NON COMMUTATIVE INTEGRATION FOR UHF ALGEBRAS WITH PRODUCT STATE

Phan Viet Thu

Department of Mathematics-Mechanics and Informatics

Collecge of Science, VNU

ABSTRACT. In this paper we shall give a proof for a lemma (Lemma 3) and a theorem (Theorem 3) stated in the paper [2] of Goldstein, S. and Viet Thu, Phan published in International Journal of Theoremtical Physics vol. 37. N_0. 1. 1998 about the construction of Lp spaces for UHF algebras. We shall also give a proof for a technical theorem (Theorem 1), as a tool for the construction.

1. Uniformly matricial, UHF algebras

A unital C^*-algebra A is called *uniformly matricial of type* $\{n_j\}, j = 1, 2, ..., n_j \in \mathbb{N}$ when there exists a sequence $\{A_j\}_{j\in\mathbb{N}}$ of C^*-subalgebras of A and *a* sequence $\{n_j\}$ of positive integers, such that for each $j \in \mathbb{N}, A_j$ is $*$isomorphic to the algebra $M_{n_j}(\mathbb{C})$ of $n_j \times n_j$ complex matrices,

$$\mathbf{1} \in A_1 \subset A_2 \subset A_3 \subset ...,$$

and $\bigcup_{j\in\mathbb{N}} A_j$ is norm dense in A. The sequece $\{A_j\}_{j\in\mathbb{N}}$ is called a *generating nest of type* $\{n_j\}$ for A. We shall also call it an *approximating sequence* for A. A uniformly matricial C^*-algebras A of type $\{n_j\}$ exists iff the sequence $\{n_j\}$ is strictly increasing and n_j divides $n_{j+1}; \forall j \in \mathbb{N}$. Moreover with these conditions A is unique (up to *isomorphism) and is a simple algebra. The uniformly matricial algebras and their representations are also called *UHF algebras* (from the terminology "uniformly hyperfinite algebras"). Which can be found in a vast literature.

2. Product states [3]

Let $\{A_i; i \in I\}$ be a family of C^*-algebras, $A = \otimes_{i\in I} A_i$ the infinite tensor product of $\{A_i; i \in I\}$, and for each $i \in I, \rho_i$ a state of $A_{(i)}$, the canonical image of A_i in A. Then there is a unique state ρ of A such that

$$\rho(a_1 a_2 ... a_n) = \rho_{i(1)}(a_1)\rho_{i(2)}(a_2)...\rho_{i(n)}(a_n),$$

where $i(1), ..., i(n)$ are distinct elements of I and $a_j \in A_{(i(j))}; j = 1, 2, ..., n$. The state ρ is denoted by $\otimes_{i\in I}\rho_i$; and such states are called product states of $\otimes_{i\in I} A_i$. Given a product

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX state ρ, the component state ρ_i are uniquely determained, since $\rho_i = \rho|A_{(i)}$. The product state is pure if and only if each ρ_i is pure and is tracial if and only if each ρ_i is tracial.

3. The inductive limit of a directed system of Banach spaces

Theorem 1. *Let $\{B_f : f \in \mathbb{F}\}$ be a family of Banach spaces, in which the index set $\mathbb{F}$ is directed by $\leqslant$. Suppose that if $f, g \in \mathbb{F}$ and $f \leqslant g$, there is an isometric linear mapping Φ_{gf} from B_f onto B_g and $\Phi_{hg}\Phi_{gf} = \Phi_{hf}$ whenever $f, g, h \in \mathbb{F}$ and $f \leqslant g \leqslant h$; then*

(i). Φ_{ff} is the identity mapping on B_f.

(ii). There is a Banach space B and for each $f \in \mathbb{F}$, an isometric linear mapping U_f from B_f into B, in such way that $U_f = U_g\Phi_{gf}$ whenever $f, g \in \mathbb{F}, f \leqslant g$ and $\cup\{U_f(B_f) : f \in \mathbb{F}\}$ is everywhere dense in B.

(iii). Suppose that A is a Banach space, V_f is an isometric linear mapping from B_f into A, for each $f \in \mathbb{F}; V_f = V_g\Phi_{gf}$ whenever $f, g \in \mathbb{F}; f \leqslant g$ and $\cup\{V_f(B_f) : f \in \mathbb{F}\}$ is everywhere dense in A. Then there exists an isometric linear mapping W from B into A such that $V_f = WU_f$ for each $f \in \mathbb{F}$

Proof. (i). Denote by $\mathbf{1}$ the identity mapping on B_f. Since Φ_{ff} is an isometric linear mapping and

$$\Phi_{ff}(\Phi_{ff} - \mathbf{1}) = \Phi_{ff}\Phi_{ff} - \Phi_{ff} = 0.$$

It follows that $\Phi_{ff} = \mathbf{1}$.

(ii). Let X be the Banach space consisting of all families $\{a_h : h \in \mathbb{F}\}$ in which $a_h \in B_h$ and $\sup\{||a_h|| : h \in \mathbb{F}\} < \infty$ (with pointwise-linear structure and the supremum norm). Let X_0 be the closed subspace of X consisting of those families $\{a_h : h \in \mathbb{F}\}$ for which the net $\{||a_h|| : h \in \mathbb{F}\}$ converges to 0 and let $Q : X \to X/X_0$ be the quotient mapping. Now for a given $f \in \mathbb{F}$, we define an isometric linear mapping U'_f from B_f into X as follows: when $a \in B_f, U'_f a$ is the family $\{a_h : h \in \mathbb{F}\}$. In which

$$a_h = \begin{cases} \Phi_{hf}a & \text{whenever} \quad h \geqslant f; h, f \in \mathbb{F} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Note that

(α) The linear mapping $QU'_f : B_f \to X/X_0$ is an isometry.

(β) $QU'_f = QU'_g\Phi_{gf}$ when $f \leqslant g; f, g \in \mathbb{F}$.

For these, suppose that $a \in B_f$. To prove (α), let $\{b_h : h \in \mathbb{F}\}$ be an element b of X_0. Given any positive real number ε, it results from the definition of X_0 that there exists an element f_0 of $\mathbb{F}$ such that $||b_k|| < \varepsilon$ whenever $h \in \mathbb{F}$ and $h \geqslant f_0$. Since $\mathbb{F}$ is directed, we can choose $g \in \mathbb{F}$ so that $g \geqslant f$ and $g \geqslant f_0$. Since $U'_f a$ is the family $\{a_h\}$ defined by (1), we have

$$||U'_f a - b|| \geqslant ||a_g - b_g|| \geqslant ||a_g|| - ||b_g|| \geqslant ||\Phi_{gf}a|| - ||b_g|| \geqslant ||a|| - \varepsilon.$$

Thus $||U'_f a - b|| \geqslant ||a||$. It follows that the distance $||QU'_f a||$ from $U'_f a$ to X_0 is not less than $||a||$. The inverse inequality is apparent and (α) is proved. For (β) note that $a \in B_f$ and $\Phi_{gf} a \in B_g$; we have to show that

$$U'_f a - U'_g \Phi_{gf} a \in X_0.$$

Now $U'_f a - U'_g \Phi_{gf} a$ is an element $\{c_h : h \in \mathbb{F}\}$ of X and we want to prove that the net $\{||c_h|| : h \in \mathbb{F}\}$ converges to 0. In fact, we have the stronger result that $||c_h|| = 0$ when $h \geqslant g (\geqslant f)$, since

$$c_h = \Phi_{hf} a - \Phi_{hg} \Phi_{gf} a = 0.$$

The range of the isometric linear mapping QU'_f is a closed subspace Y_f of the Banch space X/X_0. When $f \leqslant g$

$$Y_f = QU'_f(B_f) = QU'_g \Phi_{gf}(B_f) \subset QU'_g(B_g) = Y_g.$$

From this inclusion and since $\mathbb{F}$ is directed, it follows that the family $\{Y_f : f \in \mathbb{F}\}$ of subspaces of X/X_0 is directed by inclusion. Thus $\cup\{Y_f : f \in \mathbb{F}\}$ is a subspace B_0 of X/X_0; its closure is a Banach subspace B of X/X_0. Now we take for U_f the isometric linear mapping QU'_f from B_f into B which completes the proof of (ii).

(iii). Under the conditions set out in (iii), the mapping $V_f U_f^{-1}$ is a linear isometry from $U_f(B_f)$ onto $V_f(B_f)$; when $f \leqslant g, V_g U_g^{-1}$ extends $V_f U_f^{-1}$, since for $a \in B_f$,

$$V_g U_g^{-1}(U_f a) = V_g U_g^{-1} U_g \Phi_{gf} a = V_g \Phi_{gf} a = V_f a = V_f U_f^{-1} U_f a.$$

From this and since the family $\{U_f(B_f) : f \in \mathbb{F}\}$ is directed by inclusion, there is a linear isometry W_0 from $\cup\{U_f(B_f) : f \in \mathbb{F}\}$ onto $\cup\{V_f(B_f) : f \in \mathbb{F}\}$ such that W_0 extends $V_f U_f^{-1}$ for each $f \in \mathbb{F}$. Moreover, W_0 extends by continuity to an isometric linear mapping W from B onto A. W extends $V_f U_f^{-1}$ for each $f \in \mathbb{F}$ and thus $WU_f = V_f$.

Remark. The Theorem 1 and its proof is adapted from Kadison and Ringrose (see [3]).

Definition. In the circumstance set out in the Theorem 1, we say that the Banach spaces $\{B_f : f \in \mathbb{F}\}$ and the isometries $\{\Phi_{gf} : f \leqslant g; f, g \in \mathbb{F}\}$ together constitute a directed system of Banach spaces. The Banach space B occurring in (ii) (together with the isometries $\{U_f : f \in \mathbb{F}\}$) is called the *inductive limit* of the directed system. The effect of (iii). is to show that the construction in (ii) are unique up to isometry.

4. $L^p(A, \varphi)$ for finite discrete factors

Let M be finite discrete factor acting on H and τ a(finite) faithful normal tracial state on M (the definition and properties of these notions can be found in [3]). For $p \in [1, \infty]$, let $L^p(M, \tau)$ denote the L^p space with respect to τ as constructed in [1, 4, 5]. Recall that $||.||_p^\tau$ norm on $L^p(M, \tau)$ is difined by

$$||a||_p^\tau = \tau(|a|^p)^{1/p} \quad \text{for } a \in M, p \in [1, \infty[.$$

For $p = \infty$, put $||a||_\infty^\tau = ||a||$. Then $||.||_p^\tau$ turns $L^p(M,\tau)$ into a Banach space, moreover the Holder inequality

$$||ab||_r^\tau \leqslant ||a||_p^\tau \; ||b||_q^\tau$$

hold for all $a, b \in M$ with $p, q, r \in [1,\infty]$ such that $1/p + 1/q = 1/r$ and for each $a \in M, p \in [1,\infty[$

$$||a||_p^\tau = \sup_{||b||_q^\tau \leqslant 1} |\tau(a,b)|; q \in [1,\infty[\text{ such that } 1/p + 1/q = 1.$$

Let now φ be an arbitrary faithful (normal) state on M. There exists unique $h \in M$ such that

$$\varphi(a) = \tau(ha) \quad \text{for all } a \in M.$$

Moreover h is positive, invertible and $\tau(h) = 1$.

For all $a \in M$ and $p \in [1,\infty[$ put

$$||a||_p = \tau(|h^{1/2p}ah^{1/2p}|^p)^{1/p}.$$

For $p = \infty$, let $||a||_\infty = ||a||$. We define the belinear from

$$< a, b >= \tau(h^{1/2p}ah^{1/2p}b) \;\; \forall a, b \in M.$$

Lemma 1. For all $p \in [1,\infty]$ we have

(i) $||.||_p$ is a norm on M.

(ii) $| < a,b > | \leqslant ||a||_p ||b||_q$ where $1/p + 1/q = 1, q \in [1,\infty], a, b \in M$.

(iii) $||a||_p = \sup_{||b||_q \leqslant 1} | < a, b > |, \forall a \in M, b \in M; q \in [1,\infty]; 1/p + 1/q = 1$.

Lemma 2. If $p, s \in [1,\infty]$ and $p \leqslant s$, then $||a||_p \leqslant ||a||_s$ for all $a \in M$. (For the proof of Lemma 1 and Lemma 2: see [2]).

The norm $||.||_p$ turns M into a Banach space which we denote by $L^p(M,\varphi)$. If $\varphi = \tau$ then $L^p(M,\varphi) \equiv L^p(M,\tau)$.

Note that mapping $a \mapsto h^{1/2p}ah^{1/2p}$ defines an isometric isomorphism between $L^p(M,\varphi)$ and $L^p(M,\tau)$.

Lemma 3. For each $p \in [1,\infty]$, the Banach space $L^p(M,\varphi)$ is isometric to the Haagerup space $L^p(M)$.

Proof. We may assume that $\varphi = \tau$ and $p < \infty$. Note that, since the modular automorphism group $\{\sigma_t^\tau\}$ acts trivilly on M,

$$\mathbb{M} = M \rtimes_{\sigma\tau} \mathbb{R} \cong M \otimes L^\infty(\mathbb{R}).$$

Furthermore, the canonical trace $\overline{\tau}$ on the crossed product $\mathbb{M}$ equals $\tau \otimes e^{-s}ds$. The Haagerup space $L^p(\mathbb{M})$ consists of products $a \otimes exp((.)/p)$ where $a \in M$. Hence it is enough to show that the mapping

$$a \mapsto a \otimes exp((.)/p)$$

is an isometry. It is clear that one needs only to consider the case $p = 1$. We must show that

$$\tau(|a|) = \overline{\tau}(x_{]1,\infty[}(|a| \otimes exp(.))).$$

(see. Terp [7]). Let $|a| = \int_0^\infty \lambda de_\lambda$ be the spectral decomposition of $|a|$. We calculate:

$$\begin{aligned}
\overline{\tau}(\chi_{]1,\infty[}(|a| \otimes exp(.))) &= \int_{-\infty}^{\infty} \tau(\chi_{]e^{-s},\infty[}(|a|))e^{-s}ds \\
&= \int_0^\infty \tau(\chi_{]t,\infty[}(|a|))dt \\
&= \int_0^\infty (\int_0^\infty (\chi_{\{t<\lambda\}}d\tau(e_\lambda))dt
\end{aligned}$$

(since the indicator functions are non-negative and bounded, using the Fubini theorem, we have further)

$$\begin{aligned}
&= \int_0^\infty (\int_0^\infty (\chi_{\{t<\lambda\}}dt)d\tau(e_\lambda). \\
&= \int_0^\infty (\int_0^\lambda dt)d\tau(e_\lambda) \\
&= \int_0^\infty \lambda d\tau(e_\lambda) = \tau(|a|). \quad \square
\end{aligned}$$

5. Non commutative integration or L^p spaces for UHF algebras with product state

Theorem 2. *(Theorem 13.1.14 of [3]). Suppose that $\{A_j : j \in \mathbb{N}\}$ is a sequence of mutually commuting finite type I factors acting on a Hilbert space H. (and each containing the unit of $\mathbb{B}(H)$), A is the C^*-algebra generated by $\bigcup_j A_j$; ξ is a unit cyclic vector for A and $\omega_\xi|A$ is a product state $\otimes\rho_j$, where ρ_j is a faithful state of $A_j, j \in \mathbb{N}$. Then $\omega_\xi|A^-$ is a faithful normal state of A^- (the weak operator closure of A), the corresponding modular automorphism group $\{\sigma_t\}$ of A^- leaves each A_j invariant and $\{\sigma_t|A_j\}$ is the modular automorphism group of A_j corresponding to ρ_j.*

Proof: (The proof of Theorem 2 can be found in [3]).

Let A be a UHF C^*-algebra with a generating nest $\{A_n\}_{n\in\mathbb{N}}$, let φ be a product stste of A with respect to the sequence $\{A_n\}$. There exists the a sequence $\{B_j\}$ of mutually commuting finite type I subfactors of A (each containing unit of A)). Such that $A_n = \bigotimes_{j=1}^{n} B_j$ or, equivalently $\bigcup_{j=1}^{n} B_j$ generates A_n and $\bigcup_{j=1}^{\infty} B_j$ generates A as C^*-algebra. Denote the restriction of φ on B_j by φ_j we have

$$\varphi(b_1, b_2, ..., b_n) = \varphi(b_1)...\varphi(b_n) = \varphi_1(b_1)...\varphi_n(b_n)$$

$\forall b_j \in B_j; j = 1, 2, ..., n$. Put $\varphi^{(n)} = \varphi|A_n$ we have

$$\varphi^{(n)} = \varphi_1 \otimes ... \otimes \varphi_n$$

Theorem 3. *Let A be a UHF algebra with a generating nest $\{A_n\}, n \in \mathbb{N}$ and φ a product state on A with respect to the sequence $\{A_n\}$. Suppose that for each i; φ_i is faithful. Then for $p \in [1, \infty], L^p(A, \varphi)$ is the inductive limit of $\{L^p(A_n, \varphi^{(n)})\}$; moreover*

$$L^p(A, \varphi) \cong L^p(\pi_\varphi(A)^") = L^p(M).$$

Proof: Denote by $(H_\varphi, \pi_\varphi, \xi_\varphi)$ respectively $(H_{\varphi(n)}, \pi_{\varphi(n)}; \xi_{\varphi(n)})$ the GNS representation of the pair (A, φ) (respectively$(A_n, \varphi^{(n)})$). Let us first note that $\pi_\varphi(A)^" = M$ and $N_\infty = \{0\}$, which shows that

$$L^p(\pi_\varphi(A)^") \cong L^p(A, \varphi).(\cong L^p(M))$$

and analoguosly

$$L^p(\pi_{\varphi(n)}(A_n)^") \cong L^p(A_n, \varphi^{(n)}) \;\; \forall n \in \mathbb{N}^*; p \in [1, \infty].$$

By [3 Theorem 11.4.15. and Remark 11.4.16]. A is simple, φ is a primary stat, so π_φ is faithful and $\pi_\varphi(A)^"$ is a factor. Thus A is isometrically isomorphic to $\pi_\varphi(A)$. Upon identifying A with $\pi_\varphi(A), \varphi$ takes the form $\omega_{\xi_\varphi}|A$ for the cyclic unit vector ξ_φ. The situation remains true for each pair $(A_n, \varphi^{(n)})$ and $(H_{\varphi^{(n)}}, \pi_{\varphi^{(n)}}, \xi_{\varphi^{(n)}})$ we conclude now that $\omega_{\xi_\varphi}|\pi_\varphi(A)^"$ is faithful, hence $s_\varphi = \mathbf{1}$. It implies that $M = \pi_\varphi(A)^"$ and also $N_\infty = \{0\}$, i.e. $L^\infty(A, \varphi) \cong M$ and

$$L^p(A, \varphi) \cong L^p(M) = L^p(\pi_\varphi(A)").$$

For the pair $(A_n, \varphi^{(n)})$, by hypothesis, φ_j are faithful states of B_j; Then $\varphi^{(n)} = \bigotimes_{j=1}^{n} \varphi_j$ is a faithful state of $A_n = \bigotimes_{j=1}^{n} B_j$.

A_n are finite factor of type I; $\pi_{\varphi^{(n)}}(A_n) = \pi_{\varphi^{(n)}}(A)"$ and $\omega_{\xi_{\varphi^{(n)}}}$ are faithful on $\pi_{\varphi^{(n)}}(A_n)^"$. It implies

$$s_{\varphi^{(n)}} = \mathbf{1}; M_n = \pi_{\varphi^{(n)}}(A_n);$$

$$L^\infty(A_n, \varphi^{(n)}) \cong M_n;$$

$$L^p(A_n, \varphi^{(n)}) \cong L^p(M_n) = L^p(\pi_{\varphi^{(n)}}.(A_n)) \;\; p \in [1, \infty].$$

The modular automorphism σ_t^{φ} of $\pi_{\varphi}(A)" = M$ associated with ω_{ξ_φ} leaves each $\pi_{\varphi^{(n)}}(A_n)" = M_n$ invariant. Thus there exists a σ-weakly continuous conditional expectation E_n from M onto M_n for all $n \in \mathbb{N}$ and $L^p(M_n)$ can be canonically isometrically embeded into $L^p(M_m)$ if $n \leqslant m$. Denote this embedding by Φ_{mn}; the family $\{L^p(M_n); \Phi_{mn}; m, n \in \mathbb{N}\}$ forms a directed system of Banach spaces, with the inductive limit $\overline{\bigcup_{n=1}^{\infty} L^p(M_n)} \cong L^p(M)$. Since for each n, $L^p(M_n) \cong L^p(A_n, \varphi^{(n)})$ the family $\{L^p(A_n, \varphi^{(n)})\}$ has the same inductive limit $L^p(M)$ and from the fact that $L^p(M) \cong L^p(A, \varphi)$, it implies that the family $\{L^p(A_n, \varphi^{(n)})\}$ has the inductive limit $L^p(A, \varphi)$. □

References

1. Dixmir, J. Formes lineaires sur un aneau d' operateur, *Bull. Soc. Math. France*, **81**(1953), 3-39.
2. Goldstein, S. and Viet Thu, Phan L^p-spaces for UHF algebras, *Inter. J. of Theor. Phys.*, **37**(1998), 593-598.
3. Kadison, R. V. and Ringrose, J. R. *Fundamentals of the theory of operator algebras*, vol **I** (1983). Vol. **II**(1986), Academic Press, New York-London.
4. Nelson, E., Notes on non-commutative integration, *J. Funct. Anal.*, **15**(1974), 103-116.
5. Segal. I. E. A non-commutative extension of abstract integration, *Ann. Math.*, **57**(1953), 401-457.
6. Takesaki. M. Conditional expectations in von Neumann algrbras, *J. Funct. Ann.*, **9**(1972), 306-321.
7. Terp M. *L^p-spaces associated with von Neumann algebras*, Notes, Kϕ benhavns Universitet, Matematisk Institut, Rapport N_0**3**(1981).