CONDITIONS FOR THE APPROXIMATED ANALYTICAL SOLUTION OF A PARAMETRIC OSCILLATION PROBLEM DESCRIBED BY THE MATHIEU EQUATION

Nguyen Dang Bich

Institute of Building Science and Technology

Ngo Dinh Bao Nam

College of Sciences - Vietnam National University

Abstract: This paper presents the scientific detailed basis and the involving conditions for finding the approximated analytical solution of a parametric oscillation problem described by the Mathieu equation.

1. Introduction

The method for finding an approximated analytical solution of a parametric oscillation problem described by the Mathieu equation has been presented in [1]. However, the scientific basis and other involving conditions have not been detailly defined except for the necessary conditions. This paper investigates in details the scientific basis and the relative relationship among parameters in the method for finding the approximated solution presented in [1].

2. The scientific basis for finding the approximated analytical solution of a parametric oscillation problem

Consider a 2^{nd} order differential Mathieu equation

$$\ddot{u} + \omega^2 h(t) u = 0, \tag{2.1}$$

in which: $h(t)$ – a periodic function, $\ddot{u}$ stands for the 2^{nd} derivative of $h(t)$ with respect to t.

The problem can be stated as finding the acceptable form of $h(t)$ such that (2.1) has an analytical solution. And then we desire this analytical solution to be an approximated solution of the following equation

$$\ddot{u} + \omega^2 (k + p\cos\omega t) u = 0, \tag{2.2}$$

what related conditions must be found.

1.1. Form of function h(t)

Consider the supplementary equation

$$\dot{x} = ax^2, \tag{2.3}$$

in which: $a(t)$ – any continuous non-zero function of t.

By rearranging parameters we have

$$ax = -\frac{\dot{u}}{u} - \frac{1}{2}\frac{\dot{a}}{a}. \tag{2.4}$$

From that it yields:

$$\dot{a}x + a\dot{x} = -\frac{\ddot{u}}{u} + \frac{\dot{u}^2}{u^2} - \frac{d}{dt}\left(\frac{1}{2}\frac{\dot{a}}{a}\right). \tag{2.5}$$

Diminish $\frac{\dot{u}}{u}$ from (2.4), (2.5) we have

$$a\dot{x} = a^2x^2 + \left[-\frac{\ddot{u}}{u} - \frac{d}{dt}\left(\frac{1}{2}\frac{\dot{a}}{a}\right) + \left(\frac{1}{2}\frac{\dot{a}}{a}\right)^2\right]. \tag{2.6}$$

Replacing $\dot{x}$ determined by (2.3) into (2.6) it yields

$$\ddot{u} + \left[\frac{d}{dt}\left(\frac{1}{2}\frac{\dot{a}}{a}\right) - \left(\frac{1}{2}\frac{\dot{a}}{a}\right)^2\right]u = 0. \tag{2.7}$$

Compare (2.7), (2.1) the form of $h(t)$ can be considered as

$$\omega^2 h(t) = \frac{d}{dt}\left(\frac{1}{2}\frac{\dot{a}}{a}\right) - \left(\frac{1}{2}\frac{\dot{a}}{a}\right)^2. \tag{2.8}$$

The solution of (2.3) can be expressed as

$$ax = -\frac{a}{\int_0^t a\,dt + C_1}, \tag{2.9}$$

in which: C_1 – integral constant.

Substituting ax calculated from (2.9) into (2.4) we have

$$\frac{\dot{u}}{u} = \frac{a}{\int_0^t a\,dt + C_1} - \frac{1}{2}\frac{\dot{a}}{a}, \tag{2.10}$$

From that it yields

$$u = \frac{1}{a^{1/2}}\left[C_1 + C_2\int_0^t a\,dt\right], \tag{2.11}$$

in which: C_2 – integral constant.

It can be stated that if the function $h(t)$ has a form of (2.8), then (*1*) has an exact solution in the form of (2.11). From (2.11) and (2.8) it can be inferred that if $a(t)$ is a continuous non-zero function, then u continuously depends on $a(t)$.

Because $a(t)$ is any continuous function, then in investigation of (2.1) when $h(t)$ is a periodic function, $a(t)$ can be chosen in the form of a periodic function as

$$a(t)=\frac{(\lambda+\beta\cos\omega t)^2}{(\omega^2+\alpha\lambda+\alpha\beta\cos\omega t)^2}, \tag{2.12}$$

in which: $\frac{\lambda}{\beta}$, $\frac{\omega^2}{\alpha\beta}$ - parameters that need to be defined during the solving procedure.

Substitute (2.12) into (2.8), (2.11) we have

$$h(t)=\frac{2\gamma\omega^2}{(\lambda+\beta\cos\omega t)^2}-\frac{2\gamma\alpha+3\lambda}{\lambda+\beta\cos\omega t}+\frac{\omega^2+3\alpha\lambda+2\gamma\alpha^2}{\omega^2+\alpha\lambda+\alpha\beta\cos\omega t}, \tag{2.13}$$

where:

$$\frac{\gamma\omega^2}{\beta^2}=\frac{\lambda^2}{\beta^2}-1,\ \frac{\gamma\alpha}{\beta}=\left(\frac{\lambda^2}{\beta^2}-1\right)\frac{\alpha\beta}{\omega^2}$$

Thus, the solution of (2.1) now is of the form

$$u=\frac{\omega^2+\alpha\lambda+\alpha\beta\cos\omega t}{\lambda+\beta\cos\omega t}\left[C_1+C_2\int_0^t\frac{(\lambda+\beta\cos\omega\tau)}{(\omega^2+\alpha\lambda+\alpha\beta\cos\omega\tau)^2}d\tau\right]. \tag{2.14}$$

Formulas (2.13) and (2.14) are exact solutions presented in [1].

1.2. Approximated solution

Equation (2.2) with the condition $u \neq 0$ can be rewritten as

$$\frac{\ddot{u}}{u}+\omega^2(k+p\cos\omega t)=0. \tag{2.15}$$

Substitute (2.14) into (2.15) and denote the left hand side of (2.15) by $f(t)$ we have

$$f(t)=\omega^2\left[(k+p\cos\omega t)-\left(\frac{2\gamma\omega^2}{(\lambda+\beta\cos\omega t)^2}-\frac{2\alpha\gamma+3\lambda}{\lambda+\beta\cos\omega t}+\frac{\omega^2+3\alpha\lambda+2\gamma\alpha^2}{\omega^2+\alpha\lambda+\alpha\beta\cos\omega t}\right)\right]$$

Denote

$$g(t)=k+p\cos\omega t, \tag{2.16}$$

taking into account (2.13), the $f(t)$ function can be written as

$$f(t)=\omega^2\left[g(t)-h(t)\right] \tag{2.17}$$

If $f(t)=0\ \forall t$, then (2.14) becomes an exact solution of (2.15).

If $f(t)\approx 0\ \forall t$, that means $h(t) \approx g(t)$ with every t, then (2.14) can be considered an approximated solution of (2.15). The error of this approximated solution depends on the error of the approximation of $h(t)$ to $g(t)$.

Therefore, the scientific basis of the method is: The solution of (2.1) continuously depends on the function $h(t)$, hence when $h(t)$ is approximated by $g(t)$ with every t then the exact solution of (2.1) becomes an approximated solution of (2.15).

3. Relating conditions for parameters

3.1. Conditions in [1]

It has been stated in [1] that for $h(t)$ can be approximated by $g(t)$ with every t, the following equations and inequations should be satisfied

a. The equations

$$\frac{\omega^2}{\alpha\beta} = -\frac{k + p\dfrac{\lambda}{\beta}}{\dfrac{\dfrac{\lambda}{\beta}}{\dfrac{\lambda^2}{\beta^2} - 1} + p} \tag{3.1}$$

$$\left(\frac{\dfrac{\lambda}{\beta}}{\dfrac{\lambda^2}{\beta^2} - 1} + p\right)\left(\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} - 1\right)\left(\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} + 1\right) = \frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} \tag{3.2}$$

b. The inequations

$$\left|\frac{\lambda}{\beta}\right| > 1, \quad \left|\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta}\right| > 1, \tag{3.3}$$

$$\left[\frac{\lambda^2}{\beta^2} + \left(8 + \frac{\omega^2}{\alpha\beta}\right)\frac{\lambda}{\beta} - \left(1 - 4\frac{\omega^2}{\alpha\beta}\right)\right]\left[\frac{\lambda^2}{\beta^2} - \left(8 + \frac{\omega^2}{\alpha\beta}\right)\frac{\lambda}{\beta} - \left(1 + 4\frac{\omega^2}{\alpha\beta}\right)\right] > 0, \tag{3.4}$$

3.2. Supplementary equations

From (3.1), (3.2) it yields

$$\left(k^2 - p^2 - k\right)\frac{\lambda^2}{\beta^2} - 2p\frac{\lambda}{\beta} - \left(k^2 - p^2 + k\right) = 0, \tag{3.5}$$

$$\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} = -\frac{2p(k-1)\dfrac{\lambda}{\beta} + k^2 + p^2 - k}{\left(k^2 + p^2 - k\right)\dfrac{\lambda}{\beta} + 2pk}. \tag{3.6}$$

Denote
$$\chi = \frac{2p(k-1)\frac{\lambda}{\beta} + k^2 + p^2 - k}{\left(k^2 + p^2 - k\right)\frac{\lambda}{\beta} + 2pk}. \tag{3.7}$$

Based on (3.5) it can be proved that
$$\chi = \frac{2p(k-1)\frac{\lambda}{\beta} + k^2 + p^2 - k}{\left(k^2 + p^2 - k\right)\frac{\lambda}{\beta} + 2pk} = \frac{k^2 - p^2 - k}{k^2 - p^2 + k}\frac{\lambda}{\beta}. \tag{3.8}$$

Hence
$$\frac{\omega^2}{\alpha\beta} = -\frac{2p\left(k^2 - p^2\right)}{k^2 - p^2 + k}\frac{\lambda}{\beta}. \tag{3.9}$$

The condition $\left|\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta}\right| > 1$ can be replaced by $|\chi| > 1$ when (3.6) and (3.8) are taken into account
$$\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta} = -\chi = -\frac{k^2 - p^2 - k}{k^2 - p^2 + k}\frac{\lambda}{\beta} \tag{3.10}$$

Substitute $\frac{\lambda}{\beta}$ calculated from (3.10) into (3.5) we have
$$\left(k^2 - p^2 + k\right)\chi^2 - 2p\chi - \left(k^2 - p^2 - k\right) = 0. \tag{3.11}$$

Equations (3.5), (3.11) are the supplementary equations for finding the conditions satisfying the inequation (3.3).

3.3. The condition $\left|\frac{\lambda}{\beta}\right| > 1$

The solution of (3.5) can be written as
$$\frac{\lambda}{\beta} = \frac{p \pm \sqrt{\Delta}}{k^2 - p^2 - k}, \tag{3.12}$$

where
$$\Delta = \left(k^2 - p^2\right)\left(k^2 - p^2 - 1\right). \tag{3.13}$$

It is observed that $\Delta > 0$ when $k^2 - p^2 < 0$ or $k^2 - p^2 > 1$. (3.14)

Based on (3.12), the first condition of (3.3) leads to the following condition
$$\left|\frac{p \pm \sqrt{\Delta}}{k^2 - p^2 - k}\right| > 1. \tag{3.15}$$

From the above, it can be seen that when

$$k^2 - p^2 < 0\,; \tag{3.16}$$

equation (3.5) has one solution $\left|\frac{\lambda}{\beta}\right| > 1$, and when

$$k^2 - p^2 > 1\,,\; k > 1\,, \tag{3.17}$$

equation (3.5) has two solutions $\left|\frac{\lambda}{\beta}\right| > 1$.

3.4. The condition $\left|\frac{\omega^2}{\alpha\beta} + \frac{\lambda}{\beta}\right| > 1$ *or* $|\chi| > 1$

The solution of (3.11) can be written as

$$\chi = \frac{p \pm \sqrt{\Delta}}{k^2 - p^2 + k}. \tag{3.18}$$

The condition of $|\chi| > 1$ leads to

$$\left|\frac{p \pm \sqrt{\Delta}}{k^2 - p^2 + k}\right| > 1 \tag{3.19}$$

From the above it can be seen that when

$$k^2 - p^2 < 0\,, \tag{3.20}$$

equation (3.11) has one solution $|\chi| > 1$; and when

$$k^2 - p^2 > 1\,,\; k < -1\,, \tag{3.21}$$

equation (3.11) has two solutions $|\chi| > 1$.

3.5. Necessary and sufficient conditions for (3.3) to be satisfied simultaneously

For the two conditions in (3.3) to be satisfied simultaneously, it is essential that the conditions pairs of (3.16), (3.20) and (3.17), (3.21) must be satisfied. It is observed that there is only one condition for those requirements to be met, that is

$$k^2 - p^2 < 0\,. \tag{3.22}$$

With this condition one solution $\frac{\lambda}{\beta}$ of the equation (3.5) and one solution χ of equation (3.11) have the absolute value greater than 1.

3.6. Necessary condition for (3.4)

Based on (3.9), the condition (3.4) can be rewritten in the form

$$\frac{4}{\left(k^2-p^2+k\right)^2}\left(k^2-p^2+k+p\frac{\lambda}{\beta}-4k\frac{\lambda}{\beta}\right)\left(k^2-p^2+k+p\frac{\lambda}{\beta}+4k\frac{\lambda}{\beta}\right)>0 \tag{3.23}$$

and on (3.5), from (3.23) it yields

$$\frac{4}{\left(k^2-p^2+k\right)^2}\left[\left(k^2-p^2\right)\left(k^2-p^2-1\right)-16k^2\right]\frac{\lambda^2}{\beta^2}>0 \tag{3.24}$$

Hence, the necessary condition for (3.4) can be derived as

$$\left(k^2-p^2\right)\left(k^2-p^2-1\right)-16k^2>0 \tag{3.25}$$

It has been stated in [1] that for $h(t)$ and $g(t)$ to be approximated eachother with every t, then the formula expressing the subtraction (2.17) between $g(t)$ and $h(t)$:

$$f(y)=y^3-3\frac{\lambda}{\beta}y^2-\left(7\frac{\lambda^2}{\beta^2}+3\frac{\omega^2}{\alpha\beta}\frac{\lambda}{\beta}+2\right)y+\left(\frac{\lambda^3}{\beta^3}+\frac{\omega^2}{\alpha\beta}\frac{\lambda^2}{\beta^2}-6\frac{\lambda}{\beta}-4\frac{\omega^2}{\alpha\beta}\right) \tag{3.26}$$

cannot be vanished in the interval [-1, 1], where denote $y=\cos\omega t$.

The mentioned approximation requirement must be satisfied, then

$$f(1)f(-1)>0 \tag{3.27}$$

is the necessary condition, and $f(y)$ not vanished in the interval [-1, 1] is the sufficient condition.

The condition (3.27) can lead to the condition (3.4), so that the necessary condition (3.25) is found, and now it must be to find the sufficient condition.

3.7. Sufficient condition for (3.4)

From (3.26) it can be inferred that

$$f'(y)=\frac{df}{dy}=3\left[y^2-2\frac{\lambda}{\beta}y-\left(\frac{7}{3}\frac{\lambda^2}{\beta^2}+\frac{\omega^2}{\alpha\beta}\frac{\lambda}{\beta}+\frac{2}{3}\right)\right] \tag{3.28}$$

$$f''(y)=\frac{d^2f}{d^2y}=6\left(y-\frac{\lambda}{\beta}\right). \tag{3.29}$$

Based on the condition (3.3), from (3.29) it can be inferred that the sign of $f''(y)$ remains unchanged in the interval [-1,1], that leads to the monotone of $f'(y)$ in the interval [-1,1].

Introduce an additional condition

$$f'(1)f'(-1)>0, \tag{3.30}$$

associating with the monotone condition of $f'(y)$ in the interval [-1,1], it can be inferred that the sign $f'(y)$ remains unchanged in the interval [-1,1], therefore $f(y)$ is monotonic in the interval [-1,1].

From the condition (3.27) and the monotone condition of $f(y)$ in the interval [-1,1], it can be stated that the sign of $f(y)$ remains unchanged in the interval [-1,1], or in other words, $f(y)$ does not vanish in the interval [-1,1].

Based on (3.28), the condition (3.30) leads to

$$\left[\left(\frac{7}{3}\frac{\lambda^2}{\beta^2}+\frac{\omega^2}{\alpha\beta}\frac{\lambda}{\beta}-\frac{1}{3}\right)+2\frac{\lambda}{\beta}\right]\left[\left(\frac{7}{3}\frac{\lambda^2}{\beta^2}+\frac{\omega^2}{\alpha\beta}\frac{\lambda}{\beta}-\frac{1}{3}\right)-2\frac{\lambda}{\beta}\right]>0 \tag{3.31}$$

Associating with (3.9) it yields

$$\left|\left(\frac{4}{3}-\frac{k^2-p^2-k}{k^2-p^2+k}\right)\frac{\lambda^2}{\beta^2}-\frac{1}{3}\right|>\left|2\frac{\lambda}{\beta}\right| \tag{3.32}$$

This is the sufficient condition for (3.4) to be satisfied.

4. Conclusion

In summary, for $h(t)$, $g(t)$ to be approximated each other with every t, there are three relating conditions for parameters, that is the necessary and sufficient conditions (3.22), (3.25) and (3.32).

These conditions provide to find an approximated analytical solution to a parametric oscillation problem described by the Mathew's equation.

Acknowledgements The paper is completed with the financial support from the National Council for Natural Science.

References

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