DETERMINING THE EFFECTIVE UNIAXIAL MODULUS OF THREE-PHASE COMPOSITE MATERIAL OF ALIGNED FIBRES AND SPHERICAL PARTICLES

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Abstract. Composite material is widely used in modern structures and the life thank to its advantages. In fact, one has investigated and applied many kinds of three-phase composite material obtained by embedding spherical inclusions into the matrix phase of fibre reinforced material. Seeking solutions for the effective properties of three-phase composite including matrix phase and two other phases, which are spherical particles, has been given in [2]. Basing on algorithm introduced in [2], we have deriven three-phase problem into two two-phase problems and determined the uniaxial modulus of three-phase composite composed of matrix phase, aligned fibres and spherical inclusions. By calculating results for a specific three-phase composite, this paper has given conclusions about the influence of third phase (spherical particles) on the performance of structures.

1. Setting problem

Composite material of aligned fibres are thought to have cyclic structure, therefore, studying this kind of material leads us to considering a representative volume element among those cyclic structures. Here, representative volume element has form of a rectangular parallelepiped. According to composite cylinders model, the fibre phase is taken to be composed of infinitely long circular cylinders embedded in a continuous matrix phase. With each individual fibre of radius a, there is associated an annulus of matrix material of radius b. Each individual cylinder combination of this type is referred to as a composite cylinder. In three-phase model, one embeds spherical inclusions which are isotropic homogeneous elastic spheres of equal radii into matrix phase. Consequently, present problem can be posed as follows.

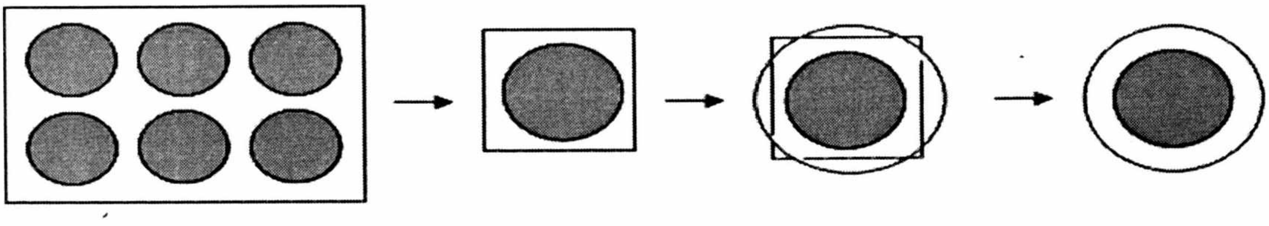

Fig. 1. The representative volume element of fibre reinforced material and composite cylinder model

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Let us consider a heterogeneous cylinder consisting of inner portion ($0 \preceq r \preceq a$) and outer portion ($a \preceq r \preceq b$). The composed materials are isotropic homogeneous elastic of properties (λ_a, μ_a) and (λ_m, μ_m), respectively. There exist an assumption that association between matrix phase and fibre phase is ideal, therefore, the uniaxial strain of two portions are the same. In this case, three-phase composite material is obtained by embedding isotropic homogeneous spheres having the same radius and elastic characteristics (λ_c, μ_c) into the continuous matrix phase of aligned fibre-reinforced material. Our present objective is that determine the effective uniaxial modulus E^*_{11} of three-phase composite as a function of the elastic properties of constituents as well as the volume fractions of the inclusions.

2. Governing relations

It is easy to recognise that investigating problem will become more convenient if governing relations are given in a cylindrical coordinate system [3].

Because of symmetry, assume the following displacement field:

$$u_r = u_r(r) \ , \ u_\theta = 0 \ , \ u_z = \varepsilon z. \tag{1}$$

Strain components are defined, respectively

$$e_{rr} = \frac{du_r}{dr} \ , \ e_{\theta\theta} = \frac{u_r}{r} \ , \ e_{zz} = \varepsilon \ . \tag{2}$$

In this case, the system of equilibrium equations has simple form

$$\frac{d\sigma_{rr}}{dr} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = 0. \tag{3}$$

By Hooke's laws, equation (3) is expressed in terms of the displacement field as follows.

$$\frac{d^2u_r}{dr^2} + \frac{1}{r}\frac{du_r}{dr} - \frac{1}{r^2}u_r. \tag{4}$$

3. Solution method

As mentioned above, governing idea for solving present three-phase problem is that converting it into two two-phase problems. Firstly, we combine original matrix phase and particle phase in order to give a new matrix phase called effective matrix phase. In fact, this effective matrix phase is a spherical particle-reinforced material of which elastic properties have been defined by some researchers, such as [1] and [5]. Then we seek solution for the effective properties of fibre-reinforced composite material composed of the effective matrix phase and aligned fibres. Method for determining the elastic moduli of aligned fibre-reinforced material has been mentioned in [1]. Basing on that method, we have specifically defined the effective uniaxial modulus of two-phase composite of aligned fibres.

It is very important to emphasize that process of converting a three-phase model into two-phase models must seriously been performed. Specifically, we can not combine initial matrix phase and the fibre phase in order to obtain the effective matrix phase. This fundamentally differ from three-phase model given in [2], where composite material is composed of matrix phase and two particle phases made of two different kinds of material.

3.1. The two-phase model

Let us consider two-phase composite consisting of isotropic matrix phase and isotropic fibre phase having properties $(\lambda_m,\ \mu_m)$ and $(\lambda_a,\ \mu_a)$, respectively. Then the effective uniaxial modulus of the two-phase composite is defined according to composite cylinders model [1] as follows.

3.1.1. Part of matrix phase

In the part of matrix phase $(a \preceq r \preceq b)$ the solution of eq. (4) is in form

$$u_r^{(2)} = A_2 r + \frac{B_2}{r}. \tag{5}$$

By Hooke's laws, stress field is defined

$$\sigma_{rr}^{(2)} = 2(\lambda_2 + \mu_2)A_2 - 2\mu_2 \frac{B_2}{r^2} + \lambda_2 \varepsilon. \tag{6}$$

After defining integration constants due to boundary and interface conditions

$$\left. \sigma_{rr}^{(2)} \right|_{r=b} = 0 \ , \ \left. \sigma_{rr}^{(2)} \right|_{r=a} = p \ , \tag{7}$$

(where p is interaction stress on the interface of fibre and matrix phases), the displacement field in the part of matrix phase is determined as follows.

$$u_r^{(2)} = \left[\frac{pa^2}{2(a^2 - b^2)(\lambda_2 + \mu_2)} - \frac{\lambda_2 \varepsilon}{2(\lambda_2 + \mu_2)} \right] r + \frac{pa^2 b^2}{(a^2 - b^2)2\mu_2} \frac{1}{r}. \tag{8}$$

3.1.2. Part of fibre phase

In this part $(0 \preceq r \preceq a)$, the displacement and stress fields have the form of

$$u_r^{(1)} = A_1 r \ , \tag{9}$$

$$\sigma_{rr}^{(1)} = 2(\lambda_1 + \mu_1)A_1 + \lambda_1 \varepsilon. \tag{10}$$

Specifying the integration constant A_1 from the interface

$$\sigma_{rr}^{(1)}\Big|_{r=a} = p \,,$$

gives us

$$u_r^{(1)} = \frac{p - \lambda_1 \varepsilon}{2(\lambda_1 + \mu_1)} r. \tag{11}$$

The interaction stress p is defined from continuity condition

$$u_r^{(1)}\Big|_{r=a} = u_r^{(2)}\Big|_{r=a} \,,$$

as follows

$$p = \frac{\mu_2(\lambda_1\mu_2 - \lambda_2\mu_1)(a^2 - b^2)\varepsilon}{\mu_2(\lambda_2 + \mu_2)(a^2 - b^2) - (\lambda_1 + \mu_1)\left[\mu_2 a^2 + (\lambda_2 + \mu_2)b^2\right]}. \tag{13}$$

3.1.3. The composite cylinders model

According to this model, the effective uniaxial module of fibre-reinforced material is determined as follows

$$E_{11} = \frac{\overline{\sigma_{zz}}}{\varepsilon} = \frac{1}{\pi b^2 \varepsilon} \iint\limits_S \sigma_{zz} dS = \frac{1}{\pi b^2 \varepsilon}\left[\iint\limits_{S_1} \sigma_{zz}^{(1)} dS + \iint\limits_{S_2} \sigma_{zz}^{(2)} dS\right], \tag{13}$$

where $S_1 = \pi a^2$, $S_2 = \pi(b^2 - a^2)$, $S = \pi b^2$ are the cross-section areas of fibre phase, matrix phase and composite cylinder, respectively.

By Hooke's laws, the uniaxial stresses of phases are defined

$$\sigma_{zz}^{(1)} = (\lambda_1 + 2\mu_1)\varepsilon + \frac{\lambda_1(p - \lambda_1\varepsilon)}{\lambda_1 + \mu_1} \,, \tag{14}$$

$$\sigma_{zz}^{(2)} = (\lambda_2 + 2\mu_2)\varepsilon + \frac{\lambda_2}{\lambda_2 + \mu_2}\left[\frac{pa^2}{a^2 - b^2} - \lambda_2\varepsilon\right]. \tag{15}$$

Introduction (14), (15) into (13) taking into account (12), we obtain the following relation

$$\begin{aligned} E_{11} &= \xi_a E_a + (1 - \xi_a)E_m \\ &+ \frac{4\xi_a(1 - \xi_a)(\nu_a - \nu_m)^2 G_m}{(1 - \xi_a)G_m\,(K_a + G_a/3)^{-1} + \xi_a G_m\,(K_m + G_m/3)^{-1} + 1} \,, \end{aligned} \tag{16}$$

where $\xi_a = a^2/b^2$ is the volume fraction of fibre phase.

Expression (16) is a formula for determining the effective uniaxial modulus of two-phase composite material of aligned cylindrical fibres.

3.2. The three-phase model

Now we embed spherical particles having the same radius and elastic properties (λ_c , μ_c) into the matrix phase of aligned fibre-reinforced material. Then we combine initial matrix phase and particle phase in order to give new matrix phase called effective matrix phase. In fact, this effective matrix phase is spherical particle-reinforced isotropic material of which properties have been determined by Hasin and Christensen [1] as follows

$$G_m^{(e)} = G_m \left[1 - \frac{15(1-\nu_m)(1-G_c/G_m)\xi_c}{7-5\nu_m+(8-10\nu_m)G_c/G_m} \right], \tag{17}$$

$$K_m^{(e)} = K_m + \frac{(K_c-K_m)\xi_c}{1+(K_c-K_m)(K_m+4G_m/3)^{-1}}, \tag{18}$$

where ξ_c is the volume fraction of particle phase.

Substituting the elastic characteristics of matrix phase in equation (16) by their effective values (17) and (18), we obtain the following relation

$$E_{11}^* = \xi_a E_a + (1-\xi_a) E_m^{(e)} + \frac{4\xi_a(1-\xi_a)\left(\nu_a - \nu_m^{(e)}\right)^2 G_m^{(e)}}{(1-\xi_a)G_m^{(e)}(K_a+G_a/3)^{-1} + \xi_a G_m^{(e)}\left(K_m^{(e)}+G_m^{(e)}/3\right)^{-1}+1}, \tag{19}$$

where

$$E_m^{(e)} = \frac{9K_m^{(e)}G_m^{(e)}}{3K_m^{(e)}+G_m^{(e)}}, \quad \nu_m^{(e)} = \frac{3K_m^{(e)}-2G_m^{(e)}}{6K_m^{(e)}-2G_m^{(e)}}.$$

Expression (19) is a formula for determining the effective uniaxial modulus of three-phase composite material composed of continuous matrix phase, aligned fibres and spherical particles. Obviously, this modulus is a function of the elastic properties and volume fractions of constituents.

For example, we consider a three-phase composite material having the following characteristics: Matrix phase is made of epoxy having properties $E_m = 0,315.10^6 (kG/cm^2)$ $\nu_m = 0,382$, whereas, fibre and particle phases are made of glass having elastic moduli $E_a = E_c = 7,4.10^6 (kG/cm^2)$, $\nu_a = \nu_c = 0,21$ in constant relation of volume fractions $\xi_a + \xi_c = 0,6$.

Calculating results for the effective uniaxial module E_{11}^* according to formula (19) are given in below table and sketched in figure 2.

ξ_α	0	0,1	0.2	0,3	0,4	0,5	0,6
$E_{11}^*.10^{-6}$	0,6842	1,3014	1,9305	2,5716	3,2247	3,8899	4,5672

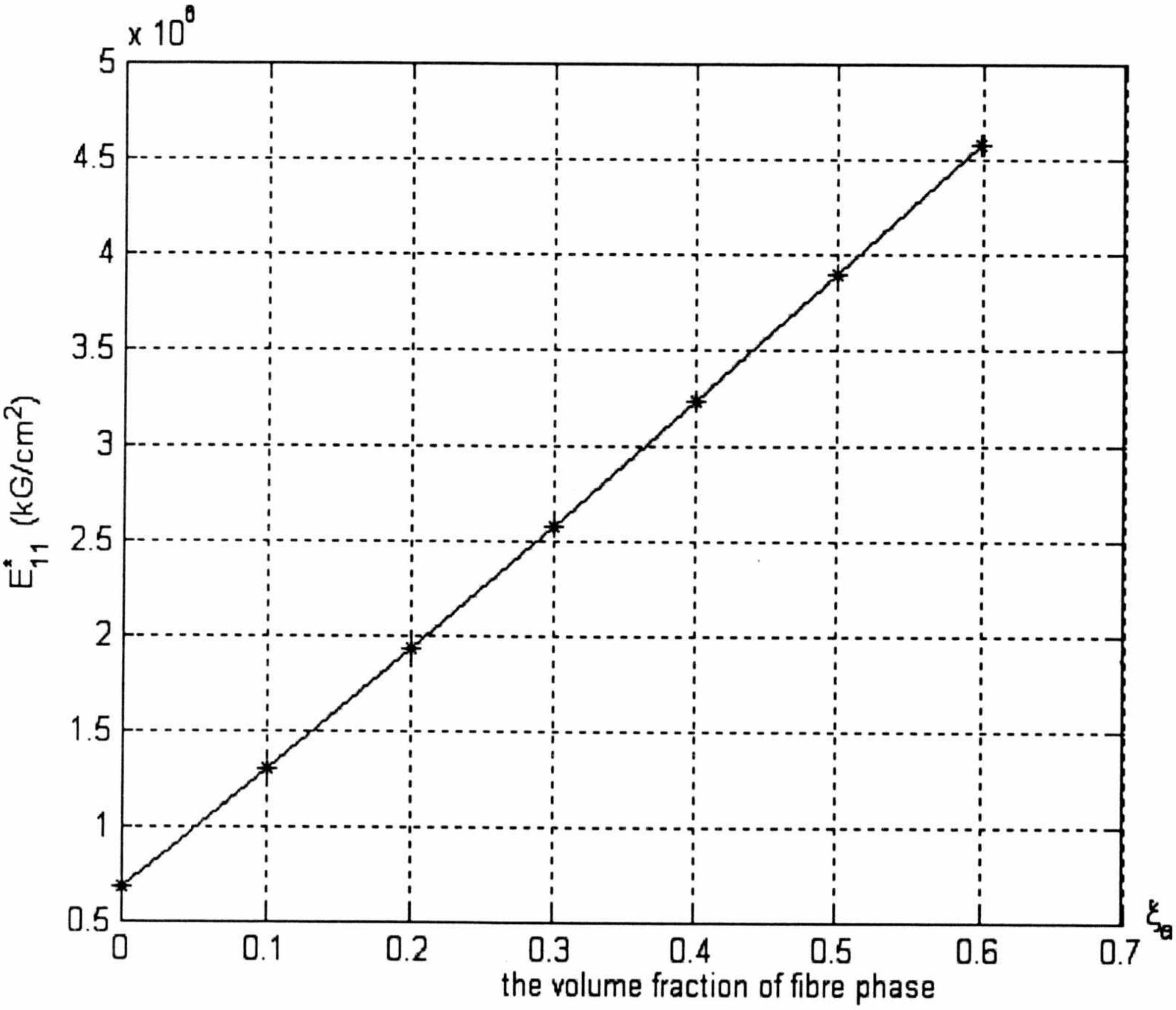

Fig. 2. The variance of the effective uniaxial modulus according to the volume fractions of constituents.

4. Conclusions

a) Converting three-phase model into two-phase models is reasonable. We have determined the effective uniaxial modulus of three-phase composite consisting of matrix phase, aligned fibres and spherical particles. Solving three-phase problem lead us to two two-phase problems. First problem is solved in order to define the effective elastic moduli of spherical particle-reinforced material. According to composite cylinders model, second one is solved for determining the effective elastic moduli of composite material composed of effective matrix phase and fibre phase.

b) Embedding spherical particles as third phase into the continuous matrix phase of aligned fibre-reinforced material to reduce the effective uniaxial modulus of this kind of material. Therefore, if structure is subjected to axial forces, it is necessary to consider embedding spherical particles into matrix phase of aligned fibre-reinforced material.

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