VNU Journal of Science, Mathematics - Physics 26 (2010) 17-27

Periodic solutions of some linear evolution systems of natural differential equations on 2-dimensional tore

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Received 3 December 2009

Abstract. In this paper we study periodic solutions of the equation

$$\frac{1}{i}\left(\frac{\partial}{\partial t}+aA\right)u(x,t)=\nu G(u-f), \tag{1}$$

with conditions

$$u_{t=0}=u_{t=b}, \quad \int_X (u(x),1)\,dx=0 \tag{2}$$

over a Riemannian manifold X, where

$$Gu(x,t)=\int_X g(x,y)u(y)dy$$

is an integral operator, $u(x,t)$ is a differential form on X, $A=i(d+\delta)$ is a natural differential operator in X. We consider the case when X is a tore Π^2. It is shown that the set of parameters (ν,b) for which the above problem admits a unique solution is a measurable set of complete measure in $\mathbb{C}\times\mathbb{R}^+$.

Keyworks and phrases: Natural differential operators, small denominators, spectrum of compact operators.

1. Introduction

Beside authors, as from A.A. Dezin (see, [1]), considered the linear differential equations on manifolds in which includes the external differential operators.

At research of such equations appear so named the small denominators, so such equations is incorrect in the classical space.

There is extensive literature on the different types of the equations, in which appear small denominators. We shall note, in particular, work of B.I. Ptashnika. (see, [2])

This work further develops part of the authors' result in [3], on the problem on the periodic solution, to the equation in the space of the smooth functions on the multidimensional tore Π^n. We shall consider one private event, when the considered manifold is 2-dimension tore Π^2 and the considered space is space of the smooth differential forms on Π^2.

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We shall note that X-n-dimension Riemannian manifold of the class C^∞ is always expected oriented and close. Let

$$\xi = \oplus_{p=0}^{n} \xi^p = \oplus_{p=0}^{n} \Lambda^p (T^* X) \otimes C$$

is the complexified cotangent bundle of manifolds X, $C^\infty(\xi)$ is the space of smooth differential forms and $H^k(\xi)$ is the Sobolev space of differential forms over X (see, [4]). By A we denote operator $i(d+\delta)$, so-called natural differential operator on manifold X, where d is the exterior differential operator and $\delta = d^*$- his formally relative to the scalar product on $C^\infty(\xi)$, that inducing by Riemannian structure on X. It is well known (see, [4], [5]) that $d+\delta$ is an elliptical differential first-order operator on X.

From the main result of the elliptical operator theories on close manifolds (see, [4]) there will be a following theorem.

Theorem 1. *In the Hilbert space $H^0(\xi)$ there is an orthonorm basis of eigenvector $\{f_m\}, m \in \mathbb{Z}$, of the operator $A = i(d+\delta)$ that correspond to the eigenvalues λ_m. Else $\lambda_m = i\mu_m$, $\mu_m \in \mathbb{R}$, $\lambda_{-m} = -\lambda_m$ and $\dfrac{1}{|\lambda_m|} \to 0$ when $m \to \infty$.*

Proof. This theorem was in [5].

The change of variable $t = b\tau$ reduces our problem to a problem with a fixed period, but with a new equation in which the coefficient of the τ-derivative is equal to $1/b$:

$$(\frac{\partial}{ib\partial\tau} + a(d+\delta))u(x, b\tau) = \nu G(u(x, b\tau) - f(x, b\tau))$$

2. Thus, in $\Pi^2 = \mathbb{R}^2/(2\mathbb{Z})^2$ problem (1)(2) turns into the problem on periodic solution of the equation

$$Lu \equiv (\frac{\partial}{ib\partial t} + a(d+\delta))u(x,t) = \nu G(u(x,t) - f(x,t)) \tag{3}$$

with the following conditions:

$$u|_{t=0} = u|_{t=1}, \quad \int_{\Pi^2} (u(x), 1) dx = 0. \tag{4}$$

Here

$$u(x,t) = \begin{pmatrix} 1 & dx^1 & dx^2 & dx^1 \wedge dx^2 \end{pmatrix} \begin{pmatrix} u_0(x,t) \\ u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix} \equiv \begin{pmatrix} u_0(x,t) \\ u_1(x,t) \\ u_2(x,t) \\ u_3(x,t) \end{pmatrix}$$

- complex form with coefficient dependent for t, $t \in [0,1]$; $a \neq 0, \nu$ are given numbers,

$$(u(x), v(x)) = u_0(x)\overline{v_0(x)} + u_1(x)\overline{v_1(x)} + u_2(x)\overline{v_2(x)} + u_3(x)\overline{v_3(x)}.$$

$$Gu(x,t) = \int_{\Pi^2} g(x,y) u(y,t) dy$$

is an integral operator on the space $L_2 = L_2(H^0(\xi), [0,1])$ with a smooth kernel

$$g(x,y) = (g_{ij}(x,y)), \quad i,j = \overline{0,3}$$

defined on $\Pi^2 \times \Pi^2$ such that

$$\int_{\Pi^2} \begin{pmatrix} g_{00}(x,y) & g_{01}(x,y) & g_{02}(x,y) & g_{03}(x,y) \end{pmatrix} dx = 0 \quad \forall y \in \Pi^2.$$

We assume that operator $\frac{1}{i}(\frac{\partial}{b\partial t} + aA) = \frac{1}{ib}\frac{\partial}{\partial t} + a(d+\delta)$ given in the differential form space $u(x,t) \in C^\infty(C^\infty(\xi), [0,1])$, with these conditions

$$u|_{t=0} = u|_{t=1}, \quad \int_{\Pi^2} (u(x), 1)dx = 0.$$

Let L -denote the closure of operation $\frac{1}{ib}\frac{\partial}{\partial t} + a(d+\delta)$ in $L_2(H^0(\xi), [0,1])$. So, an element $u \in L_2(H^0(\xi), [0,1])$ belongs to the domain $\mathcal{D}(L)$ of operator $L = \frac{1}{i}(\frac{\partial}{b\partial t} + aA)$, if and only if there is a sequence $\{u_j\} \subset C^\infty(C^\infty(\xi), [0,1])$ $u_j|_{t=0} = u_j|_{t=1}$, $\int_{\Pi^2}(u_j(x), 1)\,dx = 0$ such that $\lim u_j = u$, $\lim Lu_j = Lu$ in $L_2(H^0(\xi), [0,1])$.

Let $\mathcal{H}$-denote a subspace of space $L_2(H^0(\xi), [0,1])$

$$\mathcal{H} = \{u(x,t) \in L_2(H^0(\xi), [0,1]) \mid \int_{\Pi^2} (u(x,t), 1)\,dx = 0\}$$

We note that

$$\{\pm i\pi\sqrt{k_1^2 + k_2^2} = \pm i\pi|k|; k = (k_1, k_2) \in \mathbb{Z}^2\}$$

is the set of eigenvalue of operator $A = i(d+\delta)$ on Π^2 and eigenvectors, coresponding to

$$i\pi\eta_2\sqrt{k_1^2 + k_2^2},$$

are given by the formula:

$$f_{k\eta}(x) = e^{i\pi(k_1x^1 + k_2x^2)}\omega_{k\eta},$$

here $\omega_{k\eta} \in \bigoplus_{p=0}^2 \bigwedge^p(\mathbb{C}^2)$, $\eta = (\eta_1, \eta_2) \in \{-1, +1\}^2$ is some basic in 4− dimensional space of the complex differential forms with coefficients being constant. These coefficients depend on $k \in \mathbb{Z}^2$ and elements of this basic are numbered by parameters η. We are not show $\omega_{k\eta}$ on concrete form. (see, [6]).

Lemma 1. *The forms $e_{km\eta} = e^{i2\pi mt} f_{k\eta}(x)$, $k = (k_1, k_2) \neq 0$, are eigenform operator L that corresponds to the eigenvalues*

$$\lambda_{km\eta} = \pi\left(\frac{2m}{b} + a|k|\eta_2\right) = \frac{2m\pi}{b} + \lambda_{k\eta} \tag{5}$$

in the space $\mathcal{H}$. These forms form an orthonorm basis in given space. The domain of operator L is given by formula

$$\mathcal{D}(L) = \{\, u = \sum_{k\neq 0} u_{km\eta}e_{km\eta} \mid \sum_{k\neq 0} |\lambda_{km\eta}u_{km\eta}|^2 < \infty, \sum_{k\neq 0} |u_{km\eta}|^2 < \infty \,\}.$$

The spectrum $\sigma(L)$ operator L is the closure of the set $\Lambda = \{\lambda_{km\eta}\}$.

We note that the number of dimensions of the eigensubspace is finite and we shall not indicate exactly how many there are of them.

Lemma 2. *Let $g(x,y) \in L_2(\Pi^2 \times \Pi^2)$ and*

$$M_0 = \left(\int_{\Pi^2}\int_{\Pi^2} \|g(x,y)\|^2 dx\,dy\right)^{1/2}.$$

Then G - linear operator is bounded in $H^0(\xi)$ and his norm $||G|| \leq M_0$.

Here $\|g(x,y)\|$ - matric norm g

$$g(x,y) = (g_{ij}(x,y)), \quad i,j = \overline{0,3}$$

$||g|| = \sup\{\ ||gu||\ |\ u \in \mathbb{R}^2 \times \mathbb{R}^2,\ ||u|| \leq 1\ \}.$

Proof. If $u(x) \in H^0(\xi)$

$$||Gu(x)||^2 = ||\int_{\Pi^2} g(x,y)u(y)dy||^2 \leq \left(\int_{\Pi^2} ||g(x,y)u(y)||dy\right)^2 \leq$$

$$\left(\int_{\Pi^2} ||g(x,y)|| \cdot ||u(y)||dy\right)^2 \leq \int_{\Pi^2} ||g(x,y)||^2 dy \cdot \int_{\Pi^2} ||u(y)||^2 dy,$$

$$||Gu||^2 = \int_{\Pi^2} ||Gu(x)||^2 dx \leq$$

$$\int_{\Pi^2} \left(\int_{\Pi^2} ||g(x,y)||^2 dy \int_{\Pi^2} ||u(y)||^2 dy\right) dx,$$

$$||Gu||^2 \leq \int_{\Pi^2}\int_{\Pi^2} ||g(x,y)||^2 dxdy \int_{\Pi^2} ||u(y)||^2 dy = M_0^2 ||u||^2,$$

$$||G|| \leq M_0.$$

The lemma is proved.

Let $\widetilde{\mathcal{B}} = (-\Delta_x)^{\alpha+1}, \alpha > 0$. Then $\widetilde{\mathcal{B}}$ is M-operator in $H^0(\xi)$ and $\widetilde{\mathcal{B}} f_{k\eta} = \mu_k f_{k\eta}$, here $\mu_k = (\pi|k|)^{2+2\alpha}$ are eigenvalues. Operator $(-\Delta_x)^{\alpha+1}$ is self-conjugate. We suppose that kernel $g(x,y)$ of operator G having the following behaviour $(-\Delta_x)^{\alpha+1} g_{ij}(x,y) \in L_2(\Pi^2 \times \Pi^2)$ ($g_{ij}(.,y)$ belongs to space Sobolev $W_2^{2+2\alpha}$ for almost every $y \in \Pi^2$). Then product operator $\widetilde{\mathcal{B}} \circ G$ is integral operator $(-\Delta_x)^{1+\alpha} \circ G_{xx}$ with kernel

$$(-\Delta_x)^{1+\alpha} g(x,y) = ((-\Delta_x)^{1+\alpha} g_{ij})(x,y)),\ i,j = \overline{0,3}.$$

Let $M = \max\{||(-\Delta_x)^{1+\alpha} \circ G||, ||G||\}$.

Lemma 3. *Let $v = Gu = \sum v_{km\eta} e_{km\eta}$, then*

$$|v_{km\eta}|^2 \leq \frac{4M^2 ||u||^2}{((\pi|k|)^{2+2\alpha} + 1)^2}. \tag{6}$$

If $k \neq 0$ then

$$|v_{km\eta}|^2 \leq \frac{|\alpha_{km\eta}|^2}{((\pi|k|)^{2+2\alpha} + 1)^2},$$

here

$$\alpha_{km\eta} = \langle (-\Delta_x)^{1+\alpha} \circ Gu, e_{km\eta} \rangle_{L_2}.$$

Proof. We have

$$\alpha_{km\eta} = \langle (-\Delta_x)^{1+\alpha} \circ Gu, e_{km\eta} \rangle_{L_2} = \int_0^1 \langle (-\Delta_x)^{1+\alpha} \circ Gu; e_{km\eta} \rangle dt = \int_0^1 \langle Gu, (-\Delta_x)^{1+\alpha} e_{km\eta} \rangle dt =$$

$$\int_0^1 \langle Gu, \mu_k e_{km\eta} \rangle dt = \overline{\mu_k} \int_0^1 \langle Gu, e_{k_1,\eta_l} \rangle dt = \overline{\mu_k} \langle Gu, e_{km\eta} \rangle_{L_2} = \overline{\mu_k} v_{km\eta}.$$

Then, if $\mu_k \neq 0$ (so that $|\mu_k| \geq 1$) we have

$$|v_{km\eta}|^2 \leq \frac{4|\alpha_{km\eta}|^2}{(|\mu_k| + 1)^2}.$$

Thus, by Parseval dentity

$$\sum |\alpha_{km\eta}|^2 = ||(-\Delta_x)^{2+2\alpha} \circ Gu||^2 \leq M^2 ||u||^2.$$

So that

$$|v_{km\eta}|^2 \leq \frac{4M^2\|u\|^2}{(|\mu_k|+1)^2}.$$

In the case $\mu_k = 0$ by Parseval dentity $\sum |v_{km\eta}|^2 = \|Gu\|^2$ we have

$$|v_{km\eta}|^2 \leq \|G\|^2\|u\|^2 \leq 4\|G\|^2\|u\|^2 \leq \frac{4M^2\|u\|^2}{(|\mu_k|+1)^2}.$$

The lemma is proved.

We assume that a is real number. Then by Lemma 1, the spectrum $\sigma(L)$ lies on the real axis. The most typical and interesting is the case where the number $ab/2$ and $(ab/2)^2$ are irrational. In this case, $0 \neq \lambda_{km\eta}$ $\forall m \in \mathbb{Z}, k \in \mathbb{Z}^2, k \neq 0$ and the H.Weyl theorem (see, e.g., [7]) says that, the set of the numbers $\lambda_{km\eta}$ is everywhere dense on $\mathbb{R}$ and $\sigma(L) = \mathbb{R}$. Then in the subspace $\mathcal{H}$ the inverse operator L^{-1} is well defined , but unbounded. The expression for this inverse operator involves small denominators [8].

$$L^{-1}v(x,t) = \sum \frac{v_{km\eta}}{\lambda_{km\eta}} e_{km\eta}, \tag{7}$$

where the $v_{km\eta}$ are the Fourier coefficient of the series

$$v(x,t) = \sum_{m\in\ ,k\in\ ^2 k\neq 0} v_{km\eta}e_{km\eta}.$$

For positive numbers C, σ let $A_\sigma(C)$ denote the set of all positive b such that

$$|\lambda_{km\eta}| \geq \frac{C}{|k|^{1+\sigma}}. \tag{8}$$

for all $m \in \mathbb{Z}, k \in \mathbb{Z}^2, \eta = (\eta_1, \eta_2), \eta_{1,2} = \pm 1, k \neq 0$.

From the definition it follows that the set $A_\sigma(C)$ extends as C reduces and as σ grows. Therefore, in what follows, to prove that such a set or its part is nonempty, we require that $C > 0$ be sufficiently small and σ sufficiently large. Let A_σ denote the union of the sets $A_\sigma(C)$ over all $C > 0$. If inequality (8) is fulfilled for some b and all m, k, then it is fulfilled for $m = 0$; this provides a condition necessary for the nonemptiness of $A_\sigma(C)$:

$$C \leq |k|^{1+\sigma}|a\pi|k|| \;\forall\, k \neq 0. \tag{9}$$

We put $d = |a|\pi$ and $C \leq d/2$.

Theorem 2. *The sets $A_\sigma(C)$, A_σ are Borel. The set A_σ has complete measure, i.e., its complement to the half-line $\mathbb{R}^+$ is of zero measure.*

Proof. Obviously, the sets $A_\sigma(C)$ are closed in $\mathbb{R}^+$. The set $A_\sigma = \bigcup_{r=1}^{\infty} A_\sigma(1/r)$ - is Borel, being a countable union of closed sets. We show that A_σ has complete measure in $\mathbb{R}^+$, Suppose b, $l > 0$, $C \leq \frac{d}{2}$; we consider the complement $(0,l)\backslash A_\sigma(C)$. This set consists of all positive numbers b, for which there exist $m, k, k \neq 0$, such that

$$|\lambda_{km\eta}| < \frac{C}{|k|^{1+\sigma}}. \tag{10}$$

Solving this inequality for b, we see that, for $m, k, k \neq 0$ fixed, the number b forms an interval $I_{k,m} = (m\alpha_k, m\beta_k)$, where $m = 1, 2, 3, ...,$

$$\alpha_k = \frac{2\pi}{|a\pi|k|| + \dfrac{C}{|k|^{1+\sigma}}},\ \beta_k = \frac{2\pi}{|a\pi|k|| - \dfrac{C}{|k|^{1+\sigma}}}.$$

The length of $I_{k,m}$ is $m\delta_k$, with

$$\delta_k = \frac{4\pi C|k|^{-1-\sigma}}{|a\pi|k||^2 - C^2|k|^{-2-2\sigma}}.$$

Since $C \leq \dfrac{d}{2}$ by assumption, we have

$$\delta_k \leq \frac{16\pi C}{3|k|^{1+\sigma}|a\pi|k||^2}. \tag{11}$$

For k fixed and m varying, there is only a finite amount of intervals I_{km} that intersect the given segment $(0, l)$. Such intervals arise for the values of $m = 1, 2...,$ satisfying $m\alpha_k < l$, i.e.,

$$0 < m < \frac{l}{2\pi}(|a\pi|k|| + C|k|^{-1-\sigma}).$$

Since $C|k|^{-1-\sigma} \leq \dfrac{1}{2}|a\pi|k||$, we can write simpler restrictions on m :

$$0 < m < \frac{l}{2\pi}\frac{3}{2}|a\pi|k|| < \frac{l}{\pi}|a\pi|k||. \tag{12}$$

The measure of the intervals indicated (for $k \neq 0$ fixed) is dominated by $\delta_k\tilde{S}_k$, where $\tilde{S}_k = \tilde{S}_k(l)$ is the sum of all integers m satisfying (12). Summing an arithmetic progression, we obtain

$$\tilde{S}_k \leq \frac{l}{2\pi^2}|a\pi|k||\{l|a\pi|k|| + \pi\}. \tag{13}$$

Considering the union of the intervals in question over k and m, and using (11), we see that

$$\mu((0,l)\backslash A_\sigma(C)) \leq \sum_{k\neq 0,\, k\in {}^2} \delta_k\tilde{S}_k \leq CS(l),$$

where

$$S = S(l) = \sum_{k\neq 0,\, k\in {}^2} \frac{8l\{l|a\pi|k|| + \pi\}}{3\pi|k|^{1+\sigma}|a\pi|k||}.$$

Observe that the quantity

$$\frac{l|a\pi|k|| + \pi}{\pi|a\pi|k||}$$

is dominated by a constant D, therefore, (since $\sigma > 0$)

$$S(l) \leq \frac{8}{3}lD \sum_{k\neq 0,\, k\in {}^2} \frac{1}{|k|^{1+\sigma}} < \infty.$$

We have

$$\mu((0,l) \setminus A_\sigma) \leq \mu((0,l) \setminus A_\sigma(C)) \leq CS(l) \quad \forall C > 0.$$

It follows that $\mu((0,l) \setminus A_\sigma) = 0 \quad \forall l > 0$. Thus, $\mu((0,\infty) \setminus A_\sigma) = 0$ and A_σ- has complete measure. The theorem is proved.

Theorem 3. *Suppose $g(x,y) \in L_2(\Pi^2 \times \Pi^2)$ such that $(-\Delta_x)^{1+\alpha} g(x,y)$ is continuous on $\Pi^2 \times \Pi^2$ and*

$$\int_{\Pi^2} (\ g_{00}(x,y) \quad g_{01}(x,y) \quad g_{02}(x,y) \quad g_{03}(x,y)\ )\, dx = 0 \quad \forall y \in \Pi^2.$$

Let $0 < \sigma < 1$, and let $b \in A_\sigma(C)$. Then in the space $\mathcal{H}$ the inverse operator L^{-1} is well defined, and the operator $L^{-1} \circ G$ is compact.

Proof. Since $b \in A_\sigma(C)$, we have $\lambda_{km\eta} \neq 0 \quad \forall\, m \in \mathbb{Z}, k \in \mathbb{Z}^2,\ k \neq 0$ so that in the space $\mathcal{H}$, L^{-1} is well defined and looks like the expression in (7). Observe that $\lim \dfrac{|k|^{2+2\sigma}}{((\pi|k|)^{2+2\alpha}+1)^2} = 0$ as $|k| \to \infty$ because $0 < \sigma < 1, \alpha > 0$. Therefore, given $\varepsilon > 0$, we can find an integer $k_0 > 0$, such that $\dfrac{|k|^{2+2\sigma}}{((\pi|k|)^{2+2\alpha}+1)^2} < \dfrac{(\varepsilon C)^2}{M^2}$ for all $|k| \geq k_0$. We write

$$L^{-1}v(x,t) = Q_{k_{01}}v + Q_{k_{02}}v, \qquad v = Gu,$$

where

$$Q_{k_{01}}v = \sum_{0<|k|<k_0} \frac{v_{km\eta}}{\lambda_{km\eta}} e_{km\eta}, \quad Q_{k_{02}}v = \sum_{|k|\geq k_0} \frac{v_{km\eta}}{\lambda_{km\eta}} e_{km\eta}.$$

For the operator $Q_{k_{01}}$ we have

$$||Q_{k_{01}}v||^2 = \sum_{0<|k|<k_0} \frac{|v_{km\eta}|^2}{|\lambda_{km\eta}|^2}.$$

Observe that if $0 < |k| < k_0$, then

$$\lim_{|m|\to\infty} \frac{1}{|\frac{2m\pi}{b} + a\pi|k|\eta_2|^2} = 0.$$

Therefore, the quantity $\dfrac{1}{|\frac{2m\pi}{b} + a\pi|k|\eta_2|^2}$ is dominated by a constant $C(k_0)$. Then

$$||Q_{k_{01}}v||^2 \leq \sum |v_{km\eta}|^2 C(k_0) \leq C(k_0)||v||^2,$$

which means that $Q_{k_{01}}$ is a bounded operator.

Consider the operator $Q_{k_{02}} \circ G$. By Lemma 3 and (8), we have

$$||Q_{k_{02}}v||^2 = ||Q_{k_{02}} \circ Gu||^2 = \sum_{|k|\geq k_0} \frac{|v_{km\eta}|^2}{|\lambda_{km\eta}|^2} \leq$$

$$\sum_{|k|\geq k_0} \frac{|\alpha_{km\eta}|^2}{((\pi|k|)^{2+2\alpha}+1)^2} (\frac{1}{C})^2 |k|^{2+2\sigma} \leq (\frac{1}{C})^2 (\frac{\varepsilon C}{M})^2 \sum_{|k|\geq k_0} |\alpha_{km\eta}|^2 \leq \varepsilon^2 ||u||^2.$$

Consequently, $||Q_{k_{02}} \circ G|| \leq \varepsilon$.

Since G is compact and $Q_{k_{01}}$ is bounded, $Q_{k_{01}} \circ G$ is compact. Next, we have

$$||L^{-1} \circ G - Q_{k_{01}} \circ G|| = ||Q_{k_{02}} \circ G|| \leq \varepsilon.$$

Thus, we see that the operator $L^{-1} \circ G$ is the limit of sequence of compact operators. Therefore, it is compact itself. The theorem is proved. We denote $K = K_b = L^{-1} \circ G$.

Theorem 4. *Suppose $b \in A_\sigma(C)$. Then problem (1)(2) has the unique periodic solution with period b for all $\nu \in \mathbb{C}$, except, possibly, an at most countable discrete set of values of ν.*

Proof. Equation (3) reduces to

$$(L^{-1} \circ G - \frac{1}{\nu})u = L^{-1} \circ G(f).$$

We write $L^{-1} \circ G - \frac{1}{\nu} = K - \frac{1}{\nu}$.

Since $K = L^{-1} \circ G$ is a compact operator, its spectrum $\sigma(K)$ is at most countable, and the limit point of $\sigma(K)$ (if any) can only be zero. Therefore, the set $S = \{\nu \neq 0 \mid \frac{1}{\nu} \in \sigma(K)\}$ is at most countable and discrete, and for all $\nu \neq 0,\ \nu \notin S$ the operator $(K - \frac{1}{\nu})$ is invertible, i.e., equation (3) is uniquely solvable. The theorem is proved.

We pass to the question about the solvability of problem (1)(2) for fixed ν. We need to study the structure of the set $E \subset \mathbb{C} \times \mathbb{R}^+$, that consists of all pairs (ν, b), such that $\nu \neq 0$ and $\frac{1}{\nu} \notin \sigma(K_b)$, where $K_b = L^{-1} \circ G$.

Theorem 5. *E is a measurable set of complete measure in $\mathbb{C} \times \mathbb{R}^+$.*

For the proof, we need several auxiliary statements.

Lemma 4. *For any $\varepsilon > 0$ there exists an integer k_0 such that $||K_b - \widetilde{K}_b|| < \varepsilon$ for all $b \in A_\sigma(\frac{1}{r})$, $0 < \sigma < 1$, where $r = 1, 2, ...;$*

$$K_b u = L_b{}^{-1} v = \sum \frac{v_{km\eta}}{\lambda_{km\eta}(b)} e_{km\eta}, \quad \widetilde{K}_b u = \sum_{0<|k|<k_0} \frac{v_{km\eta}}{\lambda_{km\eta}(b)} e_{km\eta}.$$

Proof. Observe that for any $\varepsilon > 0$ there is an integer k_0 such that $\frac{|k|^{2+2\sigma}}{((\pi|k|)^{2+2\alpha}+1)^2} \leq (\frac{\varepsilon}{rM})^2 < 1$ for all $|k| \geq k_0,\ 0 < \sigma < 1, \alpha > 0$. We have

$$(K_b - \widetilde{K}_b)u = K_{k_0 b} u = \sum_{|k|\geq k_0} \frac{v_{km\eta}}{\lambda_{km\eta}(b)} e_{km\eta}$$

$$||(K_b - \widetilde{K}_b)u||^2 = ||K_{k_0 b} u||^2 = \sum_{|k|\geq k_0} |\frac{v_{km\eta}}{\lambda_{km\eta}(b)}|^2 \leq \sum_{|k|\geq k_0} \frac{r^2 \alpha^2_{km\eta} |k|^{2+2\sigma}}{((\pi|k|)^{2+2\alpha}+1)^2} \leq$$

$$r^2 (\frac{\varepsilon}{rM})^2 \sum_{|k|\geq k_0} |\alpha_{km\eta}|^2 \leq r^2 (\frac{\varepsilon}{rM})^2 M^2 ||u||^2 = \varepsilon^2 ||u||^2.$$

Thus $||K_b - \widetilde{K}_b|| = ||K_{k_0 b}|| < \varepsilon$ as required.

Lemma 5. *The operator-valued function $b \to K_b$ is continuous for $b \in A_\sigma(\frac{1}{r})$.*

Proof. Suppose $b, b + \Delta b \in A_\sigma(\frac{1}{r})$ and $\varepsilon > 0$. By Lemma 4 there exists an integer k_0 (independent of $b, b + \Delta b$) such that $||K_b - \widetilde{K}_b|| = ||K_{k_0 b}|| < \varepsilon$ and $||K_{b+\Delta b} - \widetilde{K}_{b+\Delta b}|| = ||K_{k_0(b+\Delta b)}|| < \varepsilon$. Next,

$$K_{b+\Delta b} - K_b = (\widetilde{K}_{b+\Delta b} + K_{k_0(b+\Delta b)}) - (\widetilde{K}_b + K_{k_0 b}),$$

whence we obtain

$$||K_{b+\Delta b} - K_b|| \leq ||\widetilde{K}_{b+\Delta b} - \widetilde{K}_b|| + ||K_{k_0(b+\Delta b)}|| + ||K_{k_0 b}||.$$

Considering the operators $\widetilde{K}_{b+\Delta b}, \widetilde{K}_b$, we have

$$(\widetilde{K}_{b+\Delta b} - \widetilde{K}_b)u = \sum_{0<|k|<k_0} \left(\frac{1}{\lambda_{km\eta}(b+\Delta b)} - \frac{1}{\lambda_{km\eta}(b)}\right) v_{km\eta} e_{km\eta}$$

$$||\widetilde{K}_b u - \widetilde{K}_{b+\Delta b} u||^2 = \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{0<|k|<k_0} \frac{|v_{km\eta}|^2}{|\lambda_{km\eta}(b+\Delta b)|^2} \frac{4m^2\pi^2}{|\lambda_{km\eta}(b)|^2}. \tag{14}$$

If $b+\Delta b \in A_\sigma(\frac{1}{r})$, $0 < |k| < k_0$, $0 < \sigma < 1$, then

$$\frac{|v_{km\eta}|^2}{|\lambda_{km\eta}(b+\Delta b)|^2} \leq |v_{km\eta}|^2 r^2 |k|^{2+2\sigma} \leq r^2 {k_0}^{2+2\sigma} |v_{km\eta}|^2.$$

The relation $\lim_{m\to\infty} \frac{4m^2\pi^2}{|\lambda_{km\eta}(b)|^2} = b^2$ and the condition $0 < |k| < k_0$ imply that the quantity $\frac{4m^2\pi^2}{|\lambda_{km\eta}(b)|^2} = \frac{4m^2\pi^2}{|\frac{2m\pi}{b} + a\pi|k|\eta_2|^2}$ is dominated by a constant $C(k_0)$ depending on k_0. Therefore

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{0<|k|<k_0} \frac{|v_{km\eta}|^2}{|\lambda_{km\eta}(b+\Delta b)|^2} \frac{4m^2\pi^2}{|\lambda_{km\eta}(b)|^2} \leq$$

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} \sum_{0<|k|<k_0} r^2 {k_0}^{2+2\sigma} C(k_0) |v_{km\eta}|^2 \leq$$

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} r^2 {k_0}^{2+2\sigma} C(k_0) \sum_{0<|k|<k_0} |v_{km\eta}|^2.$$

Since

$$\sum_{0<|k|<k_0} |v_{km\eta}|^2 \leq ||v||^2 \leq M^2 ||u||^2,$$

we arrive at the estimate

$$||\widetilde{K}_{b+\Delta b} - \widetilde{K}_b||^2 \leq \frac{|\Delta b|^2}{|b(b+\Delta b)|^2} M^2 r^2 {k_0}^{2+2\sigma} C(k_0).$$

We choose Δb so as to satisfy the condition

$$\frac{|\Delta b|^2}{|b(b+\Delta b)|^2} M^2 r^2 {k_0}^{2+2\sigma} C(k_0) < \varepsilon.$$

Then $||K_{b+\Delta b} - K_b|| < 3\varepsilon$. This shows that the operator-valued function $b \to K_b$ is continuous on $A_\sigma(\frac{1}{r})$. The Lemma is proved.

Lemma 6. *The spectrum $\sigma(K)$ of the compact operator K depends continuously on K in the space $Comp(\mathcal{H}_0)$ of compact operators on $\mathcal{H}_0$, in the sense that for any ε there exists $\delta > 0$ such that for all compact (and even bounded) operators B with $\|B - K\| < \delta$ we have*

$$\sigma(B) \subset \sigma(K) + V_\varepsilon(0), \quad \sigma(K) \subset \sigma(B) + V_\varepsilon(0). \tag{15}$$

Here $V_\varepsilon(0) = \{\lambda \in \mathbb{C} \mid |\lambda| < \varepsilon\}$ is the ε-neighborhood of the point 0 in $\mathbb{C}$.

Proof. Let K be a compact operator; we fix $\varepsilon > 0$. The structure of the spectrum of a compact operator shows that there exists $\varepsilon_1 < \varepsilon/2$ such that $\varepsilon_1 \neq |\lambda|$ for all $\lambda \in \sigma(K)$. Let $S = \{\lambda_1, \ldots, \lambda_k\}$ be the set of all spectrum points λ with $|\lambda| > \varepsilon_1$ and let $V = \bigcup_{\lambda \in S \cup \{0\}} V_{\varepsilon_1}(\lambda)$. Then V is neighborhood of $\sigma(K)$ and $V \subset \sigma(K) + V_\varepsilon(0)$. By the well-known property of spectra (see, e.g.,[9], Theorem 10.20) there exists $\delta > 0$ such that $\sigma(B) \subset V$ for any bounded operator B with $\|B - K\| < \delta$. Moreover (see, e.g., [9], p.293, Exercise 20), the number $\delta > 0$ can be chosen so that $\sigma(B) \cap V_{\varepsilon_1}(\lambda) \neq \emptyset$ $\forall \lambda \in S \cup \{0\}$. Then for all bounded operators B with $\|B - K\| < \delta$ the required inclusion $\sigma(K) \subset \sigma(B) + V_{2\varepsilon_1}(0) \subset \sigma(B) + V_\varepsilon(0)$ and $\sigma(B) \subset V \subset \sigma(K) + V_\varepsilon(0)$ are fulfilled. The lemma is proved.

From Lemma 6 we have the following statement.

Proposition 1. *The function $\rho(\lambda, K) = dist(\lambda, \sigma(K))$ is continuous on $\mathbb{C} \times Comp(\mathcal{H}_0)$.*

Proof. Suppose $\lambda \in \mathbb{C}$, $K \in Comp(\mathcal{H}_0)$ and $\varepsilon > 0$. By Lemma 6 there exists $\delta > 0$ such that for any operator H lying in the δ-neighborhood of K, $||H - K|| < \delta$, the inclusions (15) are fulfilled; these inclusions directly imply the estimate $|\rho(\lambda, K) - \rho(\lambda, H)| < \varepsilon$. Then for all $\mu \in \mathbb{C}$ with $|\mu - \lambda| < \varepsilon$ and all H with $||H - K|| < \delta$ we have

$$|\rho(\mu, K) - \rho(\lambda, H)| \leq |\rho(\mu, K) - \rho(\lambda, K)| + |\rho(\lambda, K) - \rho(\lambda, H)| < |\mu - \lambda| + \varepsilon < 2\varepsilon,$$

Since $\varepsilon > 0$ is arbitrary, the function $\rho(\lambda, K)$ is continuous. The proposition is proved.

Combining Proposition 1 and Lemma 5 we obtain the following fact.

Corollary 1. *The function $\rho(\lambda, b) = dist(\lambda, \sigma(K_b))$ is continuous on $(\lambda, b) \in \mathbb{C} \times A_\sigma(\frac{1}{r})$.*

Now we are ready to prove Theorem 5.

Proof of Theorem 5. By Corollary 1, the function $\rho(1/\nu, b)$ is continuous with respect to the variable $(\nu, b) \in (\mathbb{C} \setminus \{0\}) \times A_\sigma(\frac{1}{r})$. Consequently, the set

$$B_r = \{(\nu, b) \quad | \quad \rho(1/\nu, b) \neq 0, \quad b \in A_\sigma(\frac{1}{r})\}$$

is measurable, and is so the set $B = \cup_r B_r$. Clearly, $B \subset E$ and $E = B \cup B_0$, where $B_0 = E \setminus B$. Obviously, B_0 lies in the set $\mathbb{C} \times (\mathbb{R}^+ \setminus A_\sigma)$ of zero measure (recall that, by Theorem 3, A_σ has complete measure in $\mathbb{R}^+$). Since the Lebesgue measure is complete, B_0 is measurable. Thus, the set E is measurable, being the union of two measurable sets. Next, by Theorem 4, for $b \in A_\sigma$ the section $E^b = \{\nu \in \mathbb{C} \mid (\nu, b) \in E\}$ has complete measure, because its complement $\{1/\nu \mid \nu \in \sigma(K_b)\}$ is at most countable. Therefore, the set E is of full plane Lebesgue measure. The Theorem is proved.

The following important statement is a consequence of Theorem 5.

Corollary 1. *For a.e. $\nu \in \mathbb{C}$, problem (1)(2) has a unique periodic solution with almost every period $b \in \mathbb{R}^+$.*

Proof. Since the set E is measurable and has complete measure, for a.e. $\nu \in \mathbb{C}$ the section $E_\nu = \{b \in \mathbb{R}^+ \mid (\nu, b) \in E\} = \{b \in \mathbb{R}^+ \mid 1/\nu \notin \sigma(K_b)\}$ has complete measure, and for such b's problem (1)(2) has an unique periodic solution with period b. The Corollary is proved.

Aknowledgments. The work is partially supported by the study QG-10-03.

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