REPRESENTATIONS OF SOME MD_5-GROUP VIA DEFORMATION QUANTIZATION

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Abstract. The present paper is a continuation of Nguyen Viet Hai's ones [3], [4], [6], [7]. Specifically, the paper is concerned with the subclass of connected and simply connected MD_5-groups such that their MD_5-algebras $\mathcal{G}$ have the derived ideal $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] \cong \mathbb{R}^3$. We show that the ξ-representations of these MD_5-algebras result from the quantization of the Poisson bracket on the coalgebra in canonical coordinates.

Introduction

In 1980, studying the Kirillov's method of orbits (see [9]), Do Ngoc Diep introduced the class of Lie groups type MD: n-dimensional Lie group G is called an MD_n-group iff its co-adjoint orbits have zero or maximal dimension (see [2], [6]). The corresponding Lie algebra of MD_n-group are called MD_n-algebra. With $n = 4$, all MD_4-algebras were listed by Dao Van Tra in 1984 (see [15]). The description of the geometry of K-orbits of all indecomposable MD_4-groups, the topological classification of foliations formed by K-orbits of maximal dimension given by Le Anh Vu in 1990 (see [11], [12]). In 2000, the author introduced deformation quantization on K-orbits of groups $Aff(\mathbb{R}), Aff(\mathbb{C})$ (see [3], [4]). In 2001, the author also introduced quantum co-adjoint orbits of MD_4-groups and obtained all unitary irreducible representations of MD_4-groups (see [6], [7]). Until now, no complete classification of MD_n-algebras with $n \geq 5$ is known. Recently, Le Anh Vu continued study MD_5-algebras $\mathcal{G}$ in cases $\mathcal{G}^1 := [\mathcal{G}, \mathcal{G}] \equiv \mathbb{R}^k$; $k = 1, 2, 3$, (see [13]) and their MD_5-groups. In the present paper we will solve problem on deformation quantization for MD_5-groups and MD_5-algebras $\mathcal{G}$ in case $\mathcal{G}^1 \equiv \mathbb{R}^3$. The paper is organized as follows. In Section 1, we recall the co-adjoint representation, K-orbits of a Lie group, Darboux coordinates and the notion of the quantization of K-orbits. In Section 2 we list indecomposable MD_5-algebras $\mathcal{G}$ which $\mathcal{G}^1 \equiv \mathbb{R}^3$. Finally, Section 3 is devoted to the computation quantum operators of MD_5-groups corresponding to these MD_5-algebras.

1. Basic definitions and Preliminary results

1.1. The co-adjoint Representation and K-orbits of a Lie Group. Let G be a Lie group. We denote by $\mathcal{G}$ the Lie algebra of G and by $\mathcal{G}^*$ the dual space of $\mathcal{G}$. To each element $g \in G$ we associate an automorphism

$$A_g : G \longrightarrow G, \quad x \longmapsto A_g(x)| = gxg^{-1}.$$

A_g induces the tangent map $A_{g_*} : \mathcal{G} \longrightarrow \mathcal{G}, X \longmapsto A_{g_*}(X)| = \frac{d}{dt}[g.\exp(tX)g^{-1}]\,|_{t=0}$.

Definition 1.1. The action $Ad : G \longrightarrow Aut(\mathcal{G}), \quad g \longmapsto Ad(g)| = A_{g_*}$, is called the adjoint representation of G in $\mathcal{G}$. The action $K : G \longrightarrow Aut(\mathcal{G}^*), \quad g \longmapsto K_g$ such that $\langle K_g F, X\rangle := \langle F, Ad(g^{-1})X\rangle, \quad (g \in G, F \in \mathcal{G}^*, X \in \mathcal{G})$, is called the co-adjoint representation of G in $\mathcal{G}^*$.

Definition 1.2. Each orbit of the co-adjoint representation of G is called a co-adjoint orbit or a K-orbit of G.

Thus, for every $\xi \in \mathcal{G}^*$, the K-orbit containing ξ is defined as follows

$$\mathcal{O}_\xi = K(G)\xi := \{K(g)\xi| \quad g \in G, \xi \in \mathcal{G}^*\}.$$

Note that the dimension of a K-orbit of G is always even.

1.2. Darboux coordinates on the orbit $\mathcal{O}_\xi$. We let ω_ξ denote the Kirillov form on the orbit $\mathcal{O}_\xi$. It defines a symplectic structure and acts on the vectors a and b tangent to the orbit as $\omega_\xi(a,b) = \langle \xi, [\alpha,\beta]\rangle$, where $a = ad^*_\alpha\xi$ and $b = ad^*_\beta\xi$. The restriction of Poisson brackets to the orbit coincides with the Poisson bracket generated by the symplectic form ω_ξ. According to the well-known Darboux theorem, there exist local canonical coordinates (Darboux coordinates) on the orbit $\mathcal{O}_\xi$ such that the form ω_ξ becomes $\omega_\xi = dp_k \wedge dq^k; \quad k = 1, \ldots, \frac{1}{2}\dim O_\xi = \frac{n-r}{2} - s$, where s is the degeneration degree of the orbit (see [7]). Let be $F \in \mathcal{O}_\xi, F = f_i e^i$. It can be easily seen that the trasition to canoniccal Darboux coordinates $(f_i) \mapsto (p_k, q^k)$ amounts to constructing analytic functions $f_i = f_i(q,p,\xi)$ of variables (p,q) satisfying the conditions

$$f_i(0,0,\xi) = \xi_i;$$

$$\frac{\partial f_i(q,p,\xi)}{\partial p_k}\frac{\partial f_j(q,p,\xi)}{\partial q^k} - \frac{\partial f_j(q,p,\xi)}{\partial p_k}\frac{\partial f_i(q,p,\xi)}{\partial q^k} = C^l_{ij} f_l(q,p,\xi);$$

We choose the the canonical Darboux coordinates with impulse p's-coordinates. From this we can deduce that the Kirillov form ω_ξ locally are canonical and every element

$A \in \mathcal{G} = \text{Lie}G$ can be considered as a function $\tilde{A}$ on $\mathcal{O}_\xi$, linear on p's-coordinates, i.e. *There exists on each coadjoint orbit a local canonical system of Darboux coordinates, in which the Hamiltonian function* $\tilde{A} = a_i(q,p,\xi)e^i$, $A \in \mathcal{G}$, *are linear on p's impulsion coordinates and in theses coordinates,*

$$a_i(q,p,\xi) = \alpha_i^k(q)p_k + \chi_i(q,\xi); \quad \text{rank}\alpha_i^k(q) = \frac{1}{2}\dim\mathcal{O}_\xi. \tag{1}$$

1.3. The operators $\ell_i(q,\partial_q)$. We now view the transition functions $f_i(q,p;\xi)$ to local canonical coordinates as symbols of operators that are defined as follows: the variables p_k are replaced with derivatives, $p_k \to \hat{p}_k \equiv -i\hbar\frac{\partial}{\partial q^k}$, and the coordinates of a covector f_i become the linear operators

$$f_i(q,p;\xi) \to \hat{f}_i\left(q, -i\hbar\frac{\partial}{\partial q};\xi\right) \tag{2}$$

(with $\hbar$ being a positive real parameter). We require that the operators $\hat{f}_i$ satisfy the commutation relations $\frac{i}{\hbar}[\hat{f}_i,\hat{f}_j] = C_{ij}^l\hat{f}_l$. If the transition to the canonical coordinates is linear, i.e., a normal polarization exists for orbits of a given type, it is obvious that

$$\hat{f}_i = -i\hbar\alpha_i^k(q)\frac{\partial}{\partial q^k} + \chi_i(q,\xi) \tag{3}$$

With Hamiltonian function $\tilde{A} = a_i(q,p,\xi)e^i$, $A \in \mathcal{G}$, the operators $\hat{a}_i$ as shown by evidence. We introduce the operators

$$\ell_k(q,\partial_q) \equiv \frac{i}{\hbar}\hat{a}_k(q,p;\xi). \tag{4}$$

It is obvious that $[\ell_i,\ell_j] = C_{ij}^k\ell_k$

Definition 1.3. Let $f_i = f_i(q,p;\xi)$ be a transition to canonical coordinates on the orbit $\mathcal{O}_\xi$ of the Lie algebra $\mathcal{G}$. The operators $\ell_i(q,\partial_q)$ is called the representation (the ξ-representation) of the Lie algebra $\mathcal{G}$.

2. A Subclass of Indecomposable MD_5-Algebras

From now on, G will denote a connected simply-connected solvable Lie group of dimension 5. The Lie algebra of G is denoted by $\mathcal{G}$. We always choose a fixed basis (X,Y,Z,T,S) in $\mathcal{G}$. Then Lie algebra $\mathcal{G}$ isomorphic to $\mathbb{R}^5$ as a real vector space. The notation $\mathcal{G}^*$ will mean the dual space of $\mathcal{G}$. Clearly $\mathcal{G}^*$ can be identified with $\mathbb{R}^5$ by fixing in it the basis (X^*,Y^*,Z^*,T^*,S^*) dual to the basis (X,Y,Z,T,S). Note that for any MD_n - algebra $\mathcal{G}_0$ $(0 < n < 5)$, the direct sum $\mathcal{G} = \mathcal{G}_0 \oplus \mathbb{R}^{5-n}$ of $\mathcal{G}_0$ and the commutative Lie algebra $\mathbb{R}^{5-n}$ is a MD_5-algebra. It is called a *decomposable* MD_5 - algebra, the study of which can be directly reduced to the case of MD_n - algebras with $(0 < n < 5)$. Therefore, we will restrict on the case of *indecomposable* MD_5 - algebras.

2.1. List of considered indecomposable MD_5 - Algebras. We consider of solvable Lie algebras of dimension 5 which are listed in [13]: $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$, $\mathcal{G}_{5,3,2(\lambda)}$, $\mathcal{G}_{5,3,3(\lambda)}$ $\mathcal{G}_{5,3,4}$, $\mathcal{G}_{5,3,5(\lambda)}$, $\mathcal{G}_{5,3,6(\lambda)}$, $\mathcal{G}_{5,3,7}$, $\mathcal{G}_{5,3,8(\lambda,\varphi)}$. Each algebra $\mathcal{G}$ from this set has

$$\mathcal{G}^1 = [\mathcal{G},\mathcal{G}] = \mathbb{R}.Z \oplus \mathbb{R}.T \oplus \mathbb{R}.S \equiv \mathbb{R}^3;\ [X,Y] = Z;\ ad_X = 0.$$

The operator $ad_X \in \mathrm{End}(\mathcal{G}^1) \equiv \mathrm{Mat}(3,\mathbb{R})$ is given as follows:

- $\mathcal{G}_{5,3,1(\lambda_1,\lambda_2)} : ad_Y = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0,1\},\ \lambda_1 \neq \lambda_2$

- $\mathcal{G}_{5,3,2(\lambda)} : ad_Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix};$

- $\mathcal{G}_{5,3,3(\lambda)} : ad_Y = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \lambda \in \mathbb{R} \setminus \{0,1\};$

- $\mathcal{G}_{5,3,4} : ad_Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$

- $\mathcal{G}_{5,3,5(\lambda)} : ad_Y = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix};$

- $\mathcal{G}_{5,3,6(\lambda)} : ad_Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \lambda \in \mathbb{R} \setminus \{0,1\};$

- $\mathcal{G}_{5,3,7} : ad_Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix};$

- $\mathcal{G}_{5,3,8(\lambda,\varphi)} : ad_Y = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & \lambda \end{pmatrix}; \quad \lambda \in \mathbb{R} \setminus \{0\},\ \varphi \in (0,\pi).$

2.2. Remarks. We obtain a set of connected and simply-connected solvable Lie groups corresponding to the set of Lie algebras listed above. For convenience, each such Lie group is also denoted by the same indices as its Lie algebra. For example, $G_{5,3,6(\lambda)}$ is the connected and simply-connected Lie group corresponding to $\mathcal{G}_{5,3,6(\lambda)}$. We will describe quantum operators of seven exponential MD_5-groups (except for the $G_{5,3,8(\lambda,\varphi)}$) in the next section.

3. Quantum operators of the considered Lie algebras

Throughout this section, G will denote one of the groups: $G_{5,3,1(\lambda_1,\lambda_2)}$, $G_{5,3,2(\lambda)}$, $G_{5,3,3(\lambda)}$, $G_{5,3,4}$, $G_{5,3,5(\lambda)}$, $G_{5,3,6(\lambda)}$, $G_{5,3,7}$ and $\mathcal{G}$ is its Lie algebra, $\mathcal{G} =< X, Y, Z, T, S > \cong \mathbb{R}^5$. We identify its dual vector space $\mathcal{G}^*$ with $\mathbb{R}^5$ with the help of the dual basis X^*, Y^*, Z^*, T^*, S^* and with the local coordinates as $(\alpha, \beta, \gamma, \delta, \epsilon)$. Thus, the general form of an element of $\mathcal{G}$ is $U = aX + bY + cZ + dT + fS, a, b, c, d, f \in \mathbb{R}$ and the general form of an element of $\mathcal{G}^*$ is $\xi = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^* + \epsilon S^*$. Because the group G is exponential (see [2]), for $\xi \in \mathcal{G}^*$, we have

$$\mathcal{O}_\xi = \{K(\exp(U)\xi | U \in \mathcal{G}\}.$$

Using Maple 9.5, we will compute quatum operators $\ell_A(q, \partial_q)$ for each considered group (except for the $G_{5,3,8(\lambda,\varphi)}$).

3.1. Group $G = G_{5,3,1(\lambda_1,\lambda_2)}$

$$ad_Y = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}; [X, Y] = Z; \lambda_1, \lambda_2 \in \mathbb{R} - \{0, 1\},\ \lambda_1 \neq \lambda_2, ad_X = 0$$

Let $U = aX + bY + cZ + dT + fS$ be an arbitrary of $\mathcal{G}$, where $a, b, c, d, f \in \mathbb{R}$. Upon Maple 9.5, we get:

$$ad_U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -b & a - c\lambda_1 & b\lambda_1 & 0 & 0 \\ 0 & -d\lambda_2 & 0 & b\lambda_2 & 0 \\ 0 & -f & 0 & 0 & b \end{pmatrix},$$

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{(e^{b\lambda_1}-1)b}{b\lambda_1} & \frac{(a-c\lambda_1)(e^{b\lambda_1}-1)}{b\lambda_1} & e^{b\lambda_1} & 0 & 0 \\ 0 & -d\frac{(e^{b\lambda_2}-1)\lambda_2}{b\lambda_2} & 0 & e^{b\lambda_2} & 0 \\ 0 & -f\frac{(e^b-1)}{b} & 0 & 0 & e^b \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\sum_{n=1}^{\infty} \frac{b^n\lambda_1^{n-1}}{n!} & (a - c\lambda_1)\sum_{n=1}^{\infty} \frac{(b\lambda_1)^{n-1}}{n!} & e^{b\lambda_1} & 0 & 0 \\ 0 & -d\sum_{n=1}^{\infty} \frac{b^{n-1}\lambda_1^n}{n!} & 0 & e^{b\lambda_2} & 0 \\ 0 & -f\sum_{n=1}^{\infty} \frac{b^{n-1}}{n!} & 0 & 0 & e^b \end{pmatrix}$$

Thus, $\xi_U = (x, y, z, t, s)$ is given as follows:

$$x = \alpha - \gamma\frac{(e^{b\lambda_1} - 1)b}{b\lambda_1} = \alpha - \gamma\sum_{n=1}^{\infty}\frac{b^n{\lambda_1}^{n-1}}{n!};$$

$$y = \beta + \gamma\frac{(a - c\lambda_1)(e^{b\lambda_1} - 1)}{b\lambda_1} - \delta d\frac{(e^{b\lambda_2} - 1)\lambda_2}{b\lambda_2} - \epsilon f\frac{(e^b - 1)}{b}$$

$$= \beta + \gamma(a - c\lambda_1)\sum_{n=1}^{\infty}\frac{(b\lambda_1)^{n-1}}{n!} - \delta d\sum_{n=1}^{\infty}\frac{b^{n-1}{\lambda_1}^n}{n!} - \epsilon f\sum_{n=1}^{\infty}\frac{b^{n-1}}{n!};$$

$$z = \gamma e^{b\lambda_1};$$

$$t = \delta e^{b\lambda_2};$$

$$s = \epsilon e^b.$$

From this,

- If $\gamma = \delta = \epsilon = 0$ then $\mathcal{O}_\xi = \mathcal{O}^1 = \{(\alpha, \beta, 0, 0, 0)\}$, (K-orbit of dimension zero).
- The set $\gamma = \delta = 0, \epsilon \neq 0$ is a union of 2-dimensional co-adjoint orbits, which are *half-planes* $\mathcal{O}_\xi = \mathcal{O}^2 = \{(\alpha, y, 0, 0, s) | \epsilon s > 0\}$,
- The set $\gamma = 0, \delta \neq 0, \epsilon = 0$ is a union of 2-dimensional co-adjoint orbits, which are *half-planes* $\mathcal{O}_\xi = \mathcal{O}^3 = \{(\alpha, y, 0, t, 0) | \delta t > 0\}$.
- If $\gamma = 0, \delta \neq 0, \epsilon \neq 0$ then we obtain a 2-dimensional cylinder

$$\mathcal{O}_\xi = \mathcal{O}^4 = \{(\alpha, y, 0, t, s) | \epsilon^{\lambda_2} t = \delta s^{\lambda_2}, \epsilon s > 0, \delta t > 0\}.$$

- The set $\gamma \neq 0, \delta = \epsilon = 0$ is a union of 2-dimensional co-adjoint orbits, which are *half-planes* $\mathcal{O}_\xi = \mathcal{O}^5 = \{(x, y, z, 0, 0) | \lambda_1 x = \lambda_1\alpha + \gamma - z, \gamma z > 0\}$.
- If $\gamma \neq 0, \delta = 0, \epsilon \neq 0$ then we obtain a 2-dimensional cylinder

$$\mathcal{O}_\xi = \mathcal{O}^6 = \{(x, y, z, 0, s) | \lambda_1 x = \lambda_1\alpha + \gamma - z, \lambda_1 x = \lambda_1\alpha + \gamma(1 - (\frac{s}{\epsilon})^{\lambda_1}), \quad \epsilon s > 0\}.$$

- If $\gamma \neq 0, \delta \neq 0, \epsilon = 0$ then $\mathcal{O}_\xi = \mathcal{O}^7$ is a 2-dimensional cylinder.

$$\mathcal{O}^7 = \{(x, y, z, t, 0) | \lambda_1 x = \lambda_1\alpha + \gamma - z, \lambda_1 x = \lambda_1\alpha + \gamma(1 - (\frac{t}{\delta})^{\frac{\lambda_1}{\lambda_2}}), \delta t > 0\}.$$

- Last, if $\gamma \neq 0, \delta \neq 0, \epsilon \neq 0$ then we also obtain a 2-dimensional cylinder

$$\Omega_\xi = \mathcal{O}^8 = \{(x, y, z, t, s) | \lambda_1 x = \lambda_1\alpha + \gamma - z, \lambda_1 x = \lambda_1\alpha + \gamma(1 - (\frac{s}{\epsilon})^{\lambda_1}),$$
$$t = \delta(\frac{s}{\epsilon})^{\lambda_2}, \epsilon s > 0\}$$

Thus, $\mathcal{O}_\xi = \mathcal{O}^1 \cup \mathcal{O}^2 \cup \mathcal{O}^3 \cup \mathcal{O}^4 \cup \mathcal{O}^5 \cup \mathcal{O}^6 \cup \mathcal{O}^7 \cup \mathcal{O}^8$.

3.1.1. Hamiltonian functions in canonical coordinates of the orbits $\mathcal{O}_\xi$. Each element $A \in \mathcal{G}$ can be considered as the restriction of the corresponding linear functional $\tilde{A}$ onto co-adjoint orbits, considered as a subset of $\mathcal{G}^*$, $\tilde{A}(\xi) = \langle \xi, A \rangle$. It is well-known that this function is just the Hamiltonnian function, associated with the Hamiltonian vector field ξ_A, defined by the formula

$$(\xi_A f)(x) := \frac{d}{dt} f(x \exp(tA))|_{t=0}, \forall f \in C^\infty(\mathcal{O}_\xi).$$

It is well-known the relation $\xi_A(f) = \{\tilde{A}, f\}, \forall f \in C^\infty(\mathcal{O}_\xi)$. Denote by ψ the symplectomorphism from $\mathbb{R}^2$ onto $\mathcal{O}_\xi, (p,q) \mapsto \psi(p,q) \in \mathcal{O}_\xi$, we have

Proposition 3.1. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A} \circ \psi(p,q) = bp + (c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma) \bullet$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp \wedge dq$.

Proof.

1. We adapt the diffeomorphism ψ (for 2-dimensional co-adjoint orbits, only):

$$(p,q) \in \mathbb{R}^2 \mapsto \psi(p,q) = (\alpha + \gamma - \gamma e^{q\lambda_1}, p, \gamma e^{q\lambda_1}, \delta e^{q\lambda_2}, \epsilon e^q) \in \mathcal{O}_\xi \bullet$$

Element $\xi \in \mathcal{G}^*$ is of the form $\xi = \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^* + \epsilon S^*$, hence the value of the function $f_A = \tilde{A}$ on the element $A = aX + bY + cZ + dT$ is

$$\tilde{A}(\xi) = \langle \xi, A \rangle = \langle \alpha X^* + \beta Y^* + \gamma Z^* + \delta T^*, aX + bY + cZ + dT \rangle = \alpha a + \beta b + \gamma c + \delta d.$$

It follows that $\tilde{A} \circ \psi(p,q) = bp + (c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)$.

2. By a direct computation, we conclude that in the canonical coordinates the Kirillov form is the standard symplectic form $\omega = dp \wedge dq$.

The proposition is therefore proved. □

3.1.2. Representations of the group $G_{5,3,1(\lambda_1,\lambda_2)}$.

Theorem 3.2. *With $A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$, then*

$$\ell_A(q, \partial_q) = b\partial_q + \frac{i}{\hbar}[(c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)]$$

Proof. Applying directly (3), (4) we have As $\tilde{A} = bp + (c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)$ then

$$\hat{A} = -i\hbar b\partial q + (c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)$$

and from this,

$$\ell_A(q,\partial_q) = \frac{i}{\hbar}[-i\hbar b\partial q + (c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)]$$
$$= b\partial_q + \frac{i}{\hbar}[(c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)]$$

The theorem is therefore proved. □

As $G_{5,3,1(\lambda_1,\lambda_2)}$ is connected and simply connected Lie group, we obtain.

Corollary 3.3. *The irreducible unitary representations $\mathcal{T}$ of the group $G_{5,3,1(\lambda_1,\lambda_2)}$ defined by $\mathcal{T}(\exp A) := \exp(\ell_A)$; $A \in \mathcal{G}_{5,3,1(\lambda_1,\lambda_2)}$. More detail,*

$$\mathcal{T}(\exp A) = \exp\left(b\partial_q + \frac{i}{\hbar}[(c-a)\gamma e^{q\lambda_1} + d\delta e^{q\lambda_2} + f\epsilon e^q + a(\alpha+\gamma)]\right).$$

What we did here gives us more simplisity computations in this case for use the star-product (see [3], [4], [5], [6]).

Other groups are proved similarly, we get the following results (3.2-3.7).

3.2. Group $G = G_{5,3,2(\lambda)}$. $ad_Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$; $[X,Y] = Z; \lambda \in \mathbb{R} - \{0,1\}$, $ad_X = 0$. With $U = aX + bY + cZ + dT + fS \in \mathcal{G}, a, b, c, d, f \in \mathbb{R}$, upon Maple 9.5, we get:

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -e^b+1 & (a-c)\frac{(e^b-1)}{b} & e^b & 0 & 0 \\ 0 & -d\frac{(e^b-1)}{b} & 0 & e^b & 0 \\ 0 & -f\frac{\lambda(e^{b\lambda}-1)}{b\lambda} & 0 & 0 & e^{b\lambda} \end{pmatrix}$$

Proposition 3.4. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A}\circ\psi(p,q) = bp + (f\epsilon e^{q\lambda} + \gamma(c-a) + d\delta)e^q + a(\alpha+\gamma).$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp\wedge dq$.

Theorem 3.5. *$A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,2(\lambda)}$, we have*

$$\ell_A(q,\partial_q) = b\partial_q + \frac{i}{\hbar}[(f\epsilon e^{q\lambda} + \gamma(c-a) + d\delta)e^q + a(\alpha+\gamma]$$

3.3. Group $G = G_{5,3,3(\lambda)}$. $ad_Y = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $[X,Y] = Z; \lambda \in \mathbb{R} \setminus \{1\}$; $ad_X = 0$. With $U = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,3(\lambda)}$, upon Maple 9.5, we get:

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{b(e^{b\lambda}-1)}{b\lambda} & (a-c\lambda)\frac{(e^{b\lambda}-1)}{b\lambda} & e^{b\lambda} & 0 & 0 \\ 0 & -d\frac{(e^b-1)}{b} & 0 & e^b & 0 \\ 0 & -f\frac{(e^b-1)}{b} & 0 & 0 & e^b \end{pmatrix}.$$

Proposition 3.6. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A} \circ \psi(p,q) = bp + (\frac{\gamma}{\lambda} + c\gamma)e^{q\lambda} + (d\delta + f\epsilon)e^q + a\alpha - \frac{\gamma}{\lambda}$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp \wedge dq$.

Theorem 3.7. *For each $A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,3(\lambda)}$, we have*

$$\ell_A(q, \partial_q) = b\partial_q + \frac{i}{\hbar}\left[(\frac{\gamma}{\lambda} + c\gamma)e^{q\lambda} + (d\delta + f\epsilon)e^q + a\alpha - \frac{\gamma}{\lambda}\right]$$

3.4. Group $G = G_{5,3,4}$. $ad_Y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$; $[X,Y] = Z; ad_X = 0$. With $U = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,4}, a, b, c, d, f \in \mathbb{R}$, upon Maple 9.5, we get

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -e^b + 1 & \frac{(a-c)(e^b-1)}{b} & e^b & 0 & 0 \\ 0 & 0 & 0 & e^b & 0 \\ 0 & 0 & 0 & 0 & e^b \end{pmatrix}.$$

Proposition 3.8. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A} \circ \psi(p,q) = bp + (-a\alpha + c\gamma + d\delta + f\epsilon)e^q + a(\alpha + \gamma)$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp \wedge dq$.

Theorem 3.9. For each $A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,4}$, we have

$$\ell_A(q, \partial_q) = b\partial_q + \frac{i}{\hbar}\left[(-a\alpha + c\gamma + d\delta + f\epsilon)e^q + a(\alpha + \gamma)\right]$$

3.5. Group $G = G_{5,3,5(\lambda)}$. $ad_Y = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; $[X,Y] = Z; \lambda \in \mathbb{R}\backslash\{1\} ad_X = 0$.
With $U = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,5(\lambda)}$, upon Maple 9.5, we get:

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -\frac{(e^{b\lambda}-1)b}{b\lambda} & \frac{(a-c\lambda)(e^{b\lambda}-1)}{b\lambda} & e^{b\lambda} & 0 & 0 \\ 0 & -\frac{(bf-d)e^b+d}{b} & 0 & e^b & be^b \\ 0 & -\frac{f(e^b-1)}{b} & 0 & 0 & e^b \end{pmatrix}$$

Proposition 3.10. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A} \circ \psi(p,q) = bp + (-a\frac{\gamma}{\lambda} + c\gamma)e^{q\lambda} + (f\delta q + d\delta + f\epsilon)e^q + a\alpha + a\frac{\gamma}{\lambda})$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp \wedge dq$.

Theorem 3.11. For each $A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,5(\lambda)}$, we have

$$\ell_A(q, \partial_q) = b\partial_q + \frac{i}{\hbar}\left[(-a\frac{\gamma}{\lambda} + c\gamma)e^{q\lambda} + (f\delta q + d\delta + f\epsilon)e^q + a\alpha + a\frac{\gamma}{\lambda})\right]$$

3.6. Group $G = G_{5,3,6(\lambda)}$. $ad_Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \lambda \end{pmatrix}$; $[X,Y] = Z; \lambda \in \mathbb{R}\backslash\{0,1\}$ $ad_X =$ 0 With $U = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,6(\lambda)}$, upon Maple 9.5 we get:

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 - e^b & \frac{(a-c)(e^b-1)}{b} & e^b & 0 & 0 \\ -1 + (1-b)e^b & \frac{(a+d)(1-e^b)+b(a-c)e^b}{b} & be^b & e^b & 0 \\ 0 & 0 & 0 & 0 & e^{b\lambda} \end{pmatrix}$$

Proposition 3.12. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form*

$$\tilde{A}\circ\psi(p,q) = bp + (f\epsilon e^{q\lambda} + (c\alpha + cq\delta + d\delta + a - a\gamma)e^q + a(\alpha+\gamma-\delta))$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp\wedge dq$.

Theorem 3.13. For each $A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,6(\lambda)}$, we have

$$\ell_A(q,\partial_q) = b\partial_q + \frac{i}{\hbar}\left[f\epsilon e^{q\lambda} + (c\alpha + cq\delta + d\delta + a - a\gamma)e^q + a(\alpha+\gamma-\delta)\right].$$

3.7. Group $G = G_{5,3,7}$. $ad_Y = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$; $[X,Y] = Z$; $ad_X = 0$. With $U = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,7}$, upon Maple 9.5, we get

$$\exp(ad_U) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -b & a-c & 1 & 0 & 0 \\ -e^b + b + 1 & \frac{(2c-a)(e^b-1)+b(a-c)}{b} & e^b - 1 & e^b & 0 \\ e^b - be^b - 1 & \frac{(2c-a)(e^b+be^b-1)}{b} & be^b & be^b & e^b \end{pmatrix}$$

Proposition 3.14. *1. Hamiltonian function $\tilde{A}$ in canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$ is of the form* $\tilde{A}\circ\psi(p,q) =$

$$= bp + \big(c\epsilon + d\epsilon - a\delta\big)qe^q + (a\epsilon - a\delta + c\delta + d\delta + f\epsilon)e^q + a(\delta-\gamma)q + a(\alpha+\delta-\epsilon) + c(\gamma-\delta)$$

2. In the canonical coordinates (p,q) of the orbit $\mathcal{O}_\xi$, the Kirillov form ω_ξ is coincided with the standard form $dp\wedge dq$.

Theorem 3.15. For each $A = aX + bY + cZ + dT + fS \in \mathcal{G}_{5,3,7}$, we have

$$\begin{aligned}\ell_A(q,\partial_q) = b\partial_q + \frac{i}{\hbar}[&(c\epsilon + d\epsilon - a\delta)qe^q + (a\epsilon - a\delta + c\delta + d\delta + f\epsilon)e^q + a(\delta-\gamma)q \\ &+ a(\alpha+\delta-\epsilon) + c(\gamma-\delta)]\end{aligned}$$

All of them are exponential MD_5-groups. In the next paper, we will describe ξ-representasions of group $G_{5,3,8}$. This is group not exponential.

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