SYMMETRIC SPACES AND POINT-COUNTABLE COVERS

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ABSTRACT. In this paper, we prove some properties of symmetric spaces and point-countable covers in symmetric spaces.

1. Introduction

Since generalized metric spaces determined by point- countable covers were discussed by Burke, Gruenhage, Michael and Tanaka and other authors [2,3], the notion point-countable covers have drawn attention in general topology. The symmetric spaces were introduced and investigated by A.V. Arhangelskii [1], G. Gruenhage [3], Y. Tanaka [6,7,9]. In this paper, we shall consider the relations among certain spaces with a symmetric space and prove some properties of point- countable covers in the symmetric spaces.

We assume that all spaces are T_1 and regurlar. We begin at some basic definitions.

Definition 1.1. Let X be a topological space.

1) X is called a *symmetric space* if there exists a nonnegative real valued function d on $X \times X$ satisfying

a) $d(x, y) = 0$ if and only if $x = y$;

b) $d(x, y) = d(y, x)$ for every x and y in X;

c) $U \subset X$ is open if and only if for each $x \in U$, there exists $n \in \mathbb{N}$ such that $S_n(x) \subset U$, where

$$S_n(x) = \{y \in X : d(x, y) < \frac{1}{n}\}.$$

X is called a *semi-metrizable (or semi-metric) space* if we replace c) by " For $A \subset X, x \in \overline{A}$ if and only if $d(x, A) = 0$", where $d(x, A) = \inf\{d(x, a) : a \in A\}$.

2) X is called a *sequential space*, if $A \subset X$ is closed in X if and only if no sequence in A converges to a point not in A.

3) We call a subspace of X a *fan* (at a point x) if it consists of a point x, and a countably infinite family of disjoint sequences converging to x. Call a subset of a fan a *diagonal* if it is a convergent sequence meeting infinitely many of the sequences converging to x and converges to some point in the fan.

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X is call α_4-*space* if every fan at x of X has a diagonal converging to x.

Definition 1.2. Let X be a space, and $\mathcal{P}$ a cover of X. Put

$$\mathcal{P}^{<\omega} = \{\mathcal{P}' \subset \mathcal{P} : |\mathcal{P}'| < \omega\}.$$

1) $\mathcal{P}$ is a *k-network* if, whenever $K \subset U$ with K compact and U open in$X$, then

$$K \subset \cup\mathcal{F} \subset U$$

for some $\mathcal{F} \in \mathcal{P}^{<\omega}$.

2) $\mathcal{P}$ is a *network* if for every $x \in X$ and U open in X such that $x \in U$, then

$$x \in P \subset U$$

for some $P \in \mathcal{P}$.

3) $\mathcal{P}$ is a *p-k-network* if, whenever $K \subset X \setminus \{y\}$ with K compact in X, then

$$K \subset \cup\mathcal{F} \subset X \setminus \{y\}$$

for some $\mathcal{F} \in \mathcal{P}^{<\omega}$.

4) $\mathcal{P}$ is an *s-network* if it is network and for any non closed set $A \subset X$, there exists a point $x \in X$ with the property: For any neighborhood U of x, there exists $P \in \mathcal{P}$ such that $P \subset U$ and $P \cap A$ is infinite.

5) $\mathcal{P}$ is a cs^**-network* if $\{x_n\}$ is a sequence converging to $x \in X$ and U is *a* neighborhood of x, there exists $P \in \mathcal{P}$ such that

$$\{x\} \cup \{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$$

for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

6) $\mathcal{P}$ is a wcs^**-network* if $\{x_n\}$ is a sequence converging to $x \in X$ and U is *a* neighborhood of x, then there exists a $P \in \mathcal{P}$ such that

$$\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U$$

for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

7) $\mathcal{P}$ is a *p-*wcs^**-network* if $\{x_n\}$ is a sequence converging to $x \in X$ and $x \neq y$, then there exists $P \in \mathcal{P}$ such that

$$\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset X \setminus \{y\}$$

for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$.

Definition 1.3. For a space X and $x \in P \subset X$, P is called a *sequential neighborhood* at x in X if, whenever $\{x_n\}$ is a sequence converging to x in X , then $x_n \in P$ for all but finitely many $n \in \mathbb{N}$.

Definition 1.4. Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be family of subsets of X which satisfies that for each $x \in X$,

1) $\mathcal{P}_x$ is network of x in X,
2) If $U, V \in \mathcal{P}_x$, then $W \subset U \cap V$ for some $W \in \mathcal{P}_x$.

$\mathcal{P}$ is an *sn-network* for X if each element of $\mathcal{P}_x$ is a sequential neighborhood of x in X.

$\mathcal{P}$ is a *weak base* for X, if a subset G of X is open in X if and only if for each $x \in G$ there exists $P \in \mathcal{P}_x$ such that $P \subset G$.

A space X is an *snf-countable space* if X has an sn-network $\mathcal{P}$ such that each $\mathcal{P}_x$ is countable.

A space X is a *gf-countable space* if X has a weak base $\mathcal{P}$ such that $\mathcal{P}_x$ is countable for every $x \in X$.

Definition 1.5. Let X be a space. A cover $\mathcal{P}$ is called *point-countable* if for every $x \in X$, the set $\{P \in \mathcal{P} : x \in P\}$ is at most countable.

It is clear that [10]

$$\begin{array}{ccc} \text{weak base} & & \\ \downarrow & & \\ \text{sn-network} & & \\ \downarrow & & \\ \text{cs*-network} & & \\ \downarrow & & \\ \text{wcs*-network} & \longleftarrow & \text{k-network} \\ \downarrow & & \downarrow \\ \text{p-wcs*-network} & \longleftarrow & \text{p-k-network} \end{array}$$

In this paper we shall provide some partial answers for connections between kinds of network in the symmetric space.

2. The main results

Theorem 2.1. *Let X be a symmetric space. Then*

1) X is a gf-countable space;

2) X is a sequential space;

3) X is an snf-countable space;

4) X is an α_4-space.

Proof. 1) For each $x \in X$ put

$$\mathcal{P}_x = \{S_n(x) : n = 1, 2, ...\}$$

and $\mathcal{P} = \{\mathcal{P}_x : x \in X\}$. It is clear that $\mathcal{P}$ is a weak base for X. Since $\mathcal{P}_x$ is countable for every $x \in X$, X is a gf-countable space.

2) Let A be a subset of X. Assume that, if any sequence $\{x_n\}$ in A converging to x then $x \in A$. We show that A is closed. If it is not the case, then, there exists $x \in X \setminus A$ such that $S_n(x) \cap A \neq \emptyset$ for every $n \in \mathbb{N}^*$. For each $n \in \mathbb{N}^*$ choose $x_n \in S_n(x) \cap A$. Then, the sequence $\{x_n\}$ is in A and converges to x. Since $x \notin A$, we have a contradition.

Conversly, suppose that A is closed. It follows easily that, if $\{x_n\} \subset A$ is sequence converging to x, then $x \in A$. Thus X is a sequential space.

3) It is sufficient to show that $\mathcal{P}$ is an sn-network. Suppose the assertion is false. Then, there exists $P_0 \in \mathcal{P}_x$ and a sequence $\{x_n\} \subset X \setminus P_0$ with $x_n \to x$. It follows that the subset $\{x_n : n \in \mathbb{N}\}$ is not closed and hence $X \setminus \{x_n : n \in \mathbb{N}\}$ is not open. Let $y \in X \setminus \{x_n : n \in \mathbb{N}\}$.

If $y = x$ then

$$y \in P_0 \subset X \setminus \{x_n : n \in \mathbb{N}\}, P_0 \in \mathcal{P}_y.$$

Assume that $y \neq x$. Then $y \in X \setminus (\{x_n : n \in \mathbb{N}\} \cup \{x\})$. Since $\{x_n : n \in \mathbb{N}\} \cup \{x\}$ is closed, there exists $P \in \mathcal{P}_y$ such that

$$y \in P \subset X \setminus (\{x_n : n \in \mathbb{N}\} \cup \{x\}) \subset X \setminus \{x_n : n \in \mathbb{N}\}.$$

Hence $X \setminus \{x_n : n \in \mathbb{N}\}$ is open. This is a contradiction.

4) Assume that M is a fan at x in X with

$$M = \{x\} \bigcup_{n \in \mathbb{N}} \{x_{nm} : m \in \mathbb{N}\},$$

where $\{x_{nm} : m \in \mathbb{N}\}_{n \in \mathbb{N}}$ is a countable family of disjoint sequences converging to x.

Since $\mathcal{P}$ is an sn-network, $S_n(x)$ is sequential neighborhodd of x for every $n = 1, 2, \ldots$. It follows that for each $k \in \mathbb{N}$ and for each $S_n(x)$ there exists $m_{nk} \in \mathbb{N}$ such that

$$x_{km} \in S_n(x) \quad \text{for} \quad m \geqslant m_{nk}.$$

This yields

$$\{x_{km} : m \in \mathbb{N}\} \cap S_n(x) \neq \emptyset \quad \text{for all} \quad k \quad \text{and} \quad n \in \mathbb{N}.$$

Choose

$$y_n \in \{x_{nm} : n \in \mathbb{N}\} \cap S_n(x)$$

and put $C = \{y_n : n \in \mathbb{N}\}$. Then

$$C \cap \{x_{nm} : m \in \mathbb{N}\} = \{y_n\} \quad \text{for all} \quad n \in \mathbb{N}.$$

Let U be a neighborhood of x. Then there exists $n_0 \in \mathbb{N}$ such that $S_{n_0}(x) \subset U$.Hence

$$y_n \in S_n(x) \subset S_{n_0}(x) \subset U \quad \text{for all} \quad n \geqslant n_0.$$

This means that $y_n \to x$ and hence C is a diagonal of M converging to x. Thus X is an α_4−space.

Proposition 2.2. *Let X be a symmetric space. Then the following are equivalent :*

1) X is a semi-metric space;

2) For every $x \in X$ and $r > 0$, the subset

$$S_r(x) = \{y \in Y : d(x, y) < r\}$$

is a neighborhood of x.

Proof. Assume that X is a semi-metric space, $x \in X$ and $r > 0$. Then,

$$\overline{A} = \{y \in X : d(y, A) = 0\} \quad \text{for all} \quad A \subset X.$$

Put

$$E = X \setminus S_r(x).$$

Since $d(x, E) \geqslant r > 0, x \notin \overline{E}$. It follows that, there exists a open subset U in X such that

$$x \in U \subset X \setminus \overline{E}.$$

If $z \in U$, then $z \notin E$. This means $z \in S_r(x)$ and hence $U \subset S_r(x)$. Thus $S_r(x)$ is a neighborhood of x.

Conversly, assume that $S_r(x)$ is a neighborhood of x for every $x \in X$ and $r > 0$. Let A be a subset of X and $x \in \overline{A}$. Then $S_r(x) \cap A \neq \emptyset$ for all $r > 0$. Hence $d(x, A) = 0$.

Let $x \in X$ with $d(x, A) = 0$. Suppose $x \notin \overline{A}$. Then

$$x \in U \subset X \setminus \overline{A}$$

for some neighborhood U of x. It follows that, there exists $n \in \mathbb{N}$ such that

$$S_n(x) \subset U \subset X \setminus \overline{A}.$$

This yields

$$d(x, A) \geqslant d(x, \overline{A}) \geqslant \frac{1}{n} > 0$$

We have a contradiction. Hence $x \in \overline{A}$ and

$$\overline{A} = \{x \in X : d(x, A) = 0\}.$$

Thus X a semi-metric space.

For any space, the following hold:

k-network $\Rightarrow$ wcs*-network,

p-k-network $\Rightarrow$ p-wcs*-network.

The converses are false in generality case. However, we have following results for symmetric spaces.

Theorem 2.3. *Let X be a symmetric space and $\mathcal{P}$ be a point-countable cover of X. Then*

1) $\mathcal{P}$ is a k-network if and only if it is a wcs-network.*

2) $\mathcal{P}$ is a p-k-network if and only if it is a p-wcs-network.*

Proof. 1) The "only if" part is clear, so we only need to prove the "if" part. Let K be a compact subset of X and U an open set in X such that $K \subset U$. For each $x \in X$, since $\mathcal{P}$ is point-countable, we have

$$\{P \in \mathcal{P} : x \in P \subset U\} = \{P_n(x) : n \in \mathbb{N}\}.$$

We will show that K is covered by some finite subset $\mathcal{P}' \subset \{P_n(x) : x \in U, n \in \mathbb{N}\}$. If it is not the case, let $x_0 \in K$. Then, there exists $x_1 \in K \setminus P_0(x_0)$. Since

$$K \not\subset P_0(x_0) \cup P_1(x_0) \cup P_0(x_1) \cup P_1(x_1),$$

there exists

$$x_2 \in K \setminus \bigcup \{P_i(x_j) : 0 \leqslant i, j < 2\}$$

Continued applying this argument, we obtain the sequence $\{x_n\} \subset K$ such that

$$x_n \in K \setminus \cup\{P_i(x_j) : 0 \leqslant i, j < n\} \quad \text{for} \quad n = 0, 1, 2.. \tag{1}$$

By the Theorem 2.1, X is a sequential space. Since K is compact, there exists a subseqnence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \to x \in K$. As $\mathcal{P}$ a wcs*-network, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that

$$\{x_{n_{i_k}} : k \in \mathbb{N}\} \subset P \subset U$$

for some $P \in \mathcal{P}$. Then, there exist m and $x_{n_{i_j}}$ such that $P = P_m(x_{n_{i_j}})$. Put $n_0 = \max(m, n_{i_j})$. By (1),

$$x_n \notin P_m(x_{n_{i_j}}) = P \quad \text{for all} \quad n > n_0.$$

This is a contracliction. Thus $\mathcal{P}$ is a k-network.

2) The proof for 2) is similar, with U is replaced by $X \setminus \{y\}$.

Proposition 2.4. [9] *1) If $\mathcal{P}$ is an s-network in any space X, then $\mathcal{P}$ is a wcs*-network.*

2) If X is a sequential space and $\mathcal{P}$ is a wcs-network, then $\mathcal{P}$ is an s-network.*

Proof. 1) Let $\{x_n\} \subset X$ be a sequence converging to x. Without loss of generality we can assume that $x_n \neq x$ for all n. Put $A = \{x_n : n = 1, 2, 3, ...\}$. Since A is not closed and $\mathcal{P}$ is an s-network, there exists $y \in X$ with the property: For any neighborhood U of y, there exists $P \in \mathcal{P}$ such that $P \subset U$ and $P \cap A$ is infinite. Hence there exists the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\{x_{n_i}\} \subset P \subset U$$

Thus we only need to show that $y = x$. Suppose $y \neq x$. Then, since $A \cup \{x\}$ is closed, there exists the neighborhood U of y such that $U \cap (A \cup \{x\}) = \emptyset$. For each $P \in \mathcal{P}$, $P \subset U$ we have $P \cap A = \emptyset$. This is a contracdiction.

2) Let A be a not closed subset in X. Since X is a sequential space, there exists the sequence $\{x_n\} \subset A$ such that $x_n \to x \notin A$. For every neighborhood U of x, since $\mathcal{P}$ is a wcs*-network, there exists $P \in \mathcal{P}$ and the subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\{x_{n_i} : i \in \mathbb{N}\} \subset P \subset U.$$

This means that $P \cap A$ is infinite and hence $\mathcal{P}$ is an s-network.

Corollary 2.5. *The following are equivalent for a symmetric X :*

1) X has a point-countable s-network

2) X has a point-countable wsc-network*

3) X has a point-countable cs-network*

4) X has a point-countable k-network

Proof. 1) $\Leftrightarrow$ 2) by Theorem 2.1 and Proposition 2.4.

2) $\Leftrightarrow$ 4) by Theorem 2.3.

2) $\Leftrightarrow$ 3) by Theorem 2.1 and Theorem 7 in [10].

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