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# ONE RESULT OF THE CYCLIC INEQUALITY

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ABSTRACT. In this paper we present some inequalities which are obtained from comparing with

$$S(\alpha,\beta) = \sum_{k=1}^{n-2} \frac{x_k^\alpha}{(x_{k+1}+x_{k+2})^\beta} + \frac{x_{n-1}^\alpha}{(x_n+x_1)^\beta} + \frac{x_n^\alpha}{(x_1+x_2)^\beta}$$

and

$$R(\alpha,\beta) = \frac{1}{2^\beta}\sum_{k=1}^{n} x_k^{\alpha-\beta},$$

where $\alpha \in R^+, \beta \in R^+, x_i \in R^+ \quad (i=\overline{1,n}), R^+ = \{x \in R | x > 0\}$.

## I. Introduction

Some cyclic inequalities have presented under simple forms but they are really difficult to prove. The Shapiro's inequality is a very special inequality and it is suprising that many mathematicians have spent time on it. When $\alpha = \beta = 1$, we obtain the Shapiro's inequality

$$S(1,1) \geqslant R(1,1) \quad (n \geqslant 3). \tag{1.1}$$

This inequality is correct for odd intergers less than or equal to 23 and for even intergers less than or equal to 12. For all other $n$, the inequality is false. For $\alpha \in R^+, \beta \in R^+$, we will construct some inequalities between $S(\alpha,\beta)$ and $R(\alpha,\beta)$.

## II. Case $\alpha = \beta > 1$

**Theorem 1.1.** *If $x_i \in R^+ \quad (i=\overline{1,n}), \alpha > 1, n$ is an odd integer less than or equal to 23 and an even integer less than or equal to 12, then $S(\alpha,\alpha) \geqslant R(\alpha,\alpha) \quad (1.2)$.*

*Equality of (1.2) holds if and only if $x_1 = x_2 = \cdots = x_n$.*

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*Proof.* Using the inequality

$$\frac{1}{n}\sum_{i=1}^{n} a_i^{\alpha} \geqslant \Big(\frac{1}{n}\sum_{i=1}^{n} a_i\Big)^{\alpha} \quad (a_i > 0, i = \overline{1,n}), \alpha > 1$$

by (1.1) we have

$$S(\alpha,\alpha) \geqslant n\Big(\frac{S(1,1)}{n}\Big)^{\alpha} \geqslant n\Big(\frac{R(1,1)}{n}\Big)^{\alpha} = n(\frac{1}{2})^{\alpha} = R(\alpha,\alpha)$$

**III. Case $\alpha < \beta$**

In this case we obtain result.

**Theorem 1.2.** *If $\alpha \in R^+$, $\beta \in R^+$, $\alpha < \beta$ then*

*(i) The inequality $S(\alpha,\beta) \geqslant R(\alpha,\beta)$ is not true for every positive $x_i$ $(i = \overline{1,n})$*

*(ii) The inequality $S(\alpha,\beta) \leqslant R(\alpha,\beta)$ is not true for every positive $x_i$ $(i = \overline{1,n})$*

*Proof.* (*i*) Taking $x_1 = 1, x_2 = x_3 = ... = x_n = a > 0$ we obtain

$$S(\alpha,\beta) = \frac{1}{2^{\beta}}[\frac{1}{a^{\beta}} + \frac{n-3}{a^{\beta-\alpha}}] + \frac{2a^{\alpha}}{(1+a)^{\beta}},$$
$$R(\alpha,\beta) = \frac{1}{2^{\beta}}[1 + (n-1)a^{\alpha-\beta}].$$

Since $\beta > \alpha$, it follows

$$\lim_{a\to+\infty} S(\alpha,\beta) = 0$$
$$\lim_{a\to+\infty} R(\alpha,\beta) = \frac{1}{2^{\beta}}.$$

For large enough $a$ in $S(\alpha,\beta)$ and $R(\alpha,\beta)$, we have $S(\alpha,\beta) < R(\alpha,\beta)$. It follows that $S(\alpha,\beta) \geqslant R(\alpha,\beta)$ is wrong.

(*ii*) Taking $x_1 = a$, $x_2 = x_3 = ... = x_n = 1 (a > 0)$ we obtain

$$S(\alpha,\beta) = \frac{a^{\alpha}}{2^{\beta}} + \frac{n-3}{2^{\beta}} + \frac{2}{(1+a)^{\beta}},$$
$$R(\alpha,\beta) = \frac{1}{2^{\beta}}[a^{\alpha-\beta} + (n-1)].$$

Since $\beta > \alpha$, it follows

$$\lim_{a\to+\infty} S(\alpha,\beta) = +\infty$$
$$\lim_{a\to+\infty} R(\alpha,\beta) = \frac{n-1}{2^{\beta}}.$$

For large enough $a$ in $S(\alpha,\beta)$ and $R(\alpha,\beta)$, we have $S(\alpha,\beta) > R(\alpha,\beta)$. It follows that $S(\alpha,\beta) \leqslant R(\alpha,\beta)$ is wrong.

**IV. Case $\alpha > \beta$**

**Theorem 1.3.** *Given $x_i$ $(i = \overline{1,n})$ are positive numbers , $p, q$ are positive integer numbers such that $p > q$. We obtain*

$$M = \frac{x_1^p}{(x_2+x_3)^q} + \frac{x_2^p}{(x_3+x_4)^q} + \cdots + \frac{x_{n-1}^p}{(x_n+x_1)^q} + \frac{x_n^p}{(x_1+x_2)^q} \geqslant \frac{1}{2^q}(x_1^{p-q} + \cdots + x_n^{p-q}).$$

*Proof.* Lets consider the case $q \leqslant p - q \Leftrightarrow 2q \leqslant p$.

Applying the AM - GM inequality we have

$$\frac{2^{2q} \cdot {x_1}^p}{(x_2+x_3)^q} + (x_2+x_3)^q x_1^{p-2q} \geqslant 2^{q+1} x_1^{p-q}$$

Similarly, we have

$$\frac{2^{2q} x_2^p}{(x_3+x_4)^q} + (x_3+x_4)^q x_2^{p-2q} \geqslant 2^{q+1} x_2^{p-q}$$

$$\cdots$$

$$\frac{2^{2q} x_n^p}{(x_1+x_2)^q} + (x_1+x_2)^q x_n^{p-2q} \geqslant 2^{q+1} x_n^{p-q}$$

Summing all above inequalities, we have

$$2^{2q} M + x_1^{p-2q}(x_2+x_3)^q + \cdots + x_n^{p-2q}(x_1+x_2)^q \geqslant 2^{q+1}(x_1^{p-q} + x_2^{p-q} + \cdots + x_n^{p-q}). \tag{1.2}$$

Moreover,

$$x_1^{p-2q}(x_2+x_3)^q \leqslant 2^{q-1} x_1^{p-2q}(x_2^q + x_3^q).$$

We have

$$\underbrace{x_1^{p-q} + x_1^{p-q} + \cdots + x_1^{p-q}}_{p-2q \text{ terms}} + \underbrace{x_2^{p-q} + \cdots + x_2^{p-q}}_{q \text{ terms}} \geqslant (p-q) x_1^{p-2q} x_2^q, \tag{1.3}$$

and

$$\underbrace{x_1^{p-q} + x_1^{p-q} + \cdots + x_1^{p-q}}_{p-2q \text{ terms}} + \underbrace{x_3^{p-q} + \cdots + x_3^{p-q}}_{q \text{ terms}} \geqslant (p-q) x_1^{p-2q} x_3^q. \tag{1.4}$$

Taking sum of (1.3) and (1.4), we obtain

$$2(p-2q) x_1^{p-q} + q x_2^{p-q} + q x_3^{p-q} \geqslant (p-q) x_1^{p-2q}(x_2^q + x_3^q).$$

It follows that

$$x_1^{p-2q}(x_2+x_3)^q \leqslant \frac{2^{q-1}}{p-q}[2(p-2q)x_1^{p-q}+qx_2^{p-q}+qx_3^{p-q}]$$
$$\leqslant 2^{q-1}[\frac{2(p-2q)}{p-q}x_1^{p-q}+\frac{q}{p-q}x_2^{p-q}+\frac{q}{p-q}x_3^{p-q}]$$

Similarly, we obtain

$$x_2^{p-2q}(x_3+x_4)^q \leqslant 2^{q-1}[\frac{2(p-2q)}{q-q}x_2^{p-q}+\frac{q}{p-q}x_3^{p-q}+\frac{q}{p-q}x_4^{p-q}],$$
$$\ldots$$
$$x_n^{p-2q}(x_1+x_2)^q \leqslant 2^{q-1}[\frac{2(p-2q)}{q-q}x_n^{p-q}+\frac{q}{p-q}x_1^{p-q}+\frac{q}{p-q}x_2^{p-q}].$$

Taking sum of inequalities, we have

$$x_1^{p-2q}(x_2+x_3)^q+x_2^{p-2q}(x_3+x_4)^q+\cdots+x_n^{p-2q}(x_1+x_2)^q \leqslant$$
$$\leqslant 2^{q-1}[\frac{2(p-2q)}{p-q}+\frac{2q}{p-q}](x_1^{p-q}+x_2^{p-q}+\cdots+x_n^{p-q})$$
$$\Leftrightarrow x_1^{p-2q}(x_2+x_3)^q+\cdots+x_n^{p-2q}(x_1+x_2)^q \leqslant 2^q(x_1^{p-q}+x_2^{p-q}+\cdots+x_n^{p-q}). \quad (1.5)$$

Taking sum of (1.2) and (1.5) we obtain

$$2^{2q}M \geqslant 2^q(x_1^{p-q}+x_2^{p-2}+\cdots+x_n^{p-q})$$
$$\Leftrightarrow M \geqslant \frac{1}{2^q}(x_1^{p-q}+x_2^{p-q}+\cdots+x_n^{p-q})$$

*) **For the case** $q > p-q \Leftrightarrow 2q > p$.

Applying the AM - GM inequality, we have

$$q = u(p-q)+v,\ 1 \leqslant v \leqslant p-q$$

$$\frac{2^p.x_1^p}{(x_2+x_3)^q}+\underbrace{(x_2+x_3)^{p-q}+\cdots+(x_2+x_3)^{p-q}}_{u \quad \text{terms}}+2^{p-q-v}(x_2+x_3)^v x_1^{p-q-v}$$
$$\geqslant (u+2)2^{\frac{2p-q-v}{u+2}}x_2^{\frac{2p-q-v}{u+2}}$$
$$\geqslant (u+2)2^{p-q}x_1^{p-q}$$

Similarly, we have

$$\frac{2^p x_2^p}{(x_3+x_4)^q}+u(x_4+x_3)^{p-q}+2^{p-q-u}\cdot(x_3+x_4)^v x_2^{p-q-u} \geqslant (u+2)2^{p-q}x_2^{p-q}$$
$$\ldots$$
$$\frac{2^p x_n^p}{(x_1+x_2)^q}+u(x_1+x_2)^{p-q}+2^{p-q-u}(x_1+x_2)^v x_n^{p-q-v} \geqslant (u+2)2^{p-q}x_n^{p-q}$$

Taking sum of inequalities we have

$$2^pM + u[(x_2+x_3)^{p-q} + (x_3+x_4)^{p-q} + \cdots + (x_n+x_1)^{p-q} + (x_1+x_2)^{p-q}] +$$
$$+2^{p-q-v}[x_1^{p-q-v}(x_2+x_3)^v + \cdots + x_n^{p-q-v}(x_1+x_2)^v]$$
$$\geqslant (u+2)2^{p-q}(x_1^{p-q} + x_2^{p-q} + \cdots + x_n^{p-q}). \tag{1.6}$$

We have

$$u[(x_2+x_3)^{p-q} + \cdots + (x_1+x_2)^{p-q}] \leqslant$$
$$\leqslant u2^{p-q}\Big[\frac{x_2^{p-q}+x_3^{p-q}}{2} + \frac{x_3^{p-q}+x_4^{p-q}}{2} + \cdots + \frac{x_1^{p-q}+x_2^{p-q}}{2}\Big]$$
$$\leqslant u2^{p-q}(x_1^{p-q} + x_2^{p-q} + \cdots + x_n^{p-q}), \tag{1.7}$$

$$2^{p-q-v}[x_1^{p-q-v}(x_2+x_3)^v + \cdots + x_n^{p-q-v}(x_1+x_2)^v]$$
$$\leqslant 2^{p-q}[\frac{x_2^v+x_3^v}{2}\cdot {x_1}^{p-q-v} + \ldots + x_n^{p-q-v}.\frac{x_1^v+x_2^v}{2}] \tag{1.8}$$

$$\underbrace{x_1^{p-q} + x_1^{p-q}\cdots + x_1^{p-q}}_{p-q-v \quad \text{terms}} + \underbrace{x_2^{p-q} + \cdots + x_2^{p-q}}_{v \quad \text{terms}} \geqslant (p-q)x_1^{p-q-v}x_2^v,$$

and

$$\underbrace{x_1^{p-q} + x_1^{p-q}\cdots + x_1^{p-q}}_{p-q-v \quad \text{terms}} + \underbrace{x_3^{p-q} + \cdots + x_3^{p-q}}_{v \quad \text{terms}} \geqslant (p-q)x_1^{p-q-v}x_3^v$$

It follows that

$$(p-q-v)x_1^{p-q} + \frac{v}{2}x_2^{p-q} + \frac{v}{2}x_3^{p-q} \geqslant (p-q)x_1^{p-q-v}(\frac{x_2^v+x_3^v}{2})$$
$$\Leftrightarrow x_1^{p-q-v}(\frac{x_2^v+x_3^v}{2}) \leqslant \frac{p-q-v}{p-q}x_1^{p-q} + \frac{v}{2(p-q)}x_2^{p-q} + \frac{v}{2(p-q)}x_3^{p-q}.$$

Similarly, we obtain

$$x_2^{p-q-v}(\frac{x_3^v+x_4^v}{2}) \leqslant \frac{p-q-v}{p-q}x_2^{p-q} + \frac{v}{2(p-q)}x_3^{p-q} + \frac{v}{2(p-q)}x_4^{p-q},$$
$$\ldots$$
$$x_n^{p-q-v}(\frac{x_1^v+x_2^v}{2}) \leqslant \frac{p-q-v}{p-q}x_n^{p-q} + \frac{v}{2(p-q)}x_1^{p-q} + \frac{v}{2(p-q)}x_2^{p-q}.$$

Taking sum of all above $n$ inequalities we have

$$x_1^{p-q-v}(\frac{x_2^v+x_3^v}{2}) + \cdots + x_n^{p-q-v}(\frac{x_1^v+x_2^v}{2}) \leqslant [\frac{p-q-v}{p-q} + \frac{v}{p-q}](x_1^{p-q} + \cdots + x_n^{p-q})$$
$$\leqslant x_1^{p-q} + x_2^{p-q} + \cdots + x_n^{p-q}. \tag{1.9}$$

Since (1.8) and (1.9), it follows that

$$2^{p-q-v}[x_1^{p-q-v}(x_2+x_3)^v+\cdots+x_n^{p-q-v}(x_1+x_2)^v]\leqslant 2^{p-q}(x_1^{p-q}+\cdots+x_n^{p-q}). \tag{1.10}$$

Taking sum of inequalities (1.6), (1.7), (1.10) we have

$$\begin{aligned}
&2^pM \geqslant [(u+2)2^{p-q}-u2^{p-q}-2^{p-q}](x_1^{p-q}+x_2^{p-q}+\cdots+x_n^{p-q})\\
\Leftrightarrow\ &2^pM \geqslant 2^{p-q}(x_1^{p-q}+x_2^{p-q}+\cdots+x_n^{p-q})\\
\Leftrightarrow\ &M \geqslant \frac{1}{2^p}(x_1^{p-q}+x_2^{p-q}+\cdots+x_n^{p-q}).
\end{aligned}$$

In order to consider the case $\alpha,\beta$ are posotive numbers, we review the necesary inequalities

1) Given $a,b$ are positive numbers and $\alpha\geqslant 1$, we have

$$\frac{a^\alpha+b^\alpha}{2}\geqslant\left(\frac{a+b}{2}\right)^\alpha. \tag{1.11}$$

2) Given $a,b$ are positive numbers and $\alpha+\beta=1$, we have

$$\alpha a+\beta b\geqslant a^\alpha b^\beta. \tag{1.12}$$

3) Given $a,b,\alpha_1,\alpha_2$ are positive numbers and $\alpha\geqslant 1,\alpha_1+\alpha_2=1$ we have

$$(\alpha_1a+\alpha_2b)^\alpha\leqslant\alpha_1a^\alpha+\alpha_2b^\alpha. \tag{1.13}$$

**Theorem 1.4.** *Let $x_1\in R^+\quad(i=\overline{1,n}),\alpha\in R^+,\beta\in R^+$ with $1\leqslant\beta\leqslant\frac{\alpha}{2}$, we have*

$$P=\frac{x_1^\alpha}{(x_2+x_3)^\beta}+\frac{x_2^\alpha}{(x_3+x_4)^\beta}+\cdots+\frac{x_n^\alpha}{(x_1+x_2)^\beta}\geqslant\frac{1}{2^\beta}(x_1^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta}).$$

*Proof.* We have

$$\begin{aligned}
&\frac{2^{2\beta}x_1^\alpha}{(x_2+x_3)^\beta}+(x_2+x_3)^\beta x_1^{\alpha-2\beta}\geqslant 2^{\beta+1}x_1^{\alpha-\beta}\\
&\frac{2^{2\beta}x_2^\alpha}{(x_3+x_4)^\beta}+(x_3+x_4)^\beta x_2^{\alpha-2\beta}\geqslant 2^{\beta+1}x_2^{\alpha-\beta}\\
&\dots\\
&\frac{2^{2\beta}x_n^\alpha}{(x_1+x_2)^\beta}+(x_1+x_2)^\beta x_n^{\alpha-2\beta}\geqslant 2^{\beta+1}x_n^{\alpha-\beta}.
\end{aligned}$$

Taking sum of all above $n$ inequalities, we have

$$2^{2\beta}P+x_1^{\alpha-2\beta}(x_2+x_3)^\beta+\cdots+x_n^{\alpha-2\beta}(x_1+x_2)^\beta\geqslant 2^{\beta+1}(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta}). \tag{1.14}$$

Since $\beta \geqslant 1$, apply the inequality (1.11) we have

$$x_1^{\alpha-2\beta}(x_2+x_3)^\beta \leqslant 2^{\beta-1}x_1^{\alpha-2\beta}(x_2^\beta+x_3^\beta).$$

Since $\dfrac{\alpha-2\beta}{\alpha-\beta}+\dfrac{\beta}{\alpha-\beta}=1$, apply the inequality (1.12) we have

$$\frac{\alpha-2\beta}{\alpha-\beta}x_1+\frac{\beta}{\alpha-\beta}x_2 \geqslant x_1^{\frac{\alpha-2\beta}{\alpha-\beta}}x_2^{\frac{\beta}{\alpha-\beta}}$$

$$\Leftrightarrow x_1^{\alpha-2\beta}x_2^\beta \leqslant \Big(\frac{\alpha-2\beta}{\alpha-\beta}x_1+\frac{\beta}{\alpha-\beta}x_2\Big)^{\alpha-\beta}$$

Since $1 \leqslant \beta \leqslant \dfrac{\alpha}{2} \to 1 \leqslant \beta \leqslant \alpha-\beta$ and apply the inequality (1.13) we have

$$x_1^{\alpha-2\beta}x_2^\beta \leqslant \Big(\frac{\alpha-2\beta}{\alpha-\beta}x_1+\frac{\beta}{\alpha-\beta}x_2\Big)^{\alpha-\beta} \leqslant \frac{\alpha-2\beta}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_2^{\alpha-\beta}$$

Similarly, we have

$$x_1^{\alpha-2\beta}x_3^\beta \leqslant \frac{\alpha-2\beta}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_3^{\alpha-\beta}$$

It follows that

$$2^{\beta-1}x_1^{\alpha-2\beta}(x_2^\beta+x_3^\beta) \leqslant 2^{\beta-1}[\frac{2(\alpha-2\beta)}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_2^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_3^{\alpha-\beta}]$$

It follows that

$$x_1^{\alpha-2\beta}(x_2+x_3)^\beta \leqslant 2^{\beta-1}[\frac{2(\alpha-2\beta)}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_2^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_3^{\alpha-\beta}].$$

Similarly, we obtain

$$x_2^{\alpha-2\beta}(x_3+x_4)^\beta \leqslant 2^{\beta-1}[\frac{2(\alpha-2\beta)}{\alpha-\beta}x_2^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_3^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_4^{\alpha-\beta}]$$

$$\ldots$$

$$x_n^{\alpha-2\beta}(x_1+x_2)^\beta \leqslant 2^{\beta-1}[\frac{2(\alpha-2\beta)}{\alpha-\beta}x_n^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{\beta}{\alpha-\beta}x_2^{\alpha-\beta}].$$

Taking sum of all above inequalities, we obtain

$$\begin{aligned}
&x_1^{\alpha-2\beta}(x_2+x_3)^\beta+\cdots+x_n^{\alpha-2\beta}(x_1+x_2)^\beta\\
\leqslant\, &2^{\beta-1}[\frac{2(\alpha-2\beta)}{\alpha-\beta}+\frac{2\beta}{\alpha-\beta}](x_1^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta})\\
\leqslant\, &2^\beta(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta}).
\end{aligned} \tag{1.15}$$

Adding (1.14) and (1.15) we have

$$2^{2\beta}P \geqslant (2^{\beta+1}-2^\beta)(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta})$$

$$P \geqslant \frac{1}{2^\beta}(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta}).$$

**Theorem 1.5.** *Let $x_i \in R^+ \quad (i = \overline{1, n}), \alpha \in R^+, \beta \in R^+$ satisfy conditions*

$$\beta > \frac{\alpha}{2} \geqslant \frac{1}{2}(\beta + 1)$$

*and $\beta = u(\alpha - \beta) + v$ where $u$ is positive integer and $v \geqslant 1$. We have*

$$P = \frac{x_1^\alpha}{(x_2 + x_3)^\beta} + \frac{x_2^\alpha}{(x_3 + x_4)^\beta} + \cdots + \frac{x_n^\alpha}{(x_1 + x_2)^\beta} \geqslant \frac{1}{2^\beta}(x_1^{\alpha-\beta} + \cdots + x_n^{\alpha-\beta}).$$

*Proof.* Applying AM - GM inequality, we have

$$\frac{2^\alpha x_1^\alpha}{(x_2 + x_3)^\beta} + \underbrace{(x_2 + x_3)^{\alpha-\beta} + \cdots + (x_2 + x_3)^{\alpha-\beta}}_{u \text{ terms}} + 2^{\alpha-\beta-v}(x_2 + x_3)^v x_1^{\alpha-\beta-v}$$
$$\geqslant (u + 2)2^{\frac{2\alpha-\beta-v}{u+2}} x_1^{\frac{2\alpha-\beta-v}{u+2}} = (u + 2)2^{\alpha-\beta} x_1^{\alpha-\beta}.$$

Similarly, we have

$$\frac{2^\alpha x_2^\alpha}{(x_3 + x_4)^\beta} + u(x_3 + x_4)^{\alpha-\beta} + 2^{\alpha-\beta-v}(x_3 + x_4)^v x_2^{\alpha-\beta-v} \geqslant (u + 2)2^{\alpha-\beta} {x_2}^{\alpha-\beta}$$
$$\cdots$$
$$\frac{2^\alpha x_n^\alpha}{(x_1 + x_2)^\beta} + u(x_1 + x_2)^{\alpha-\beta} + 2^{\alpha-\beta-v}(x_1 + x_2)^v x_n^{\alpha-\beta-v} \geqslant (u + 2)2^{\alpha-\beta} {x_n}^{\alpha-\beta}.$$

Taking sum of the above $n$ inequalities we have

$$2^\alpha P + u[(x_2 + x_3)^{\alpha-\beta} + \cdots + (x_1 + x_2)^{\alpha-\beta}] + 2^{\alpha-\beta-v}[x_1^{\alpha-\beta-v}(x_2 + x_3)^v + \cdots +$$
$$\cdots + x_n^{\alpha-\beta-v}(x_1 + x_2)] \geqslant (u + 2)2^{\alpha-\beta}(x_1^{\alpha-\beta} + x_2^{\alpha-\beta} + \cdots + x_n^{\alpha-\beta}). \qquad (1.16)$$

Since the assumption $\frac{\alpha}{2} > \frac{1}{2}(\beta + 1) \Leftrightarrow \alpha - \beta > 1$, we have

$$u[(x_2 + x_3)^{\alpha-\beta} + \cdots + (x_1 + x_2)^{\alpha-\beta}] \leqslant u2^{\alpha-\beta}[\frac{x_2^{\alpha-\beta} + x_3^{\alpha-\beta}}{2} + \cdots + \frac{x_1^{\alpha-\beta} + x_2^{\alpha-\beta}}{2}]$$
$$\leqslant u2^{\alpha-\beta}(x_1^{\alpha-\beta} + x_2^{\alpha-\beta} + \cdots + x_n^{\alpha-\beta}). \qquad (1.17)$$

Since $v \geqslant 1$, we have

$$2^{\alpha-\beta-v}[x_1^{\alpha-\beta-v}(x_2 + x_3)^v + \cdots + x_n^{\alpha-\beta-v}(x_1 + x_2)^v]$$
$$\leqslant 2^{\alpha-\beta}[x_1^{\alpha-\beta-v}\frac{x_2^v + x_3^v}{2} + \cdots + x_n^{\alpha-\beta-v}\frac{x_1^v + x_2^v}{2}]. \qquad (1.18)$$

Applying the inequality (1.12) and (1.13), we have

$$x_1^{\alpha-\beta-v}x_2^v \leqslant \Big(\frac{\alpha-\beta-v}{\alpha-\beta}x_1+\frac{v}{\alpha-\beta}x_2\Big)^{\alpha-\beta}$$
$$\leqslant \frac{\alpha-\beta-v}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{v}{\alpha-\beta}x_2^{\alpha-\beta}.$$

Similarly, we have

$$x_1^{\alpha-\beta-v}x_3^v \leqslant \frac{\alpha-\beta-v}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{v}{\alpha-\beta}x_3^{\alpha-\beta}.$$

Taking sum of 2 inequalities, it follows that

$$x_1^{\alpha-\beta-v}(x_2^v+x_3^v) \leqslant \frac{2(\alpha-\beta-v)}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{v}{\alpha-\beta}x_2^{\alpha-\beta}+\frac{v}{\alpha-\beta}x_3^{\alpha-\beta}$$
$$\Rightarrow 2^{\alpha-\beta}x_1^{\alpha-\beta-v}\frac{x_2^v+x_3^v}{2} \leqslant 2^{\alpha-\beta}[\frac{\alpha-\beta-v}{\alpha-\beta}x_1^{\alpha-\beta}+\frac{v}{2(\alpha-\beta)}x_2^{\alpha-\beta}+\frac{v}{2(\alpha-\beta)}x_3^{\alpha-\beta}].$$

Similarly, we have

$$2^{\alpha-\beta}x_2^{\alpha-\beta-v}\frac{x_3^v+x_4^v}{2} \leqslant 2^{\alpha-\beta}[\frac{\alpha-\beta-v}{\alpha-\beta}x_2^{\alpha-\beta}+\frac{v}{2(\alpha-\beta)}x_3^{\alpha-\beta}+\frac{v}{2(\alpha-\beta)}x_4^{\alpha-\beta}]$$

$$\dots$$

$$2^{\alpha-\beta}x_n^{\alpha-\beta-v}\frac{x_1^v+x_2^v}{2} \leqslant 2^{\alpha-\beta}[\frac{\alpha-\beta-v}{\alpha-\beta}x_n^{\alpha-\beta}+\frac{v}{2(\alpha-\beta)}x_1^{\alpha-\beta}+\frac{v}{2(\alpha-\beta)}x_2^{\alpha-\beta}].$$

Taking sum of $n$ inequalities and applying (1.18), we obtain

$$2^{\alpha-\beta-v}[x_1^{\alpha-\beta-v}(x_2+x_3)^v+\cdots+x_n^{\alpha-\beta-v}(x_1+x_2)^v]$$
$$\leqslant 2^{\alpha-\beta}[\frac{\alpha-\beta-v}{\alpha-\beta}+\frac{v}{\alpha-\beta}](x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta})$$
$$\leqslant 2^{\alpha-\beta}(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta}). \tag{1.19}$$

Since the inequality (1.16), (1.17), (1.19), it follows

$$2^{\alpha}P \geqslant [(u+2)2^{\alpha-\beta}-u2^{\alpha-\beta}-2^{\alpha-\beta}](x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta})$$
$$\geqslant 2^{\alpha-\beta}(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta})$$
$$\Leftrightarrow P \geqslant \frac{1}{2^{\beta}}(x_1^{\alpha-\beta}+x_2^{\alpha-\beta}+\cdots+x_n^{\alpha-\beta}).$$

## References

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