SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF A CLASS OF RATIONAL RECURSIVE SEQUENCES

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ABSTRACT. We study the convergence of positive recursive sequence $x_{n+1} = f(x_n, x_{n-1})$. Here $f(x, y)$ is a positive continuous function of two variables. Our results are applicable for rational recursive sequences $x_{n+1} = \frac{Ax_n+B}{x_n+ax_{n-1}+b}$ and $x_{n+1} = \frac{Ax_{n-1}+B}{x_{n-1}+ax_n+b}$. Ladas has conjectured that the first sequence would always converge while we prove that the second may be 2-periodic.

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1. Introduction and Preliminaries

For application we should compute with numbers. We should also find global convergent numerical algorithms. The first well-known numerical algorithm is the Newton-iteration to find roots of real functions. But Newton-iteration is locally convergent only. This is not so good, because in the practice only global convergent algorithms are applicable. The other very bad thing in computational algorithms is the periodicity. In this case, the computers are unable to give us approximated results, although the running time is over. Hence, at the end of the 20th century there are more and more interests in the investigating nonlinear difference equations. For example, to solve (approximated) the equation $f(x, x) = x$ in the set of positive numbers we let $x_0, x_1 > 0$ be given and $x_{n+1} = f(x_n, x_{n-1})$ for $n = 1, 2, \cdots$. We wish that the recursive sequence $\{x_n\}_n$ converges rapidly to a root of the equation $f(x, x) = x$. But in the practice unpleasant things would occur: Or the periodicity or the convergence not so rapid.

G. Ladas and more authors give several problems and conjectures involving the convergence and the periodicity of positive rational recursive sequences. In the following we will deal with this problem systematically.

Consider the following difference equation

$$x_{n+1} = f(x_n, x_{n-1}), \qquad \text{for} \quad n = 1, 2, \cdots, \quad (x_0, x_1 > 0 \text{ are given}). \tag{1}$$

Here f is a continuous function on $[0,\infty)^2$ and taking values in the $(0,\infty)$. If this sequence converges to a positive number ℓ then we must have

$$\ell = f(\ell, \ell). \tag{2}$$

Therefore, we assume that there is a unique positive number ℓ such that (2) holds. Clearly this is not sufficient. We should assume more conditions. First of all we have

Lemma 1. *If every solution of* (1) *is convergent to a positive number ℓ, then the following system*

$$x = f(y, x),$$
$$y = f(x, y)$$

has a unique (positive) solution $x = y = \ell$.

Proof: Let (x, y) be a positive solution of the above system. Consider the difference equation (1) with $x_1 = x$ and $x_0 = y$. Then $x_2 = f(x,y) = y$ and $x_3 = f(x_2, x_1) = f(y, x) = x$. By induction, we obtain $x_{2k} = y$ and $x_{2k+1} = x$. But by our assumption the sequence $\{x_n\}$ is convergent to ℓ, hence $x = y = \ell$. The proof is complete.

The following Lemma will show that the condition of Lemma 1 is sufficient if the function $f(x,y)$ is bounded and decreasing in the variable x and increasing in the variable y.

Lemma 2. *Assume that the function $f(x,y)$ is decreasing in the variable x for each $y > 0$ and increasing in the variable y for each $x > 0$. Suppose further that*

$$M := \sup_{x,y \geq 0} f(x,y) < \infty,$$

and the system

$$\alpha = f(\beta, \alpha),$$
$$\beta = f(\alpha, \beta)$$

has the only solution $\alpha = \beta = \ell$. Then every positive solution of (1) *converges to ℓ.*

Proof: Clearly, $x_{n+1} = f(x_n, x_{n-1}) < M$ for ali $n = 2, 3, \cdots$. Without loss of generality we assume $x_n < M$ for all $n = 0, 1, \cdots$. Consider the following system of difference equations

$$\alpha_{n+1} = f(\beta_n, \alpha_n),$$
$$\beta_{n+1} = f(\alpha_n, \beta_n)$$

for $n = 0, 1, \cdots$. Here we let $\alpha_0 = 0, \beta_0 = M$. Clearly,

$$\alpha_o < x_n < \beta_0 \qquad \text{for all} \quad n = 0, 1, \cdots .$$

The function $f(x, y)$ is decreasing in x and increasing in y, hence

$$x_{n+2} = f(x_{n+1}, x_n) < f(\alpha_0, \beta_0) = \beta_1$$

and similarly,

$$x_{n+2} = f(x_{n+1}, x_n) > f(\beta_0, \alpha_0) = \alpha_1 \qquad \text{for all} \quad n = 0, 1, \cdots .$$

By induction we can see that

$$\alpha_k < x_{n+2k} < \beta_k \qquad \text{for all} \quad k, n = 0, 1, \cdots .$$

On the other hand note that $\alpha_0 < \alpha_1$ and $\beta_0 > \beta_1$. Since the function $f(x, y)$ is increasing in x and decreasing in y, we get

$$\alpha_2 = f(\beta_1, \alpha_1) > f(\beta_0, \alpha_0) = \alpha_1$$

and similarly $\beta_2 < \beta_1$. By induction we can see that the sequence $\{\alpha_k\}$ is increasing and the sequence $\{\beta_k\}$ is decreasing. Let α be the limit of the $\{\alpha_n\}$ and let β be the limit of $\{\beta_n\}$. Then α and β satisfy the following system

$$\alpha = f(\beta, \alpha),$$
$$\beta = f(\alpha, \beta).$$

Our assumption assures that $\alpha = \beta = \ell$. The proof is complete.

Remark. In some cases the function $f(x, y)$ is decreasing in the variable y only. The following Lemma will give another sufficient condition for the convergence of the recursive sequence (1).

Lemma 3. *Assume that the function $f(x, y)$ is increasing in the variable x for each $y > 0$ and decreasing in the variable y for each $x > 0$. Suppose further that*

$$M := \sup_{x,y \geq 0} f(x, y) < \infty,$$

and the system

$$\alpha = f(\alpha, \beta),$$
$$\beta = f(\beta, \alpha)$$

has the only solution $\alpha = \beta = \ell$. Then every positive solution of (1) converges to ℓ.

Proof: See [2].

2. Positive rational recursive sequences

Now consider the positive rational recursive sequence

$$x_{n+1} = \frac{Ax_n + B}{x_n + ax_{n-1} + b}, \qquad \text{for } n = 1, 2, \cdots, \qquad (x_0, x_1 > 0 \text{ are given}). \tag{3}$$

Here A, B and a, b are positive parameters. G. Ladas has conjectured that this sequence always converges. In [2] we prove this conjecture with small restriction on these parameters. It is also proved that the recursive sequence (3) is not periodic with minimal period 2 or 3. We have

$$f(x,y) = \frac{Ax + B}{x + ay + b}.$$

Note that the function $f(x,y)$ is decreasing in the variable y and non-monotone decreasing in the variable x for each $y > 0$, so we cannot apply Theorems of [1,3]. Now consider the equation $\ell = f(\ell, \ell)$. This has the only positive solution

$$\ell = \frac{\sqrt{(A-b)^2 + 4B(a+1)} + (A-b)}{2(a+1)}.$$

Combining Lemma 4 with [2, Theorem 1] we have:

Theorem 1. *Assume that $A \geq B/b$. If one of the following conditions holds, then the above conjecture of Ladas is true:*

(i) $A \leqslant b$;

(ii) $A > b$ and $a \leqslant 1$;

(iii) $A > b, a > 1$ and $(A-b)^2 \leqslant 4B/(a-1)$.

Otherwise, for every recursive sequence (3) we have

$$\alpha \leqslant \liminf_{n\to\infty} x_n \leqslant \ell \leqslant \limsup_{n\to\infty} x_n \leqslant \beta,$$

where

$$\alpha = \frac{1}{2}\left((A-b) - \sqrt{(A-b)^2 - \frac{4B}{a-1}}\right)$$

$$\beta = \frac{1}{2}\left((A-b) + \sqrt{(A-b)^2 - \frac{4B}{a-1}}\right).$$

The following theorem is also proved in [2]:

Theorem 2. *If $A < b$, then the above conjecture of Ladas is true.*

Now we assume $A = b$ and try to prove Ladas' conjecture in this case. Unfortunately, at this time we always have to restrict on A, B and a.

Theorem 3. *Assume $A = b$ and $a < 1$. If $B < 4A^2/(a+1)$, then the above conjecture of Ladas is true.*

Proof: First note that if $B \leqslant A^2$, we can apply the case (i) of Theorem 1. Therefore, without loss of generality we assume that $B > A^2$. On the other hand we have

$$\ell = \sqrt{\frac{B}{a+1}} \tag{4}$$

and

$$|x_{n+1} - \ell| = \frac{|(A-\ell)(x_n - \ell) - a\ell(x_{n-1} - \ell)|}{x_n + ax_{n-1} + A}. \tag{5}$$

Let $\delta_n = |x_n - \ell|$. We get

$$\delta_{n+1} \leqslant \frac{|A-\ell|\delta_n + a\ell\delta_{n-1}}{A}.$$

Now consider the following linear difference equation

$$y_{n+1} = \frac{|A-\ell|y_n + a\ell y_{n-1}}{A} \qquad \text{for } n = 1, 2, \cdots, \quad (y_0 = \delta_0, \quad y_1 = \delta_1).$$

Trivially $\delta_n \leqslant y_n$ for all $n = 0, 1, \cdots$ and y_n has the following form

$$y_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n,$$

where $\lambda_{1,2}$ are roots of the following equation

$$A\lambda^2 = |A - \ell|\lambda + a\ell. \tag{6}$$

We prove that two roots of the equation (6) having absolute values less than 1 (and consequently $y_n \to 0$ as $n \to \infty$). This is equivalently to show

$$A > |A - \ell| + a\ell.$$

To end this we consider two possible cases: If $B < A^2(a+1)$, then we have $A > \ell$ and consequently $|A - \ell| + a\ell = A + (a-1)\ell < A$ (because $a < 1$). The second case is $B \geq A^2(a+1)$. We have $A \leqslant \ell$ and $|A-\ell|+a\ell = (a+1)\ell - A = \sqrt{B(a+1)} - A < 2A - A = A$ (because $B < 4A^2/(a+1)$). The proof is complete.

Theorem 4. *Assume $A = b$ and $1 \leqslant a \leqslant 2$. If $B < 9A^2/(a+1)$, then the above conjecture of Ladas is true.*

Proof: Note as before that if $B \leqslant A^2$, we can apply the case (i) of Theorem 1. Therefore, assume without loss of generality that $B > A^2$. Consider the function

$$f(x,y) = \frac{Ax+B}{x+ay+A}.$$

We have

$$\sup_{x,y>0} f(x,y) = \frac{B}{A} \quad \text{and} \quad \frac{\partial}{\partial x} f(x,y) = \frac{aAy + (A^2 - B)}{(x+ay+A)^2}.$$

Therefore,

$$\frac{\partial}{\partial x} f(x, B/A) > 0 \qquad (\text{because} \quad a \geq 1)$$

and consequently

$$\inf_{x,y\in(0,B/A)} f(x,y) = f(0,B/A) = \frac{AB}{A^2+aB} > \frac{AB}{(a+1)B} = \frac{A}{a+1}.$$

Note that $x_{n+1} = f(x_n, x_{n-1})$, so

$$x_n > \frac{A}{a+1} \qquad \text{for} \quad n = 4, 5, \cdots.$$

On the other hand, putting $\delta_n = |x_n - \ell|$, it follows from (5) that

$$\delta_{n+1} \leqslant \frac{|A-\ell|\delta_n + a\ell\delta_{n-1}}{2A}.$$

Now consider the following linear difference equation

$$y_{n+1} = \frac{|A-\ell|y_n + a\ell y_{n-1}}{2A} \qquad \text{for } n = 5, 6, \cdots, \quad (y_4 = \delta_4, \quad y_5 = \delta_5).$$

Trivially $\delta_n \leqslant y_n$ for all $n = 4, 5, \cdots$ and y_n has the following form

$$y_n = \alpha_1 \lambda_1^n + \alpha_2 \lambda_2^n,$$

where $\lambda_{1,2}$ are roots of the following equation

$$2A\lambda^2 = |A-\ell|\lambda + a\ell. \tag{7}$$

We prove that two roots of the equation (7) having absolute values less than 1 (and consequently $y_n \to 0$ as $n \to \infty$). This is equivalently to show

$$2A > |A-\ell| + a\ell.$$

To this end consider two possible cases: If $B < (a+1)A^2$, then we have $A > \ell = \sqrt{B/(a+1)}$ and consequently $|A-\ell| + a\ell = A + (a-1)\ell \leqslant A + \ell < 2A$ (because $a \leqslant 2$).

The second case is $B \geq (a+1)A^2$. We have $A \leqslant \ell$ and $|A-\ell| + a\ell = (a+1)\ell - A = \sqrt{B(a+1)} - A < 3A - A = 2A$ (because $B < 9A^2/(a+1)$). The proof is complete.

Due to Ladas' problem we consider the following recursive sequence

$$x_{n+1} = \frac{Ax_{n-1} + B}{x_{n-1} + ax_n + b}, \qquad \text{for } n = 1, 2, \cdots, \qquad (x_0, x_1 > 0 \text{ are given}). \tag{8}$$

Here A, B and a, b are positive parameters. We have

$$f(x,y) = \frac{Ay + B}{y + ax + b}.$$

Now consider the equation $\ell = f(\ell, \ell)$. This has the only positive solution

$$\ell = \frac{\sqrt{(A-b)^2 + 4B(a+1)} + (A-b)}{2(a+1)}.$$

An elementary computation gives

$$M := \sup_{x,y \geq 0} f(x,y) = \max\{A, \frac{B}{b}\}$$

On the other hand,

$$\frac{\partial}{\partial y} f(x,y) = \frac{aAx + (Ab - B)}{(y + ax + b)^2}.$$

Note that the function $f(x,y)$ is decreasing in the variable x and if $A \geq B/b$ this function is increasing in the variable x. We should solve the system

$$\alpha = f(\beta, \alpha)$$
$$\beta = f(\alpha, \beta)$$

to obtain $\alpha = \beta$. This requires some restrictions on parameters A, B, a and b. First the above system is equivalent to

$$\alpha^2 + a\alpha\beta + b\alpha = A\alpha + B,$$
$$\beta^2 + a\alpha\beta + b\beta = A\beta + B.$$

Taking the difference between these equations we obtain

$$(\alpha - \beta)(\alpha + \beta + b) = A(\alpha - \beta).$$

If $A \leqslant b$ we should have $\alpha = \beta$. Now we assume $A > b$ and $\alpha + \beta = A - b$. Now taking the sum of equations of (11) we obtain

$$(\alpha^2 + \beta^2) + 2a\alpha\beta + b(\alpha + \beta) = A(\alpha + \beta) + 2B,$$

or equivalently,

$$(\alpha+\beta)^2+2(a-1)\alpha\beta=(A-b)(\alpha+\beta)+2B.$$

Replace $\alpha+\beta=A-b$ we obtain

$$(a-1)\alpha\beta=B.$$

Therefore, if $a\leqslant 1$ this is a contradiction. Or equivalently, the recursive sequence (8) is convergent if $a\leqslant 1$. Now let $a>1$. We have

$$\alpha\beta=\frac{B}{a-1}.$$

If $\alpha\neq\beta$ we should have $(\alpha+\beta)^2>4\alpha\beta$. Hence the recursive sequence (8) is convergent if

$$(A-b)^2\leqslant\frac{4B}{a-1}.$$

To sum up we obtain

Theorem 5. *Assume that $A\geq B/b$. If one of the following conditions holds, then the recursive sequence (8) is convergent:*

(i) $A\leqslant b$;

(ii) $A>b$ *and* $a\leqslant 1$;

(iii)$A>b, a>1$ *and* $(A-b)^2\leqslant 4B/(a-1)$.

Remark. If conditions (i)-(iii) of the above theorem do not hold, then there is a 2-periodic solution of the equation (8). Indeed, let

$$\alpha=\frac{1}{2}\left((A-b)-\sqrt{(A-b)^2-\frac{4B}{a-1}}\right)$$
$$\beta=\frac{1}{2}\left((A-b)+\sqrt{(A-b)^2-\frac{4B}{a-1}}\right)$$

then the solution with $x_0=\alpha$ and $x_1=\beta$ is 2-periodic (not convergent). In contrast with Ladas' conjecture the recursive sequence (8) may be not convergent.

Next we prove the recursive sequence (8) is convergent with only one restriction that $A<b$. To this end let

$$H(x,y,u,v)=\frac{Ay+B}{v+au+b}.$$

Note that $H(x,y,x,y) = f(x,y)$. Moreover, $H(x,y,u,v)$ is monotone increasing in variables x, y and decreasing in variables u, v. We consider the following system of difference equations

$$u_{n+1} = H(u_n, u_{n-1}, \lambda_n, \lambda_{n-1})$$
$$\lambda_{n+1} = H(\lambda_n, \lambda_{n-1}, u_n, u_{n-1}) \quad \text{for} \quad n = 1, 2, \cdots.$$

Here, we let

$$\lambda_0 = \lambda_1 = 0$$
$$u_0 = u_1 = M + \frac{B}{b-A}.$$

Clearly, $x_{n+1} = f(x_n, x_{n-1}) \leqslant M = \sup_{x,y>0} f(x,y)$ for all $n = 1, 2, \cdots$. Hence, we assume without loss of generality that $x_0, x_1 \leqslant M$. So we have

$$u_0 \geqslant u_1 \geqslant u_2$$
$$\lambda_0 \leqslant \lambda_1 \leqslant \lambda_2$$
$$\lambda_0 \leqslant x_0 \leqslant u_0$$
$$\lambda_1 \leqslant x_1 \leqslant u_1.$$

By induction, we can prove that $\{\lambda_n\}$ is monotone nondecreasing, $\{u_n\}$ is monotone nonincreasing and $\lambda_n \leqslant x_n \leqslant u_n$ for $n = 1, 2, \cdots$. Let λ be the limit of $\{\lambda_n\}$ and let u be the limit of $\{u_n\}$. Then

$$u = \frac{Au + B}{(a+1)\lambda + b}$$
$$\lambda = \frac{A\lambda + B}{(a+1)u + b}.$$

By our assumption $A < b$, the above system has the only positive solution $u = \lambda = \ell$. We obtain

Theorem 6. *If $A < b$, the recursive sequence* (8) *is always convergent.*

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