

Non-commutative chern characters of the C^* -algebras of the spheres

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Received 8 September 2009

Abstract. We propose in this paper the construction of non-commutative Chern characters of the C^* -algebras of spheres and quantum spheres. The final computation gives us clear relation with the ordinary $\mathbb{Z}/(2)$ -graded Chern characters of torsion or their normalizers.

Keywords: Characters of the C^* -algebras.

1. Introduction

For compact Lie groups the Chern character $ch : K^*(G) \otimes \mathbb{Q} \longrightarrow H_{DR}^*(G; \mathbb{Q})$ were constructed. In [4]-[5] we computed the non-commutative Chern characters of compact Lie group C^* -algebras and of compact quantum groups, which are also homomorphisms from quantum K -groups into entire current periodic cyclic homology of group C^* -algebras (resp., of C^* -algebra quantum groups), $ch_{C^*} : K_*(C^*(G)) \longrightarrow HE_*(C^*(G))$, (resp., $ch_{C^*} : K_*(C_\varepsilon^*(G)) \longrightarrow HE_*(C_\varepsilon^*(G))$). We obtained also the corresponding algebraic version $ch_{alg} : K_*(C^*(G)) \longrightarrow HP_*(C^*(G))$, which coincides with the Fedosov-Cuntz- Quillen formula for Chern characters [5]. When $A = C_\varepsilon^*(G)$ we first computed the K -groups of $C_\varepsilon^*(G)$ and the $HE_*(C_\varepsilon^*(G))$. Thereafter we computed the Chern character $ch_{C^*} : K_*(C_\varepsilon^*(G)) \longrightarrow HE_*(C_\varepsilon^*(G))$ as an isomorphism modulo torsions.

Using the results from [4]-[5], in this paper we compute the non-commutative Chern characters $ch_{C^*} : K_*(A) \longrightarrow HE_*(A)$, for two cases $A = C^*(S^n)$, the C^* -algebra of spheres and $A = C_\varepsilon^*(S^n)$, the C^* -algebras of quantum spheres. For compact groups $G = O(n+1)$, the Chern character $ch : K^*(S^n) \otimes \mathbb{Q} \longrightarrow H_{DR}^*(S^n; \mathbb{Q})$ of the sphere $S^n = O(n+1)/O(n)$ is an isomorphism (se. [15]). In the paper, we describe two Chern character homomorphisms

$$ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n)),$$

and

$$ch_{C^*} : K_*(C_\varepsilon^*(S^n)) \longrightarrow HE_*(C_\varepsilon^*(S^n)).$$

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Finally, we show that there is a commutative diagram

$$\begin{array}{ccc}
 K_*(C^*(S^n)) & \xrightarrow{ch_{C^*}} & HE_*(C^*(S^n)) \\
 \cong \downarrow & & \downarrow \cong \\
 K_*(\mathbb{C}(\mathcal{N}_{T_n})) & \xrightarrow{ch_{C^*}} & HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \\
 \cong \downarrow & & \downarrow \cong \\
 K^*(\mathcal{N}_{T_n}) & \xrightarrow{ch} & HE_{DR}^*(\mathcal{N}_{T_n})
 \end{array}$$

(Similarly, for $A = C_\epsilon^*(S^n)$, we have an analogous commutative diagram with $W \times S^1$ of place of $W \times S^n$), from which we deduce that ch_{C^*} is an isomorphism modulo torsions.

We now briefly review the structure of the paper. In section 1, we compute the Chern character of the C^* -algebras of spheres. The computation of Chern character of $C^*(S^n)$ is based on two crucial points:

i) Because the sphere $S^n = O(n + 1)/O(n)$ is a homogeneous space and C^* -algebra of S^n is the transformation group C^* -algebra, following J.Parker [10], we have:

$$C^*(S^n) \cong C^*((O(n)) \otimes \mathcal{K}(L^2(S^n))).$$

ii) Using the stability property theorem K_* and HE_* in [5], we reduce it to the computation of C^* -algebras of subgroup $O(n)$ in $O(n + 1)$ group.

In section 2, we compute the Chern character of C^* -algebras of quantum spheres. For quantum sphere S^n , we define the compact quantum C^* -algebras $C_\epsilon^*(S^n)$, where ϵ is a positive real number. Thereafter, we prove that:

$$C_\epsilon^*(S^n) \simeq \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt,$$

where $\mathcal{K}(H_{\omega,t})$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space $H_{\omega,t}$ and W is the Weyl of a maximal torus \mathbf{T}_n in $SO(n)$.

Similar to Section 1, we first compute the $K_*(C_\epsilon^*(S^n))$ and $HE_*(C_\epsilon^*(S^n))$, and we prove that

$$ch_{C^*} : K_*(C_\epsilon^*(S^n)) \longrightarrow HE_*(C_\epsilon^*(S^n))$$

is a isomorphism modulo torsion.

Notes on Notation: For any compact space X , we write $K^*(X)$ for the $\mathbb{Z}/(2)$ - graded topological K -theory of X . We use Swan's theorem to identify $K^*(X)$ with $\mathbb{Z}/(2)$ - graded $K^*(\mathbb{C}(X))$. For any involution Banach algebra A , $K_*(A)$, $HE_*(A)$ and $HP_*(A)$ are $\mathbb{Z}/(2)$ - graded algebraic or topological K -groups of A , entire cyclic homology, and periodic cyclic homology of A , respectively. If \mathbf{T} is a maximal torus of a compact group G , with the corresponding Weyl group W , write $\mathbb{C}(\mathbf{T})$ for the algebra of complex valued functions on \mathbf{T} . We use the standard notation from the root theory such as P, P^+ for the positive highest weights, etc... We denote by $\mathcal{N}_{\mathbf{T}}$ the normalizer of \mathbf{T} in G , by \mathbb{N} the set of natural numbers, \mathbb{R} the field of real numbers and \mathbb{C} the field of complex numbers, $\ell_A^2(\mathbb{N})$ the standard ℓ^2 space of square integrable sequences of elements from A , and finally by $C_\epsilon^*(G)$ we denote the compact quantum algebras, $C^*(G)$ the C^* -algebra of G .

2. Non-commutative Chern characters of C^* -algebras of spheres.

In this section, we compute non-commutative Chern characters of C^* -algebras of spheres. Let A be an involution Banach algebra. We construct the non-commutative Chern characters $ch_{C^*} : K_*(A) \rightarrow HE_*(A)$, and show in [4] that for C^* -algebra $C^*(G)$ of compact Lie groups G , the Chern character ch_{C^*} is an isomorphism.

Proposition 2.1. ([5], Theorem 2.6) *Let H be a separable Hilbert space and B an arbitrary Banach space. We have*

$$\begin{aligned} K_*(\mathcal{K}(H)) &\cong K_*(\mathbb{C}); \\ K_*(B \otimes \mathcal{K}(H)) &\cong K_*(B) \\ HE_*(\mathcal{K}(H)) &\cong HE_*(\mathbb{C}); \\ HE_*(B \otimes \mathcal{K}(H)) &\cong HE_*(B), \end{aligned}$$

where $\mathcal{K}(H)$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space H .

Proposition 2.2. ([5], Theorem 3.1) *Let A be an involution Banach algebra with unity. There is a Chern character homomorphism*

$$ch_{C^*} : K_*(A) \rightarrow HE_*(A).$$

Proposition 2.3. ([5], Theorem 3.2) *Let G be an compact group and \mathbf{T} a fixed maximal torus of G with Weyl $W := \mathcal{N}_{\mathbf{T}}/\mathbf{T}$. Then the Chern character $ch_{C^*} : K_*(C^*(G)) \rightarrow HE_*(C^*(G))$, is an isomorphism modulo torsions, i.e.*

$$ch_{C^*} : K_*(C^*(G)) \otimes \mathbb{C} \xrightarrow{\cong} HE_*(C^*(G)),$$

which can be identified with the classical Chern character

$$ch_{C^*} : K_*(C(\mathcal{N}_{\mathbf{T}})) \rightarrow HE_*(C(\mathcal{N}_{\mathbf{T}})),$$

that is also an isomorphic modulo torsion, i.e

$$ch : K_*(\mathcal{N}_{\mathbf{T}}) \otimes \mathbb{C} \xrightarrow{\cong} H_{DR}^*(\mathcal{N}_{\mathbf{T}}).$$

Now, for $S^n = O(n+1)/O(n)$, where $O(n), O(n+1)$ are the orthogonal matrix groups. We denote by \mathbf{T}_n a fixed maximal torus of $O(n)$ and $\mathcal{N}_{\mathbf{T}_n}$ the normalizer of \mathbf{T}_n in $O(n)$. Following Proposition 1.2, there a natural Chern character $ch_{C^*} : K_*(C(S^n)) \rightarrow HE_*(C(S^n))$. Now, we compute first $K_*(C(S^n))$ and then $HE_*(C(S^n))$ of C^* -algebra of the sphere S^n .

Proposition 2.4.

$$HE_*(C(S^n)) \cong H_{DR}^W(\mathbf{T}_n).$$

Proof. We have

$$\begin{aligned} HE_*(C(S^n)) &= HE_*(C(O(n+1)/O(n))) \\ &\cong HE_*(C^*(O(n)) \otimes \mathcal{K}(L^2(O(n+1)/O(n)))) \end{aligned}$$

(in virtue of, the $\mathcal{K}(L^2(O(n+1)/O(n)))$) is a C^* -algebra compact operators in a separable Hilbert space $L^2(O(n+1)/O(n))$

$$\begin{aligned} &\cong HE_*(C(O(n))) \quad (\text{by Proposition 1.1}) \\ &\cong HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \quad (\text{see [5]}). \end{aligned}$$

Thus, we have $HE_*(C^*(S^n)) \cong HE_*(\mathbb{C}(\mathcal{N}_{T_n}))$.

Apart from that, because $\mathbb{C}(\mathcal{N}_{T_n})$ is then commutative C^* -algebra, by a Cuntz- Quillen's result [1], we have an isomorphism

$$HP_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong H_{DR}^*(\mathcal{N}_{T_n}).$$

Moreover, by a result of Khalkhali [8],[9], we have

$$HP_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong HE_*(\mathbb{C}(\mathcal{N}_{T_n})).$$

We have, hence

$$\begin{aligned} HE_*(C^*(S^n)) &\cong HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong HP_*(\mathbb{C}(\mathcal{N}_{T_n})) \\ &\cong H_{DR}^*(\mathcal{N}_{T_n}) \cong H_{DR}^W(\mathcal{N}_{T_n}) \quad (\text{by [15]}). \end{aligned}$$

Remark 1. Because $H_{DR}^W(\mathcal{N}_{T_n})$ is the de Rham cohomology of T_n , invariant under the action of the Weyl group W , following Watanabe [15], we have a canonical isomorphism $H_{DR}^W(T_n) \cong H^*(SO(n)) = \Lambda(x_3, x_7, \dots, x_{2i+3})$, where $x_{2i+3} = \sigma^*(p_i) \in H^{2n+3}(SO(n))$ and $\sigma^* : H^*(BSO(n), R) \rightarrow H^*(SO(n), R)$ for a commutative ring R with a unit $1 \in R$, and $p_i = \sigma_i(t_1^2, t_2^2, \dots, t_i^2) \in H^*(BT_n\mathbb{Z})$ the Pontryagin classes.

Thus, we have

$$HE_*(C^*(S^n)) = \Lambda(x_3, x_7, \dots, x_{2i+3}).$$

Proposition 2.5.

$$K_*(C(S^n)) \cong K^*(\mathcal{H}_{T_n}).$$

Proof. We have

$$\begin{aligned} K_*(C(S^n)) &= K_*(C(O(n+1)/O(n))) \\ &\cong K_*(C^*(O(n)) \otimes \mathcal{K}(L^2(O(n+1)/O(n)))) \quad (\text{see [10]}) \\ &\cong K_*(C^*(O(n))) \quad (\text{by Proposition 1.1}) \\ &\cong K_*(\mathbb{C}(\mathcal{N}_{T_n})) \\ &\cong K_*(\mathcal{N}_{T_n}) \quad (\text{by Lemma 3.3, from [5]}). \end{aligned}$$

Thus, $K_*(C(S^n)) \cong K_*(\mathcal{N}_{T_n})$.

Remark 2. Following Lemma 4.2 from [5], we have

$$\begin{aligned} K_*(\mathcal{N}_{T_n}) &\cong K^*(SO(n+1))/\text{Tor} \\ &= \Lambda(\beta(\lambda_1), \dots, \beta(\lambda_{n-3}, \varepsilon_{n+1})), \end{aligned}$$

where $\beta : R(SO(n)) \rightarrow \tilde{K}^{-1}(SO(n))$ be the homomorphism of Abelian groups assigning to each representation $\rho : SO(n) \rightarrow U(n+1)$ the homotopic class $\beta(\rho) = [i_n \rho] \in [SO(n), U] = \tilde{K}^{-1}(SO(n))$, where $i_n : U(n+1) \rightarrow U$ is the canonical one, $U(n+1)$ and U by the n -th and infinite unitary groups respectively and $\varepsilon_{n+1} \in K^{-1}(SO(n+1))$. We have, finally

$$K^*(C^*(S^n)) \cong \Lambda(\beta(\lambda_1), \dots, \beta(\lambda_{n-3}, \varepsilon_{n+1})).$$

Moreover, the Chern character of $SU(n + 1)$ was computed in [14], for all $n \geq 1$. Let us recall the result. Define a function

$$\phi : \mathbb{N} \times \mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{Z},$$

given by

$$\phi(n, k, q) = \sum_{i=1}^k (-1)^{i-1} \binom{n}{k-1} i^{q-1}.$$

Theorem 2.6. *Let \mathbf{T}_n be a fixed maximal torus of $O(n)$ and T the fixed maximal torus of $SO(n)$, with Weyl groups $W := \mathcal{N}_{\mathbf{T}}/\mathbf{T}$, the Chern character of $C^*(S^n)$*

$$ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$$

is an isomorphism, given by

$$ch_{C^*}(\beta(\lambda_k)) = \sum_{i=1}^n ((-1)^{i-1} 2 / (2i - 1)!) \phi(2n + 1, k, 2i) x_{2i+1} \quad (k=1, \dots, n-1);$$

$$ch_{C^*}(\varepsilon_{n+1}) = \sum_{i=1}^n ((-1)^{i-1} 2 / (2i - 1)!) \left(\frac{1}{2^n} \sum_{i=1}^n \phi(2n + 1, k, 2i) \right) x_{2i+1}.$$

Proof. By Proposition 1.5, we have

$$K_*(C^*(S^n)) \cong K_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong K^*(\mathcal{N}_{T_n})$$

and

$$HE_*(C^*(S^n)) \cong HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \cong H_{DR}^*(\mathcal{N}_{T_n}) \quad (\text{by Proposition 1.4}).$$

Now, consider the commutative diagram

$$\begin{array}{ccc} K_*(C^*(S^n)) & \xrightarrow{ch_{C^*}} & HE_*(C^*(S^n)) \\ \cong \downarrow & & \downarrow \cong \\ K_*(\mathbb{C}(\mathcal{N}_{T_n})) & \xrightarrow{ch_{CQ}} & HE_*(\mathbb{C}(\mathcal{N}_{T_n})) \\ \cong \downarrow & & \downarrow \cong \\ K^*(\mathcal{N}_{T_n}) & \xrightarrow{ch} & H_{DR}^*(\mathcal{N}_{T_n}). \end{array}$$

Moreover, by the results of Watanabe [15], the Chern character $ch : K^*(\mathcal{N}_{T_n}) \otimes \mathbb{C} \longrightarrow H_{DR}^*(\mathcal{N}_{T_n})$ is an isomorphism

Thus, $ch_{C^*} : K_*(C^*(S^n)) \longrightarrow HE_*(C^*(S^n))$ is an isomorphism (Proposition 1.4 and 1.5), given by

$$ch_{C^*}(\beta(\lambda_k)) = \sum_{i=1}^n ((-1)^{i-1} 2 / (2i - 1)!) \phi(2n + 1, k, 2i) x_{2i+1} \quad (k=1, \dots, n-1);$$

$$ch_{C^*}(\varepsilon_{n+1}) = \sum_{i=1}^n ((-1)^{i-1} 2 / (2i - 1)!) \left(\frac{1}{2^n} \sum_{i=1}^n \phi(2n + 1, k, 2i) \right) x_{2i+1},$$

where

$$\begin{aligned} K^*(C^*(S^n)) &\cong \Lambda(\beta(\lambda_1), \dots, \beta(\lambda_{n-3}, \varepsilon_{n+1})) \\ HE_*(C^*(S^n)) &\cong \Lambda(x_3, x_7, \dots, x_{2i+3}). \end{aligned}$$

3. Non-commutative Chern characters of C^* -algebras of quantum spheres

In this section, we at first recall definition and main properties of compact quantum spheres and their representations. More precisely, for S^n , we define $C_\varepsilon^*(S^n)$, the C^* -algebras of compact quantum spheres as the C^* -completion of the $*$ -algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the C^* -norm, where $\mathcal{F}_\varepsilon(S^n)$ is the quantized Hopf subalgebra of the Hopf algebra, dual to the quantized universal enveloping algebra $U(\mathcal{G})$, generated by matrix elements of the $U(\mathcal{G})$ modules of type 1 (see [3]). We prove that

$$C_\varepsilon^*(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt,$$

where $\mathcal{K}(H_{\omega,t})$ is the elementary algebra of compact operators in a separable infinite dimensional Hilbert space $H_{\omega,t}$ and W is the Weyl group of S^n with respect to a maximal torus \mathbf{T} .

After that, we first compute the K -groups $K_*(C_\varepsilon^*(S^n))$ and the $HE_*(C_\varepsilon^*(S^n))$, respectively. Thereafter we define the Chern character of C^* -algebras quantum spheres, as a homomorphism from $K_*(C_\varepsilon^*(S^n))$ to $HE_*(C_\varepsilon^*(S^n))$, and we prove that $ch_{C^*} : K_*(C_\varepsilon^*(S^n)) \rightarrow HE_*(C_\varepsilon^*(S^n))$ is an isomorphism modulo torsion.

Let G be a complex algebraic group with Lie algebra $\mathcal{G} = \text{Lie}G$ and ε is real number, $\varepsilon \neq -1$.

Definition 3.1. ([3], Definition 13.1). *The quantized function algebra $\mathcal{F}_\varepsilon(G)$ is the subalgebra of the Hopf algebra dual to $U_\varepsilon(\mathcal{G})$, generated by the matrix elements of the finite-dimensional $U_\varepsilon(\mathcal{G})$ -modules of type 1.*

For compact quantum groups the unitary representations of $\mathcal{F}_\varepsilon(G)$ are parameterized by pairs (ω, t) , where t is an element of a fixed maximal torus of the compact real form of G and ω is a element of the Weyl group W of \mathbf{T} in G .

Let $\lambda \in P^+$, $V_\varepsilon(\lambda)$ be the irreducible $U_\varepsilon(\mathcal{G})$ -module of type 1 with the highest weight λ . Then $V_\varepsilon(\lambda)$ admits a positive definite hermitian form (\cdot, \cdot) such that $xv_1, v_2 = (v_1, x^*v_2)$ for all $v_1, v_2 \in V_\varepsilon(\lambda)$, $x \in U(\mathcal{G})$. Let $\{v_\mu^\nu\}$ be an orthogonal basis for weight space $V_\varepsilon(\lambda)_\mu$ $\mu \in P^+$. Then $\bigcup \{v_\mu^\nu\}$ is an orthogonal basis for $V_\varepsilon(\lambda)$. Let $C_{\nu,s,\mu,r}^\lambda(x) = (xv_\mu^r, v_\nu^s)$ be the associated matrix elements of $V_\varepsilon(\lambda)$. Then the matrix elements $C_{\nu,s,\mu,r}^\lambda$ (where λ runs through P^+ , while (μ, r) and (ν, s) runs independently through the index set of a basis of $V_\varepsilon(\lambda)$) form a basis of $\mathcal{F}_\varepsilon(G)$ (see [3]).

Now very irreducible $*$ -representation of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ is equivalent to a representation belonging to one of the following two families, each of which is parameterized by $S^1 = \{t \in \mathbb{C} \mid |t| = 1\}$

- i) the family of one-dimensional representations T_t
- ii) the family π_t of representations in $\ell^2(\mathbb{N})$ (see [3]).

Moreover, there exists a surjective homomorphism $\mathcal{F}_\varepsilon(G) \rightarrow \mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ induced by the natural inclusion $SL_2\mathbb{C} \hookrightarrow G$ and by composing the representation π_{-1} of $\mathcal{F}_\varepsilon(SL_2\mathbb{C})$ with this homomorphism, we obtain a representation of $\mathcal{F}_\varepsilon(G)$ in $\ell^2(\mathbb{N})$ denoted by π_{s_i} , where s_i appears in the reduced decomposition $\omega = s_{i_1} s_{i_2} \dots s_{i_k}$. More precisely, $\pi_{s_i} : \mathcal{F}_\varepsilon(G) \rightarrow \mathcal{L}(\ell^2(\mathbb{N}))$ is of class CCR (see [11]), i.e, its image is dense in the ideal of compact operators $\mathcal{L}(\ell^2(\mathbb{N}))$.

Then representation T_t is one-dimensional and is of the form

$$T_t(C_{\nu,s,\mu,r}^\lambda) = \delta_{r,s} \delta_{\mu,\nu} \exp(2\pi\sqrt{-1}\mu(x)),$$

if $t = \exp(2\pi\sqrt{-1}\mu(x) \in \mathbf{T}$, for $x \in \text{Lie}\mathbf{T}$ (see [3]).

Proposition 3.1. ([3], 13.1.7). *Every irreducible unitary representation of $\mathcal{F}_\epsilon(G)$ on a separable Hilbert space is the completion of a unitarizable highest weight representation. Moreover, two such representation are equivalent if and only if they have the same highest weight.*

Proposition 3.2. ([3],13.1.9) *Let $\omega = s_{i_1}, s_{i_2}, \dots, s_{i_k}$ be a reduced decomposition of an element ω of the Weyl group W of G . Then*

- i) *The Hilbert space tensor product $\rho_{\omega,t} = \pi_{s_{i_1}} \otimes \pi_{s_{i_2}} \otimes \dots \otimes \pi_{s_{i_k}} \otimes T_t$ is an irreducible $*$ -representation of $\mathcal{F}_\epsilon(G)$ which is associated to the Schubert cell S_ω ;*
- ii) *Up to equivalence, the representation $\rho_{\omega,t}$ does not depend on the choice of the reduced decomposition of ω ;*
- iii) *Every irreducible $*$ -representation of $\mathcal{F}_\epsilon(G)$ is equivalent to some $\rho_{\omega,t}$.*

The sphere S^n , can be realized as the orbit under the action of the compact group $SU(n+1)$ of the highest weight vector v_0 in its natural $(n+1)$ -dimensional representation V of $SU(n+1)$. If $t_{r,s}$, $0 \leq r, s \leq n$, are the matrix entries of V , the algebra of functions on the orbit is generated by the entries in the "first column" t_{s0} and their complex conjugates. In fact,

$$\mathcal{F}(S^n) := \mathbb{C}[t_{00}, \dots, t_{n0}, \bar{t}_{00}, \dots, \bar{t}_{n0}] / \sim,$$

where " \sim " is the following equivalence relation

$$t_{s0} \sim \bar{t}_{s0} \iff \sum_{s=0}^n t_{s0} \bar{t}_{s0} = 1.$$

Proposition 3.3. ([3], 13.2.6). *The $*$ -structure on Hopf algebra $\mathcal{F}_\epsilon(SL_2(\mathbb{C}))$, is given by*

$$t_{rs}^* = (-\epsilon)^{r-s} q \det(\widehat{T}_{rs}),$$

where \widehat{T}_{rs} is the matrix obtained by removing the r^{th} row and the s^{th} column from T .

Definition 3.2. ([3],13.2.7). *The $*$ -subalgebra of $\mathcal{F}_\epsilon(SL_{n+1}(\mathbb{C}))$ generated by he elements t_{s0} and t_{s0}^* , for $s = 0, \dots, n$, is called the quantized algebra of functions on the sphere S^n , and is denoted by $\mathcal{F}_\epsilon(S^n)$. It is a quantum $SL_{n+1}(\mathbb{C})$ -space.*

We set $z_s = t_{s0}$ from now on. Using Proposition 2.4, it is easy to see that the following relations hold in $\mathcal{F}_\epsilon(S^n)$:

$$\begin{cases} z_r \cdot z_s = \epsilon^{-1} z_s z_r \text{ if } r < s \\ z_r \cdot z_s^* = \epsilon^{-1} z_s^* z_r \text{ if } r \neq s \\ z_r \cdot z_r^* - z_r^* \cdot z_r + (\epsilon^{-2} + 1) \sum_{s>r} z_s \cdot z_s^* = 0 \\ \sum_{s=0}^n z_s \cdot z_s^* = 0. \end{cases} \tag{CP}$$

Hence, $\mathcal{F}_\epsilon(S^n)$ has (CP) as its defining relations. The construction of irreducible $*$ -representation of $\mathcal{F}_\epsilon(S^n)$, is given by.

Theorem 3.4. ([3],13.2.9). *Every irreducible $*$ -representation of $\mathcal{F}_\epsilon(S^n)$ is equivalent exactly to one of the following:*

- i) *the one-dimensional representation $\rho_{0,t}$ $t \in S^1$, given by $\rho_{0,t}(z_0^*) = t^{-1}$ and $\rho_{0,t}(z_r^*) = 0$ if $r > 0$,*

ii) the representation $\rho_{0,t}, 1 \leq r \leq n, t \in S^1$, on the Hilbert space tensor product $\ell^2(\mathbb{N})^{\otimes r}$, give by

$$\rho_{r,t}(z_s^*)(e_{k_1} \otimes \dots \otimes e_{k_r}) = \begin{cases} \varepsilon^{\binom{r-1}{i=1} k_i+s} (1 - \varepsilon^{-2(k_{s+1}+1)})^{-2} e_{k_1} \otimes \dots \otimes e_{k_s} \otimes e_{k_{s+1}} + 1 \otimes e_{k_{s+2}} \otimes \dots \otimes e_{k_s} & \text{if } s < r \\ t^{-1} \varepsilon^{\binom{r}{j=1} k_j+r} e_{k_1} \otimes \dots \otimes e_{k_r} & \text{if } s = r \\ 0 & \text{if } s > r. \end{cases}$$

The representation $\rho_{0,t}$ is equivalent to the restriction of the representation T_t of $\mathcal{F}_\varepsilon(SL_{n+1}(\mathbb{C}))$ (cf.2.3); and or $r > 0, \rho_{r,t}$ is equivalent to the restriction of $\pi_{s_1} \otimes \dots \otimes \pi_{s_r} \otimes T_t$.

From Theorem 2.6, we have

$$\bigcap_{(\omega,t) \in W \times T} \ker \rho_{\omega,t} = \{0\},$$

i.e. the representation $\bigoplus_{\omega \in W} \int_T^{\oplus} \rho_{\omega,t} dt$ is faithful and

$$\dim \rho_{\omega,t} = \begin{cases} 1 & \text{if } \omega = e \\ 0 & \text{if } \omega \neq e. \end{cases}$$

We recall now the definition of compact quantum of spheres C^* - algebra.

Definition 3.3. The C^* -algebraic compact quantum sphere $C_\varepsilon^*(S^n)$ is the C^* -completion of the C^* -algebra $\mathcal{F}_\varepsilon(S^n)$ with respect to the C^* -norm

$$\|f\| = \sup_{\rho} \|\rho(f)\|, \quad f \in \mathcal{F}_\varepsilon(S^n)$$

where ρ runs through the $*$ -representations of $\mathcal{F}_\varepsilon(S^n)$ (cf., Theorem 2.6) and the norm on the right-hand side is the operator.

It suffices to show that $\|f\|$ is finite for all $f \in \mathcal{F}_\varepsilon(S^n)$, for it is clear that $\|\cdot\|$ is a C^* -norm, i.e. $\|f \cdot f^*\| = \|f\|^2$. We now prove that following result about the structure of compact quantum C^* -algebra of sphere S^n .

Theorem 3.5. With notation as above, we have

$$C_\varepsilon^*(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt,$$

where $\mathbb{C}(S^1)$ is the algebra of complex valued continuous functions on S^1 and $\mathcal{K}(H)$ ideal of compact operators in a separable Hilbert space H .

Proof. Let $\omega = s_{i_1} \cdot s_{i_2} \dots s_{i_k}$ be a reduced decomposition of the element $\omega \in W$ into a product of reflections. Then by Proposition 2.6, for $r > 0$, the representation $\rho_{\omega,t}$ is equivalent to the restriction of $\pi_{s_{i_1}} \otimes \dots \otimes \pi_{s_{i_k}} \otimes T_t$, where π_{s_i} is the composition of the homomorphism of $\mathcal{F}_\varepsilon(G)$ onto $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ and the representation π_{-1} of $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ in the Hilbert space $\ell^2(\mathbb{N})^{\otimes r}$; and the family of one-dimensional representations T_t , given by

$$T_t(a) = t, T_t(b) = T_t(c) = 0, T_t(d) = t^{-1},$$

where $t \in S^1$ and a, b, c, d are give by: Algebra $\mathcal{F}_\varepsilon(SL_2(\mathbb{C}))$ is generated by the matrix elements of type $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Hence, by construction the representation $\rho_{\omega,t} = \pi_{s_{i_1}} \otimes \dots \otimes \pi_{s_{i_k}} \otimes T_t$. Thus, we have

$$\pi_{s_i} : C_\varepsilon^*(S^n) \longrightarrow C_\varepsilon^*(SL_2(\mathbb{C})) \xrightarrow{\pi_{-1}} \mathcal{L}(\ell^2(\mathbb{N})^{\otimes r}).$$

Now, π_{s_i} is CCR (see, [11]) and so, we have $\pi_{s_i}(C_\epsilon^*(S^n) \cong \mathcal{K}(H_{\omega,t})$. Moreover $T_t(C_\epsilon^*(S^n)) \cong \mathbb{C}$.
Hence,

$$\begin{aligned} \rho_{\omega,t}(C_\epsilon^*(S^n)) &= (\pi_{s_{i_1}} \otimes \dots \otimes \pi_{s_{i_k}} \otimes T_t)(C_\epsilon^*(S^n)) \\ &= \pi_{s_{i_1}}(C_\epsilon^*(S^n)) \otimes \dots \otimes \pi_{s_{i_k}}(C_\epsilon^*(S^n)) \otimes T_t(C_\epsilon^*(S^n)) \\ &\cong \mathcal{K}(H_{s_{i_1}}) \otimes \dots \otimes \mathcal{K}(H_{s_{i_k}}) \otimes \mathbb{C} \\ &\cong \mathcal{K}(H_{\omega,t}), \end{aligned}$$

where $H_{\omega,t} = H_{s_1} \otimes \dots \otimes H_{s_i} \otimes \mathbb{C}$.

Thus, $\rho_{\omega,t}(C_\epsilon^*(S^n)) = \mathcal{K}(H_{\omega,t})$.

Hence,

$$\bigoplus_{\omega \in W} \int_{S^1}^{\oplus} \rho_{\omega,t}(C_\epsilon^*(S^n)) \cong \bigoplus_{\omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt.$$

Now, recall a result of S. Sakai from [11]: Let A be a commutative C^* -algebra and B be a C^* -algebra. Then, $C_0(\Omega, B) \cong A \otimes B$, where Ω is the spectrum space of A .

Applying this result, for $B = \mathcal{K}(H_{\omega,t}) \cong \mathcal{K}$ and $A = \mathbb{C}(W \times S^1)$ be a commutative C^* -algebra. Thus, we have

$$C_\epsilon^*(S^n) \cong \mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt.$$

Now, we first compute the $K_*(C_\epsilon^*(S^n))$ and the $HE_*(C_\epsilon^*(S^n))$ of C^* -algebra of quantum sphere S^n .

Proposition 3.6.

$$HE_*(C_\epsilon^*(S^n)) \cong H_{DR}^*(W \times S^1).$$

Proof. We have

$$\begin{aligned} HE_*(C_\epsilon^*(S^n)) &= HE_*(\mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt) \\ &= HE_*(\mathbb{C}(S^1) \oplus HE_*(\bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt)) \\ &\cong HE_*(\mathbb{C}(W \times S^1) \otimes \mathcal{K}) \quad (\text{by Proposition 1.1}) \\ &\cong HE_*(\mathbb{C}(W \times S^1)). \end{aligned}$$

Since $C(W \times S^1)$ is a commutative C^* -algebra, by Proposition 1.5 §1, we have

$$HE_*(C_\epsilon^*(S^n)) \cong HE_*(\mathbb{C}(W \times S^1)) \cong H_{DR}^*(W \times S^1):$$

Proposition 3.7.

$$K_*(C_\epsilon^*(S^n)) \cong K^*(W \times S^1).$$

Proof. We have

$$\begin{aligned} K_*(C_\epsilon^*(S^n)) &= K_*(\mathbb{C}(S^1) \oplus \bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt) \\ &= K_*(\mathbb{C}(S^1) \oplus K_*(\bigoplus_{e \neq \omega \in W} \int_{S^1}^{\oplus} \mathcal{K}(H_{\omega,t}) dt)) \\ &\cong K_*(\mathbb{C}(W \times S^1) \otimes \mathcal{K}) \quad (\text{by Proposition 1.1}) \\ &\cong K_*(\mathbb{C}(W \times S^1)). \end{aligned}$$

In result of Proposition 1.5, §1, we have

$$K_*(\mathbb{C}(W \times S^1)) \cong K_*(W \times S^1).$$

Theorem 3.8. *With notation above, the Chern character of C^* -algebra of quantum sphere $C^*_{\epsilon}(S^n)$*

$$ch_{C^*} : K_*(C^*_{\epsilon}(S^n)) \longrightarrow HE_*(C^*_{\epsilon}(S^n))$$

is an isomorphism.

Proof. By Proposition 2.9 and 2.10, we have

$$\begin{aligned} HE_*(C_\epsilon^*(S^n)) &\cong HE_*(\mathbb{C}(W \times S^1)) \cong H_{DR}^*(W \times S^1), \\ K_*(C_\epsilon^*(S^n)) &\cong K_*(\mathbb{C}(W \times S^1)) \cong K^*(W \times S^1). \end{aligned}$$

Now, consider the commutative diagram

$$\begin{array}{ccc} K_*(C_\epsilon^*(S^n)) & \xrightarrow{ch_{C_\epsilon^*}} & HE_*(C_\epsilon^*(S^n)) \\ \cong \downarrow & & \downarrow \cong \\ K_*(\mathbb{C}(W \times S^1)) & \xrightarrow{ch_{C^*}} & HE_*(\mathbb{C}(W \times S^1)) \\ \cong \downarrow & & \downarrow \cong \\ K^*(W \times S^1) & \xrightarrow{ch} & H_{DR}^*(W \times S^1). \end{array}$$

Moreover, following Watanabe [15], the $ch : K^*(W \times S^1) \otimes \mathbb{C} \longrightarrow H_{DR}^*(W \times S^1)$ is an isomorphism.

Thus, $ch_{C^*} : K_*(C^*_{\epsilon}(S^n)) \longrightarrow HE_*(C^*_{\epsilon}(S^n))$ is an isomorphism.

Acknowledgment. The author would like to thank Professor Do Ngoc Diep for his guidance and encouragement during this paper.

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