SOME PROPERTIES ALMOST-PERIODIC SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Let $\mathbb{R}$ be the set of all real numbers, E be a Banach space and B be the space of all almost-periodic functions $f : \mathbb{R} \to E$. We consider the equation

$$\dot{x} = Ax + f(t), \tag{1}$$

where A is a bounded linear operator on E. As well-known that (see [1]), the equation (1) has an unique solution $\varphi(t) \in B$ for each $f \in B$ if and only if there exists $R(i\omega) = (i\omega I - A)^{-1}$ for all $\omega \in \mathbb{R}$, further this solution has the form

$$\varphi(t) = \int_{-\infty}^{\infty} G(t-s)f(s)ds, \tag{2}$$

where $G(u)$ is the Green function defined by

$$G(u) = \begin{cases} e^{uA}P_+ & \text{for} \quad u < 0 \\ e^{uA}P_- & \text{for} \quad u > 0. \end{cases} \tag{3}$$

In this paper, we show the formula and the resolvent of the spectrum of the operator K defined by the right side of the formula (2).

For any $\alpha \in \mathbb{R}$, by B_α we denote the Banach space of almost-periodic functions whose spectrum belong to α . Let K_α be the restriction of K on B_α, we will prove that the operator K_α is completely continuous if and only if E is a finite dimensionnal space and α has no cluster points.

1. Introduction

It is well-known that B is a Banach space equiped with the norm $\|f\| = \sup_{t\in\mathbb{R}} |f(t)|$. We denote by P_+ and P_- the spectral projections, corresponding the spectral set of A, which lies in a right and in a left half planes, respectively. We consider the operator K, which takes B into itself defined by

$$(Kf)(t) = \int_{-\infty}^{+\infty} G(t-s)f(s)ds.$$

Proposition 1.1. (see [1]) *If* $f(t) = \sum f_\omega(t)e^{i\omega t}$ *for* $t \in \mathbb{R}$, *where* f_ω *is Fourier coefficient of the function* f, *then*

$$(Kf)(t) = \sum R(i\omega, A)(t) f_\omega e^{i\omega t} \quad \text{for all} \quad t \in \mathbb{R}. \tag{4}$$

2. Main results

Theorem 2.1. *The spectrum* $\sigma(K)$ *of the operator* K *coincides with the closure of the set*

$$\sigma = \left\{ \frac{1}{i\omega - \xi} \ ; \ \omega \in \mathbb{R}, \xi \in \sigma(A) \right\}.$$

Moreover, the resolvent of the operator K *has the form*

$$R(Z, K) = \frac{1}{z} I + \frac{1}{z^2} K(A + \frac{1}{z} I), z \in \sigma(A). \tag{5}$$

Proof. The first we prove that $\sigma \subset \sigma(K)$. If $Z \notin \sigma(K)$, then the equation $(zI - K)f = g$ has an unique solution for each $f \in B$ By the proposition 1.1, we have

$$(zI - K)f(t) = \sum (zI - R(i\omega, A))(t) \ f_\omega(t)e^{i\omega t}. \tag{6}$$

Hence, the equation $(zI - R(i\omega, A))f_\omega = g_\omega$ has an unique solution for each $\omega \in \mathbb{R}$. This the operator $zI - R(i\omega, A)$ is invertible. By virtue of the Spectral Mapping theorem(See[2]) we get $z \neq \dfrac{1}{i\omega - \xi}$ for $\xi \in \sigma(A)$ therefore $\sigma(K) \supset \sigma$. Conversely, suppose that $z \in \sigma$ then $z \neq 0$. The spectral mapping theorem implies that the operator $A + \dfrac{1}{z} I$ is regular. Denote by S the operator, wich defined by the right side of the formula (5), then the formula (4) means

$$(Sf)(t) = \sum \left\{ \frac{1}{z} I + \frac{1}{z^2} R(i\omega, A + \frac{1}{z}) \right\} (t) f_\omega(t) e^{i\omega t} \tag{7}$$

On the other hand

$$\begin{aligned} \{zI - (i\omega I - A)^{-1}\} &= \{z(i\omega I - A)^{-1}(i\omega I - A) - (i\omega I - A)^{-1}\}^{-1} \\ &= (i\omega I - A)(i\omega I - zA - I)^{-1} \\ &= \frac{1}{z}(i\omega I - A - \frac{1}{z} I + \frac{1}{z} I)(i\omega I - A - \frac{1}{z} I)^{-}1 \\ &= \frac{1}{z} I + \frac{1}{z}(i\omega I - A - \frac{1}{z} I)^{-1}. \end{aligned} \tag{8}$$

Combining (6) and (7) we get

$$(zI - K)Sf(t) = S(zI - K)f(t) = \sum_{w} e^{i\omega t} \text{ for all } t \in B.$$

Since the almost-periodic functions are unique, we have

$$(zI - K)S = S(zI - K) = I.$$

Therefore $\sigma(K) \subset \sigma$. The theorem is completely proved.

Lemma 2.2. *Suppose that E is a finite dimensional space and the set $\alpha \subset \mathbb{R}$ has no cluster points. Then the collective B of $\mathcal{B}$ is compact if and only if B is uniformly bounded and equally continuous.*

Proof. The "if" part is clear. To prove "nly if" part we consider the continuous real-value function $a(t)$ such that $a(t) = 1$ for $|t| \leqslant 1$, $a(t) = 2$ for $|t| > 2$ and $a(t)$ is linear for otherwise t. Set

$$a_n(t) = a(\epsilon_n t), \quad \text{where} \quad \epsilon_n > 0 \quad \text{and} \quad \epsilon_n \to 0.$$

The Fourier transform $\hat{a}_n$ of function is integrable and

$$\varphi_n = \int_{-\infty}^{+\infty} \hat{a}_n(s)\varphi(t+s)ds, \varphi \in B$$

is trigonometric polynomial. The spectrum of this polynomial belong to $\alpha \cap (-\frac{2}{\epsilon_n}, \frac{2}{\epsilon_n})$. It is well-known that (see[3]).

$$\|\varphi_n\| \leqslant c\|\varphi\| \tag{9}$$

and

$$\|\varphi . \varphi_n\| \leqslant \frac{4\|\varphi\|}{\pi N} + c\delta_4(2n\epsilon_n), \; N > 0. \tag{10}$$

Since B is equally continuous and by virtue of inequalities (9) and (10), it follows that the collective of polynomial $B_n = \varphi_n$ is uniformly bounded i.e B_n is compact. Since collective B is equally continuous and (12), for any $\epsilon > 0$), there exist number $n(\epsilon)$ and $\varphi \in B$ such that $\|\varphi - \varphi_n\| < \epsilon, n > n(\epsilon)$.

The set B is compact results from B_n is ϵ-compact net. The lemma is proved.

Theorem 2.3. *The operator K_α is completely continuous if and only if the space E is a finite dimensional and α has no cluster points.*

Proof. The "if" part. Suppose that K_α is completely continuous. Denote by D the bounded supset of E and

$$\mathcal{D} = \{f(t) = (i\omega I - A)xe^{i\omega t}, \; x \in D\}.$$

Hence $\mathcal{D}$ is the bounded subset of B_α and the collective $K_\alpha \mathcal{D} = \{xe^{i\omega t}, x \in D\}$ is compact in B_α.

If $x_1 e^{i\omega t}, x_2 e^{i\omega t}, ..., x_p e^{i\omega t}$ is ϵ-net of $K_\alpha \mathcal{D}$, then $x_1, x_2, ..., x_p$ is ϵ-net of D. Since B is compact, the Ritze theorem implies that E is a finite dimensional.

Suppose ω_0 is a cluster point of the set α and $\{\omega_n\}$ is the sequence of the diferent points of the set α, which convergences to ω_0. It follows that the sequence $\dfrac{1}{i\omega_n - \xi}$ of the different spectral points of the operator K_α convergences to $\dfrac{1}{i\omega_0 - \xi} \neq 0$, which is a contradition because K_α is the completely continuous. Therefore the set α has no cluster points.

Thus, the "if" part have been proved. To prove the "only if" part we will show that the operator K tranfer from the collective of the uniformly bounded function $\mathcal{D} \subset B$ to the collective of the function, which is uniformly bounded and equally continuous. In fact, it easily seen that this collective is uniformly bounded. We show that this collective is equally continuous. From (3), we have

$$\|G(u)\| \leqslant \mu e^{-\gamma|u|}, u \in \mathbb{R},\ \mu > 0, \gamma > 0$$

By (2), (3) we obtain

$$\begin{aligned} \|Kf(t+h) - Kf(t)\| &\leqslant \| \int_{-\infty}^{t} \{G(t+h-s) - G(t-s)f(s)ds\| \\ &+ \| \int_{-\infty}^{t} \{G(t+h-s) - G(t-s)f(s)ds\| \\ &\leqslant \frac{2M}{\gamma}\{\|e^{hA} - I\| + (I - e^{\gamma h})\}\|f\|, \end{aligned} \tag{11}$$

which deduce that the collective $K\mathcal{D}$ is equally continuous. By virtue of Lemma 2.2 the collective $k_\alpha \mathcal{D}$ is compact. Consequently, K_α is a completely continuous operator theorem is completely proved.

References

1. Krein. M. H, *Lectures on the theory of stability of solutions of diiferential equations in Banach sapces,* Kiev, 1964 (Russian).

3 Dunford. N and Schwatz. J,*Linear operators,* Moscow, 1982, Vol.1 (Russian).

2 Levintan. B.M, *Almost-periodic function GITTL,* Moscow, 1953 (Russian).