = \sum_{0<k\leq k_0} (\frac{1}{\lambda_{km}(b+\Delta b)} - \frac{1}{\lambda_{km}(b)})v_{km}e_{km}$$

$$||\widetilde{K_b}u - \widetilde{K_{b+\Delta b}}u||^2 = \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{1\leq k\leq k_0} \frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b)|^2} \frac{4m^2\pi^2}{|\lambda_{km}(b)|^2}. \tag{15}$$

If $b+\Delta b \in A_\sigma(\frac{1}{r})$, $\quad 1 \leq k \leq k_0, \quad 0 < \sigma < 1$, then

$$\frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b)|^2} \leq |v_{km}|^2 r^2 k^{2+2\sigma} \leq r^2 k_0^4 |v_{km}|^2.$$

The relation $\lim_{m\to\infty} \frac{4m^2\pi^2}{|\lambda_{km}(b)|^2} = b^2$ and the condition $1 \leq k \leq k_0$ imply that the quantity $\frac{4m^2\pi^2}{|\lambda_{km}(b)|^2} = \frac{4m^2\pi^2}{|\frac{2m\pi}{b} + ak(k+n-1)|^2}$ is dominated by a constant $C(k_0)$ depending on k_0. Therefore

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{1<k\leq k_0} \frac{|v_{km}|^2}{|\lambda_{km}(b+\Delta b)|^2} \frac{4m^2\pi^2}{|\lambda_{km}(b)|^2} \leq$$

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{1\leq k\leq k_0} r^2 k_0{}^4 C(k_0)|v_{km}|^2 \leq$$

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} r^2 k_0{}^4 C(k_0) \sum_{1\leq k\leq k_0} |v_{km}|^2.$$

Since

$$\sum_{1\leq k\leq k_0} |v_{km}|^2 \leq ||v||^2 \leq M^2||u||^2,$$

we arrive at the estimate

$$||\widetilde{K_{b+\Delta b}} - \widetilde{K_b}||^2 \leq \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} M^2 r^2 {k_0}^4 C(k_0).$$

We choose Δb so as to satisfy the condition

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} M^2 r^2 {k_0}^4 C(k_0) < \varepsilon.$$

Then $||K_{b+\Delta b} - K_b|| < 3\varepsilon$. This shows that the operator-valued function $b \to K_b$ is continuous on $A_\sigma(\frac{1}{r})$. The Lemma is proved.

Lemma 6. *The spectrum $\sigma(K)$ of the compact operator K depends continuously on K in the space $Comp(\mathcal{H}_0)$ of compact operators on $\mathcal{H}_0$, in the sense that for any ε there exists $\delta > 0$ such that for all compact (and even bounded) operators B with $\|B - K\| < \delta$ we have*

$$\sigma(B) \subset \sigma(K) + V_\varepsilon(0), \quad \sigma(K) \subset \sigma(B) + V_\varepsilon(0). \tag{16}$$

Here $V_\varepsilon(0) = \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$ is the ε-neighborhood of the point 0 in $\mathbb{C}$.

Proof. Let K be a compact operator; we fix $\varepsilon > 0$. The structure of the spectrum of a compact operator shows that there exists $\varepsilon_1 < \varepsilon/2$ such that $\varepsilon_1 \neq |\lambda|$ for all $\lambda \in \sigma(K)$. Let $S = \{\lambda_1, \ldots, \lambda_k\}$ be the set of all spectrum points λ with $|\lambda| > \varepsilon_1$ and let $V = \bigcup_{\lambda \in S \cup \{0\}} V_{\varepsilon_1}(\lambda)$. Then V is neighborhood of $\sigma(K)$ and $V \subset \sigma(K) + V_\varepsilon(0)$. By the well-known property of spectra (see, e.g.,[5], Theorem 10.20) there exists $\delta > 0$ such that $\sigma(B) \subset V$ for any bounded operator B with $\|B - K\| < \delta$. Moreover (see, e.g., [5], p.293, Exercise 20), the number $\delta > 0$ can be chosen so that $\sigma(B) \cap V_{\varepsilon_1}(\lambda) \neq \emptyset$ $\forall \lambda \in S \cup \{0\}$. Then for all bounded operators B with $\|B - K\| < \delta$ the required inclusions $\sigma(K) \subset \sigma(B) + V_{2\varepsilon_1}(0) \subset \sigma(B) + V_\varepsilon(0)$ and $\sigma(B) \subset V \subset \sigma(K) + V_\varepsilon(0)$ are fulfilled. The lemma is proved.

It is easy to deduce the following statement from Lemma 6.

Proposition 1. *The function $\rho(\lambda, K) = dist(\lambda, \sigma(K))$ is continuous on $\mathbb{C} \times Comp(\mathcal{H}_0)$.*

Proof. Suppose $\lambda \in \mathbb{C}$, $K \in Comp(\mathcal{H}_0)$ and $\varepsilon > 0$. By Lemma 6, there exists $\delta > 0$ such that for any operator H lying in the δ-neighborhood of K, $||H - K|| < \delta$, the inclusions (16) are fulfilled; these inclusions directly imply the estimate $|\rho(\lambda, K) - \rho(\lambda, H)| < \varepsilon$. Then for all $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \varepsilon$ and all H with $||H - K|| < \delta$ we have

$$|\rho(\mu, K) - \rho(\lambda, H)| \leq |\rho(\mu, K) - \rho(\lambda, K)| + |\rho(\lambda, K) - \rho(\lambda, H)| < |\mu - \lambda| + \varepsilon < 2\varepsilon,$$

Since $\varepsilon > 0$ is arbitrary, the function $\rho(\lambda, K)$ is continuous. The proposition is proved.

Combining Proposition 1 and Lemma 5 we obtain the following fact.

Corollary 1. *The function $\rho(\lambda, b) = dist(\lambda, \sigma(K_b))$ is continuous on $(\lambda, b) \in \mathbb{C} \times A_\sigma(\frac{1}{r})$.*

Now we are ready to prove Theorem 4.

Proof of Theorem 4. By Corollary 1, the function $\rho(1/\nu, b)$ is continuous with respect to the variable $(\nu, b) \in (\mathbb{C} \setminus \{0\}) \times A_\sigma(\frac{1}{r})$. Consequently, the set

$$B_r = \{(\nu, b) \quad | \quad \rho(1/\nu, b) \neq 0, \quad b \in A_\sigma(\frac{1}{r})\}$$

is measurable, and so is the set $B = \cup_r B_r$. Clearly, $B \subset E$ and $E = B \cup B_0$, where $B_0 = E \setminus B$. Obviously, B_0 lies in the set $\mathbb{C} \times (\mathbb{R}^+ \setminus A_\sigma)$ of zero measure (recall that, by Theorem 1, A_σ has full measure in $\mathbb{R}^+$). Since the Lebesgue measure is complete, B_0 is measurable. Thus, the set E is measurable, being the union of two measurable sets. Next, by Theorem 3, for $b \in A_\sigma$ the section $E^b = \{\nu \in \mathbb{C} \mid (\nu, b) \in E\}$ has full measure, because its complement $\{1/\nu \mid \nu \in \sigma(K_b)\}$ is at most countable. Therefore, the set E is of full plane Lebesgue measure. The Theorem is proved.

The following important statement is a consequence of Theorem 4.

Corollary 2. *For a.e. $\nu \in \mathbb{C}$, problem (1), (2) has a unique periodic solution with almost every period $b \in \mathbb{R}^+$.*

Proof. Since the set E is measurable and has full measure, for a.e. $\nu \in \mathbb{C}$ the section $E_\nu = \{b \in \mathbb{R}^+ \mid (\nu, b) \in E\} = \{b \in \mathbb{R}^+ \mid 1/\nu \notin \sigma(K_b)\}$ has full measure, and for such b's problem (1), (2) has unique periodic solution with period b. The Corollary is proved.

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