QUANTIZATION OF AXIAL VECTOR FIELD

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ABSTRACT. The possibility of constructing the Hamiltonian quantization for axial electrodynamics with anomalies in a four - dimensional space is studied. It is shown that in this theory the Jacobi identity for operators of the Hamilton , of a time component of the current and of the very field is broken. The usual quantum theory is consistent only for a zero magnetic field.

1. Introduction

The absence of anomalies in the gauge theories represents one of the fruitful principles for constructing physical theories $[1-3]$. At the same time these is an opinion $[4-10]$ that the gauge theories with anomalies con be be considered as physical ones. However, up to now despite numerous attempts $[7-11]$ these is no consistent quantization of the theories with anomalies . In the present paper , we shall study the possibility of Hamiltonian quantization $[12-13]$ of such gauge theories with anomalies.

2. The formulation

Let us consider the classical theory of massless free fermions in a four dimensional space time.

$$S = \int d^x \mathcal{L}_0(x); \mathcal{L}_0 = \overline{\psi} i\hat{\partial}\psi; (\hat{\partial} = \gamma_\mu \partial_\mu; \overline{\psi} = \psi^+ \gamma_0) \tag{1}$$

We demand that the theory (2.1) should be invariant with respect to the axial gauge transformation

$$\psi(x)^\beta = e^{i\gamma_5 \beta(x)} \psi(x) \tag{2}$$

According to the classical principle of the local gauge invariance the invariance of the theory (2.1) is achieved by introducing the fermion interaction with an axial vector field

$$\begin{aligned} S &= \int d^4x \overline{\psi} i\hat{D}\psi \equiv \int d^4x \left(\overline{\psi} i\hat{\partial}\psi + J^5_\mu A_\mu\right), \\ i\hat{D} &= \gamma_\mu(i\partial_\mu + \gamma_5 A_\mu), J^5_\mu = \overline{\psi}\gamma_\mu\gamma_5\psi, \end{aligned} \tag{3}$$

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whose transformation

$$A - \mu \to A_\mu^\beta(x) = A_\mu(x) + \partial_\mu \beta(x), \tag{4}$$

compensate the transformations (2.2)

However, if the fermion fields are quantum and satisfy the commutation relation

$$\left[\psi_\alpha^+(\overrightarrow{x}, t), \psi_\beta(\overrightarrow{y}, t)\right] = \delta_{\alpha\beta}\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right),$$

so that their axial transformation are made by generator

$$\begin{aligned} \psi^\beta(x) &= U\psi(x)U^{-1}, \\ U &= \exp\left\{iQ_5(\beta)\right\}, Q_5(\beta) = \int d^3x J_0^5(x)\beta(x), \end{aligned} \tag{5}$$

then the "classical" principle of the local gauge invariance is broken.

What is the physical of this breaking?

The quantum fermions differ from classical ones by the Dirac sea (continuum) that aries from the requirement for the quantum Hamiltonian being positive. In the external (classical) axial vector field (2.3) the Dirac sea is rearranged so that the current commutators become anomalous $[14 - 17]$.

$$\begin{aligned} i\left[J_0^5(\overrightarrow{x}, t), J_0^5(\overrightarrow{y}, t)\right] &= -\frac{1}{6\pi^2} B_k(y)\frac{\partial}{\partial x_k}\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right), \\ i\left[J_i^5(\overrightarrow{x}, t), J_0^5(\overrightarrow{y}, t)\right] &= -\frac{1}{6\pi^2}\epsilon_{ik\ell} A_k(y)\frac{\partial}{\partial x_\ell}\delta^3\left(\overrightarrow{x} - \overrightarrow{y}\right), \\ B_k(y) = \epsilon_{kij}\partial_i A_j(y), \quad & A_k(y) = \frac{\partial}{\partial t} A_k(y). \end{aligned} \tag{6}$$

Solving by these commutators the Heisenberg equation for

$$\frac{\partial}{\partial t} J_0^5(x) = i\left[H, J_0^5(x)\right], \tag{7}$$

where H is the Hamiltonian of the theory

$$H = \int d^3x \left[\overline{\psi} i\gamma_i \partial_i \psi - J_\mu^5 A_\mu\right],$$

we get the anomalous divergence of the axial current

$$\begin{aligned} \partial_\mu J_\mu^5(x) &= \frac{1}{48\pi^2} F_{\mu\nu}\widetilde{F}_{\mu\nu}, \\ \widetilde{F}_{\mu\nu} &= \epsilon_{\mu\nu\alpha\beta} F_{\alpha\beta}. \end{aligned} \tag{8}$$

Formula (2.8) is consistent with the calculation of the anomalous triangle diagram [14 – 17]. From eq.(2.8) we see that the action (2.3) for the quantum fermions is non-invariant under the axial gauge transformation (2.5) and acquires the extra term

$$\Delta S = \int d^4x \beta(x) \partial_\mu J^5_\mu(x) \equiv \int d^4x \beta(x) \frac{1}{48\pi^2} F_{\mu\nu} \widetilde{F}_{\mu\nu}. \quad (9)$$

3. Quantization of axial - vector field

We consider the interaction of massless quantum fermions with an external axial - vector field

$$\mathcal{L}(x) = -\frac{1}{4} F^2_{\mu\nu} + \overline{\psi} \gamma_\mu \left(i\partial_\mu + A_\mu \gamma_5 \right) \psi. \quad (10)$$

As it is pointed above, this lagrangian is not invariant with respect to transformation (2.4), (2.5) (see eq.(2.9)). We can restore the symmetry of the theory (2.3) by introducing into the Lagrangian an extra term whose transformation compensates the anomalous reaction (2.9) of the initial action (2.3).

For example, we choose the following extra term [1]

$$\mathcal{L}_{mod}(x) = \mathcal{L}(x) . \Delta\mathcal{L}(x), \Delta\mathcal{L}(x) = -\frac{1}{\Box} \left(\partial_\mu A_\mu \right) \frac{F_{\alpha\beta} \widetilde{F}_{\alpha\beta}}{48\pi^2}, \quad (11)$$

or a classical equivalent

$$\begin{aligned} \Delta\mathcal{L}(x) &= \partial_\mu \chi \left(\partial_\mu \phi + A_\mu \right) + \phi \frac{F_{\alpha\beta} \widetilde{F}_{\alpha\beta}}{48\pi^2}, \\ \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi)} &= 0 \Rightarrow \phi = -\frac{1}{\Box} \left(\partial_\mu A_\mu \right). \end{aligned} \quad (12)$$

For quantization of the axial - vector field we choose transverse variables which are fixed by transformations

$$\begin{aligned} A^T_\mu &= A_\mu + \partial_\mu \theta^T(A), \\ \psi^T &= e^{i\theta^T(A)} \psi, \\ \phi^T &= \phi - \theta^T(A), \theta^T(A) = -\frac{1}{\partial^2_k} (\partial_i A_i), \\ \chi^T &= \chi. \end{aligned} \quad (13)$$

The transverse physical field (3.4) are nonlocal (gauge invariant) functionals of the initial fields A, ψ, ϕ [12 – 13]. The transverse variables are convenient for the Hamiltonian quantization and are only variables consistent with the classical equation [17] for the time component of the field (A_0).

Due to the nonlocality (3.4) the gauge of the variables is not fixed and follows the time - axis rotation in the course of the relativistic transformations [12 – 13].Upon passing to the transverse variables we have only one nondynamic variable A_0^T and the constraint equation

$$\frac{\delta S}{\delta A_0^T} = 0 \Rightarrow \partial_k^2 A_0^T = \left[-J_0^{5T} + \partial_0 \chi^T + \frac{\partial_i \phi^T B_i^T}{6\pi^2} \right]. \tag{14}$$

The Hamiltonian of the theory (3.2), (3.3) has the form

$$\begin{aligned} H =& \int d^3x T_{00}, \\ T_{00} =& \dot{A}_k^T E_k^T + \dot{\phi}^T \pi_\phi^T + \dot{\chi}^T \pi_\chi^T + \dot{\psi}^T \pi_\psi^T - \mathcal{L}_{mod}^T \\ =& \frac{1}{2}\left[\left(\dot{A}_k^T\right)^2 + (\partial_k A_0^T)^2 + (B_k^T)^2 \right] + \pi_\phi^T \pi_\chi^T - A_i^T J_i^{5T} \\ &+ \overline{\psi}^T i\gamma_i \partial_i \psi^T + \partial_i \chi^T \partial_i \phi^T, \end{aligned} \tag{15}$$

where the canonical conjugate momenta $E_k^T, \pi_\phi^T, \pi_\chi^T, \pi_\psi^T$ are given by the following formulae

$$E_k^T = \dot{A}_k^T + \pi_{ki}\left(\frac{\phi^T B_i^T}{6\pi^2}\right), \left(\pi_{ki} = \delta_{ki} - \delta_k \frac{1}{\partial^2}\partial_i\right), \tag{16}$$

$$\pi_\chi^T = \partial_0 \phi^T + A_0^T, \pi_\phi^T = \partial_0 \chi^T, \pi_\psi^T = i\psi^T, \tag{17}$$

$$A_0^T = \frac{1}{\partial_k^2}\left(-J_0^{5T} + \pi_\phi^T + \frac{\partial_i \phi^T B_i^T}{6\pi^2} \right). \tag{18}$$

Foe the boson operators we choose the usual commutation relations

$$\begin{aligned} & i\left[E_k^T(\overrightarrow{x}, t), A_j^T(\overrightarrow{y}, t)\right] = \left(\delta_{kj} - \partial_k \frac{1}{\partial^2}\partial_j\right)\delta^3(\overrightarrow{x} - \overrightarrow{y}), \\ & i\left[\pi_\phi^T(\overrightarrow{x}, t), \phi^T(\overrightarrow{y}, t)\right] = i\left[\pi_\chi^T(\overrightarrow{x}, t), \chi^T(\overrightarrow{y}, t)\right] = \delta^3(\overrightarrow{x} - \overrightarrow{y}), \\ & \left[J_0^5(\overrightarrow{x}, t), A_i^T(\overrightarrow{y}, t)\right] = 0. \end{aligned} \tag{19}$$

Then we check the divergence of the axial current with the help of the Heisenberg equation

$$\frac{\partial}{\partial t} J_0^{5T}(x) = i\left[H, J_0^{5T}\right]. \tag{20}$$

We get an expression of the type (2.8) with a new term

$$\partial_\mu J_\mu^{5T}(x) = \frac{1}{48\pi^2} F_{\alpha\beta}\widetilde{F}_{\alpha\beta} + \int d^3y \left[\frac{1}{2}\left(\dot{A}_k^T\right)^2(y), J_0^{5T}(x)\right], \tag{21}$$

that appears owing to the axial field quantization. This term can easily be calculated

$$\begin{aligned} \int d^3y \left[\frac{1}{2}\left(\dot{A}_k^T\right)^2(y), J_0^{5T}(x)\right] =& \int d^3y \frac{1}{2}\Big\{\dot{A}_k^T(y)\left[\dot{A}_k^T(y), J_0^{5T}(x)\right] \\ &+ \left[\dot{A}_k^T(y), J_0^{5T}(x)\right]\dot{A}_k(y)\Big\} = -\frac{1}{12\pi^2}\frac{\partial}{\partial t}\left(A_k^T B_k^T\right). \end{aligned} \tag{22}$$

Thus, instead of eq.(2.8) we get

$$\frac{\partial}{\partial t}\left[J_0^{5T}(x) - \frac{1}{12\pi^2}\left(A_k^T(x)B_k^T(x)\right)\right] = \partial_i J_i^{5T}(x) + \frac{1}{48\pi^2}F_{\alpha\beta}\widetilde{F}_{\alpha\beta}. \qquad (23)$$

The usual choice of the commutators (3.10) leads also to nonzero Jacobi brackets

$$\begin{aligned}\mathcal{L}\left(A_k^T(x), H, J_0^{5T}(y)\right) =& \Big\{ \left[A_k^T(x), \left[H, J_0^{5T}(y)\right]\right] \\ &+ \left[J_0^{5T}(y), \left[A_k^T(x), H\right]\right] + \left[H, \left[J_0^{5T}(y), A_k^T(x)\right]\right]\Big\} \\ =& \frac{1}{6\pi^2}B_k(x)\delta^3(\overrightarrow{x} - \overrightarrow{y}).\end{aligned}$$

Violation the Jacobi identities for the very current is noted in ref. [9 − 11]. Thus, the ordinary quantum theory with axial anomalies is not consistent for the quantum gauge fields. In any case the application of the ordinary part integral is problematic [6 − 8].

4. Conclusions

We have attempted to construct the Hamiltonian operator formalism for the Abelian axial gauge theory. It is shown that the ordinary commutation relation for the gauge fields breaks the Jacobi identity for operators, of the Hamiltonian, of the time component of the very field. These operators do not satisfy the associativity law respect to their multiplications and cannot be represented by linear operators in the Hilbert space.

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