ON PSEUDO-OPEN S-IMAGES AND PERFECT IMAGES OF FRÉCHET HEREDITARILY DETERMINED SPACES

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ABSTRACT. In this paper we prove a mapping theorem on Fréchet spaces with a locally countable k-network and give a partial answer for the question posed by G. Gruenhage, E. Michael and Y. Tanaka.

1. Introduction

Let X be a topological space, and $\mathcal{P}$ be a cover of X. We say that X is *determined* by $\mathcal{P}$, or $\mathcal{P}$ *determines* X, if $U \subset X$ is open (closed) in X if and only if $U \cap P$ is relatively open (respectively, closed) in P for every $P \in \mathcal{P}$.

$\mathcal{P}$ is a *k-network*, if whenever $K \subset U$ with K compact and U open in X, then $K \subset \cup\mathcal{F} \subset U$ for a certain finite collection $\mathcal{F} \subset \mathcal{P}$. $\mathcal{P}$ is a *network*, if $x \in U$ with U open in X, then $x \in P \subset U$ for some $P \in \mathcal{P}$.

A collection $\mathcal{P}$ of subsets of X is *star-countable* (respectively, *point-countable*), if every $P \in \mathcal{P}$ (respectively, single point) meets only countable many members of $\mathcal{P}$. A collection $\mathcal{P}$ of subsets of X is *locally countable*, if every $x \in X$ there is a neighborhood V of x such that V meets only countable many members of $\mathcal{P}$.

Note that every star-countable collection or every locally countable collection is point-countable.

A space X is a *sequential space*, if every $A \subset X$ is closed in X if and only if no sequence in A converges to a point not in A.

A space X is *Fréchet*, if for every $A \subset X$ and $x \in \overline{A}$ there is a sequence $\{x_n\} \subset A$ such that $x_n \to x$.

A space X is a *k-space*, if every $A \subset X$ is closed in X if and only if $A \cap K$ is relatively closed in K for every compact $K \subset X$.

A space X is a *σ-space* if it has a σ-locally finite network.

A space X has *countable tightness* (abbrev. $t(X) \leqslant \omega$), if, whenever $x \in \overline{A}$ in X, then $x \in \overline{C}$ for some countable $C \subset A$.

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A space X is a *countably bi-k-space* if, whenever (A_n) is a decreasing sequence of subsets of X with a common cluster point x, then there exists a decreasing sequence (B_n) of subsets of X such that $x \in \overline{(A_n \cap B_n)}$ for all $n \in \mathbb{N}$, the set $K = \bigcap_{n \in \mathbb{N}} B_n$ is compact, and each open U containing K contains some B_n.

Note that every Fréchet space is a sequential space and every sequential Hausdorff space is a k-space, every sequential space has countable tightness, locally compact spaces and first countable spaces are countably bi-k-space, and every countably bi-k-space is a k-space.

We say that a map $f : X \to Y$ is *perfect* if f is a closed map and $f^{-1}(y)$ is a compact subspace of X for every $y \in Y$. A map $f : X \to Y$ is *pseudo-open* if, for each $y \in Y$, $y \in \text{Int} f(U)$ whenever U is an open subset of X containing $f^{-1}(y)$. A map $f : X \to Y$ is *Lindelöf* if every $f^{-1}(y)$ is Lindelöf. A map $f : X \to Y$ is *a s-map* if $f^{-1}(y)$ is separable for each $y \in Y$. A map $f : X \to Y$ is *compact-covering* if every compact $K \subset Y$ is an image of a compact subset $C \subset X$. A map $f : X \to Y$ is *compact-covering* if every compact $K \subset Y$ is an image of a compact subset $C \subset X$. A map $f : X \to Y$ is *sequence-covering* if every convergent sequence (including its limit) $S \subset Y$ is an image of a compact subset $C \subset X$.

Note that every closed map or every open map is pseudo-open, every pseudo-open map is quotient, and if $f : X \to Y$ is a quotient map from X onto a Fréchet space Y, then f is pseudo-open. Every compact-covering map is sequence-covering, and every sequence-covering map onto a Hausdorff sequential space is quotient.

In [3] G. Gruenhage, E. Michael and Y. Tanaka raised the following question

Question. Is a Fréchet space having a point-countable cover $\mathcal{P}$ such that each open $U \subset X$ is determined by $\{P \in \mathcal{P} : P \subset U\}$ preserved by pseudo-open s-maps or perfect maps?

In [5] S. Lin and C. Liu gave a partial answer for the above question.

In this paper we prove a mapping theorem on Fréchet spaces with a locally countable k-network and give an another partial answer for the above question.

We assume that spaces are regular T_1, and all maps are continuous and onto.

2. Preliminaries

For a cover $\mathcal{P}$ of X, we consider the following conditions (A) - (E), which are labelled (1.1) - (1.6), respectively in [3].

(A) X has a point-countable cover $\mathcal{P}$ such that every open set $U \subset X$ determined by $\{P \in \mathcal{P} : P \subset U\}$.

(B) X has a point-countable cover $\mathcal{P}$ such that if $x \in U$ with U open in X, then $x \in (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset U$ for some finite subfamily $\mathcal{F}$ of $\mathcal{P}$.

$(B)_p$ X has a point-countable cover $\mathcal{P}$ such that if $x \in X \setminus \{p\}$ with p is a point in X, then $x \in (\cup\mathcal{F})^o \subset \cup\mathcal{F} \subset X \setminus \{p\}$ for some finite subfamily $\mathcal{F}$ of $\mathcal{P}$.

(C) X has a point-countable cover $\mathcal{P}$ such that every open set $U \subset X$ determined by collection $\{P \in \mathcal{P} : P \subset U\}^*$, where $\mathcal{U}^* = \{\cup\mathcal{F} : \mathcal{F}$ is a finite subfamily of $\mathcal{U}\}$.

$(C)_p$ X has a point-countable cover $\mathcal{P}$ such that for every point $p \in X$, the set $X \setminus \{p\}$ determined by collection $\{P \in \mathcal{P} : P \subset (X \setminus \{p\})\}^*$.

(D) X has a point-countable k-network.

$(D)_p$ X has a point-countable k-network $\mathcal{P}$ such that if K is compact and $K \subset X \setminus \{p\}$, then $K \subset \cup\mathcal{F} \subset X \setminus \{p\}$ for some finite subfamily $\mathcal{F}$ of $\mathcal{P}$.

(E) X has a point-countable closed k-network.

Now we recall some results which will be used in the sequel

Lemma 2.1 ([1[). *The following properties of a space X are equivalent*

(i) X has a point-countable base;

(ii) X is a k-space satisfying (B);

(iii) $t(X) \leqslant \omega$ and X satisfies (B).

Lemma 2.2 ([3]). *For a space X, we have the following diagram*

$$\begin{array}{ccccc}
 & & (B) & \Longrightarrow & (B)_p \\
 & & \Downarrow\Uparrow_{(2)} & & \Downarrow\Uparrow_{(2)} \\
(A) & \overset{(1)}{\Longleftarrow\!\Longrightarrow} & (C) & \Longrightarrow & (C)_p \\
 & & \Downarrow\Uparrow_{(3)} & & \Downarrow\Uparrow_{(3)} \\
 & & (D) & \Longrightarrow & (D)_p.
\end{array}$$

(1) A cover $\mathcal{P}$ of X is closed,

(2) X is a countably bi-k-space, (3) X is a k-space

Lemma 2.3 ([9[). *Every k-space with a star-countable k-network is a paracompact σ-space.*

Lemma 2.4 ([2]). *Every separable paracompact space is a Lindelöf space.*

Lemma 2.5 ([7]). *If $f : X \to Y$ is a pseudo-open map, and X is a Fréchet space, then so is Y.*

Lemma 2.6 ([3]). *For a space X the following statements are equivalent*

(a) X is a sequence-covering quotient s-image of a metric space;

(b) X is a quotient s-image of a metric space;

(c) X is a k-space satisfying (A).

Remark 2.7. We write

(d) X is a k-space satisfying (E);

(e) X is a compact-covering quotient s-image of a metric space.

Then we have (d) $\Rightarrow$ [(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)] , (e) $\Rightarrow$ [(a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)], and (d) $\Rightarrow$ (e) hold.

Lemma 2.8 ([3]). *Suppose that X is a space satisfying (D) and $f : X \to Y$ is a map. Then either (i) or (ii) implies that Y is a space satisfying (D).*

(i) f is a quotient s-map and X is a Fréchet space;

(ii) f is a perfect map.

Lemma 2.9 ([4]). *Let X be a Fréchet space. Then the following statements are equivalent*

(i) X has a star-countable closed k-network;

(ii) X has a locally countable k-network;

(iii) X has a point-countable separable closed k-network;

(iv) X is a locally separable space satisfying (D);

(v) X has a σ-locally finite closed Lindelöf k-network.

3. The main Results

Lemma 3.1. *Let X be a space having a locally countable k-network. Then for every $x \in X$ there is a Lindelöf neighborhood V of x.*

Proof. Let $\mathcal{P}$ be a locally countable k-network for X. For $x \in X$ there is an open neighbourhood V of x such that V meets only countable many elements of $\mathcal{P}$. Denote $\mathcal{P}_x = \{P \in \mathcal{P} : P \subset V\}$. Then $\mathcal{P}_x$ is countable and $V = \cup\{P : P \in \mathcal{P}_x\}$. Let $\mathcal{U}$ be an any open cover of V. For $y \in V$ there exists $U \in \mathcal{U}$ such that $y \in U$. Since $\mathcal{P}$ is a locally countable k-network for X, there is $P \in \mathcal{P}$ satisfying $y \in P \subset U \cap V$. For $P \in \mathcal{P}_x$ put a

$U_P \in \mathcal{U}$ such that $P \subset U_P$. Since $\mathcal{P}_x$ is countable and $V = \cup\{P : P \in \mathcal{P}_x\}$, it implies that the family $\mathcal{U}_x = \{U_P \in \mathcal{U} : P \in \mathcal{P}_x\}$ is a countable cover of X. Hence, V is Lindelöf.

Lemma 3.2. *Let X be a Fréchet space having a locally countable k-network. Then the following conditions are equivalent*

(i) $f : X \to Y$ is a Lindelöf map;

(ii) $f : X \to Y$ is a s-map.

Proof. (i) $\Rightarrow$ (ii). Suppose that $f : X \to Y$ is a Lindelöf map, and X is a Fréchet space having a locally countable k-network $\mathcal{P}$. For every $y \in Y$, put any $z \in f^{-1}(y)$, by Lemma 3.1 there is an open Lindelöf neighborhood V_z of z such that V_z meets only countably many elements of $\mathcal{P}$. The family $\{V_z : z \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$. Because $f^{-1}(y)$ is Lindelöf, there exists a countable family $\{V_{z_k} : k \geq 1\}$ covering $f^{-1}(y)$ for every $y \in Y$. Putting $U = \bigcup_{k=1}^{\infty} V_{z_k}$ we have $f^{-1}(y) \subset U$, and $\mathcal{Q} = \{P \in \mathcal{P} : P \subset U\}$ is countable. Then it is easy to show that $\mathcal{Q}$ is a countable-network in U. Because every space with a countable-network is hereditarily separable and $f^{-1}(y) \subset U$, it follows that $f^{-1}(y)$ separable. Thus f is a s-map.

(ii) $\Rightarrow$ (i). Suppose that $f : X \to Y$ is a s-map, and X is a Fréchet space having a locally countable k-network. As well-known that every Fréchet space is a k-space. Then by Lemma 2.9 and Lemma 2.3, X is a paracompact σ-space. Since f is continuous, for every $y \in Y$, we have $f^{-1}(y)$ is closed, it implies that $f^{-1}(y)$ is a paracompact subspace of X. Because f is a s-map, by Lemma 2.4, it follows that $f^{-1}(y)$ is Lindelöf. Hence f is a Lindelöf map.

Lemma 3.3. *Let $f : X \to Y$ be a pseudo-open Lindelöf map (or a pseudo-open s-map, or a perfect map), and X a Fréchet space having a locally countable k-network. Then Y is a locally separable space.*

Proof. Let $f : X \to Y$ be a pseudo-open Lindelöf map, and X a Fréchet space having a locally countable k-network. By Lemma 2.9 it implies that X is a locally separable space. For every $y \in Y$, we take $z \in f^{-1}(y)$. Since X is a locally separable space, there exists an open neighborhood V_z of z such that V_z is separable. The family $\{V_z : z \in f^{-1}(y)\}$ is an open cover of $f^{-1}(y)$. Because $f^{-1}(y)$ is Lindelöf, there exists a countable family $\{V_{z_k} : k \geq 1\}$ covering $f^{-1}(y)$. Denoting $U = \bigcup_{k=1}^{\infty} V_{z_k}$ we have $f^{-1}(y) \subset U$ and U is separable. Because f is continuous, it implies that $f(U)$ is a separable subset of Y. Since f is pseudo-open, we get $y \in \mathrm{Int} f(U)$. Thus, $f(U)$ is a separable neighborhood of y, and Y is a locally separable space.

Because every perfect map is pseudo-open Lindelöf and it follows from Lemma 3.2 that the theorem is true for a pseudo-open s-map, or a perfect map.

Theorem 3.4. *For a k-space X we have*

(i) $(E) \Rightarrow (A)$ holds;

(ii) The converse implication is true if X is a locally separable Fréchet space.

Proof. Firstly we shall prove the first assertion. Suppose X is a k-space, and $\mathcal{P}$ is a point-countable closed k-network for X satisfying (E), then we shall prove that $\mathcal{P}$ satisfies (A). Let U be open in X, and let $A \subset U$ such that $A \cap P$ is closed in P for every $P \in \mathcal{P}$ with $P \subset U$, and suppose that A is not closed in U. Then because U is open in X, U is a k-space, so we have $A \cap K_0$ is not closed in K_0 for some compact $K_0 \subset U$. Since $\mathcal{P}$ is a k-network for X, there exists a finite $\mathcal{F} \subset \mathcal{P}$ such that $K_0 \subset \cup\mathcal{F} \subset U$. On the other hand, cover $\mathcal{P}$ is closed. This implies that there exists a $P \in \mathcal{F}$ such that $A \cap P$ is not closed in P. This is a contradiction. Hence we have (A), and $(E) \Rightarrow (A)$ holds.

We now prove the second assertion. Suppose X is a locally separable Fréchet space satisfying (A). Since X satisfies (A), it follows from Lemma 2.2 that X satisfies (D). By Lemma 2.9 it implies that X satisfies (E).

By Lemma 2.1, Lemma 2.2, Lemma 2.9 and Theorem 3.4, we obtain the following

Corollary 3.5. *For a space X, we have the following diagram*

$$
\begin{array}{ccccccc}
 & & (A) & \Longleftarrow (4) & (B) & \Longrightarrow & (B)_p \\
 & & & & \Downarrow\Uparrow_{(2)} & & \Downarrow\Uparrow_{(2)} \\
(E) & \underset{(3)\Longrightarrow}{\Longleftarrow (5)} & (A) & \underset{\Longrightarrow}{\Longleftarrow (1)} & (C) & \Longrightarrow & (C)_p \\
 & & & & \Downarrow\Uparrow_{(3)} & & \Downarrow\Uparrow_{(3)} \\
 & & (A) & \Longleftarrow (5) & (D) & \Longrightarrow & (D)_p.
\end{array}
$$

(1) A cover $\mathcal{P}$ of X is closed or X is a countably bi-k-space,

(2) X is a countably bi-k-space, (3) X is a k-space,

(4) X is a k-space, or $t(X) \leqslant \omega$, (5) X is a locally separable Fréchet space

By Remark 2.7, Lemma 2.9, and using the proof presented in (ii) of Theorem 3.4 we obtain the following

Corollary 3.6. *Let X be a locally separable Fréchet space. Then the following statements are equivalent*

(a) X is a sequence-covering quotient s-image of a metric space;

(b) X is a quotient s-image of a metric space;

(c) *X is a space satisfying (A);*

(d) *X is a space satisfying (E);*

(e) *X is a compact-covering quotient s-image of a metric space.*

(f) *X has a star-countable closed k-network;*

(g) *X has a locally countable k-network;*

(h) *X has a point-countable separable closed k-network;*

(k) *X is a space satisfying (D);*

(l) *X has a σ-locally finite closed Lindelöf k-network.*

We now have a mapping theorem for Fréchet spaces having a locally countable k-network

Theorem 3.7. *Let $f : X \to Y$ be a pseudo-open Lindelöf map (or a pseudo-open s-map, or a perfect map). If X is a Fréchet space having a locally countable k-network, then so does Y.*

Proof. Because every perfect map is a pseudo-open Lindelöf map, and X is a Fréchet space having a locally countable k-network, by Lemma 3.2 we suppose that $f : X \to Y$ is a pseudo-open s-map. Since X is Fréchet, and f is pseudo-open, it follows from Lemma 2.5 that Y is a Fréchet space. Because every locally countable k-network is a point-countable k-network, and every pseudo-open map is quotient, by Lemma 2.8(i) we get that Y has a point-countable k-network.

From Lemma 3.3 it follows that Y is a locally separable space. Hence, Y is a locally separable Fréchet space satisfying (D). By Corollary 3.6, it implies that Y has a locally countable k-network.

From the above theorem we obtain the following corollary

Corollary 3.8. *Let $f : X \to Y$ be a pseudo-open Lindelöf map (or a pseudo-open s-map, or a perfect map). If X is a Fréchet space satisfying one of the following, then so doing Y, respectively.*

(a) *X has a locally countable k-network;*

(b) *X has a star-countable closed k-network;*

(c) *X is a locally separable space satisfying (D);*

(d) *X has a σ-locally finite closed Lindelöf k-network;*

(e) *X has a point-countable separable closed k-network.*

Definition 3.9. A space X is called a *Fréchet hereditarily determined space* (abbrev. *FHD-space*), if X is Fréchet and satisfies (A).

Remark. (i) Every metric space is a FHD-space.

(ii) Every subspace of a FHD-space is a FHD-space.

(iii) If X is a FHD-space, and if $f : X \to Y$ is an open s-map or a pseudo-open map with countable fibers, then so is Y.

Now we give a partial answer for the question in §1.

Theorem 3.10. *If X is a locally separable FHD-space, and $f : X \to Y$ is a pseudo-open Lindelöf map (or a pseudo-open s-map , or a perfect map), then Y is a locally separable FHD-space.*

Proof. Because every perfect map is pseudo-open Lindelöf s-map, we can suppose that X is a FHD-space and $f : X \to Y$ is a pseudo-open s-map or a pseudo-open Lindelöf map. Since X is Fréchet, and $f : X \to Y$ is a pseudo-open map, it follows from Lemma 2.5 that Y is Fréchet. On the other hand, since X is a Fréchet space satisfying (A), by Corollary 3.6 it implies that X is a locally separable space satisfying (D). It follows from Corollary 3.8 that Y is a locally separable space satisfying (D). Using Corollary 3.6 again we obtain Y is a space satisfying (A). Hence, Y is a locally separable FDH-space.

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