SHARPENING MEAN-FORM AND CYCLIC INEQUALITY

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ABSTRACT. In this paper, we construct some sharpening forms of the mean form and sharpening some types of cyclic inequalities.

1. Introduction

In the international conference 1996, Zivojin Mijakovic and Milan Mijakovic presented a method to sharpen AM-GM inequality in their paper. Starting with the AM-GM inequality

$$G_n(a) = \Big(\prod_{k=1}^{k=n} a_k\Big)^{1/n} \leqslant \frac{1}{n}\sum_{k=1}^{k=n} a_k = A_n(a),$$

where $a_i \quad (i = \overline{1,n})$ are positive numbers, they created some stronger inequalities

$$G_n(a) \leqslant G(a,\alpha) \leqslant A_n(a),$$

where

$$G(a,\alpha) = \Big(\prod_{k=1}^{k=n}(a_k+\alpha)\Big)^{1/n} - \alpha$$

is a non-decreasing monotonic function. By virtur of the results, they created many infinitely symmetric expressions, that depend on parameter α, between $G_n(a)$ and $A_n(a)$. In this paper, we will sharpen some types of the inequalities.

1) $\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^r\Big)^{\frac{1}{r}} \geqslant \frac{1}{n}\sum_{k=1}^{n} a_k \quad$ with $r > 1, a_k > 0 (k = \overline{1,n})$

and $\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^r\Big)^{\frac{1}{r}} \leqslant \frac{1}{n}\sum_{k=1}^{n} a_k \quad$ with $0 < r < 1, (k = \overline{1,n})$

2) For $p > q > 1$, we have

$\Big(\frac{1}{n}\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}} \geqslant \Big(\frac{1}{n}\sum_{k=1}^{n} a_k^q\Big)^{\frac{1}{q}} \quad$ with $a_k > 0 (k = \overline{1,n})$

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3) $$\sum_{k=1}^{n-1} \frac{a_k^2}{a_k + a_{k+1}} + \frac{a_n^2}{a_n + a_1} \geqslant \frac{1}{2}\sum_{k=1}^{n} a_k \quad \text{with } a_k > 0 (k = \overline{1,n})$$

4) $$\sum_{k=1}^{n-2} \frac{a_k^2}{a_{k+1} + \beta a_{k+2}} + \frac{a_{n-1}^2}{a_n + \beta a_1} + \frac{a_n^2}{a_1 + \beta a_2} \geqslant \frac{1}{1+\beta}\sum_{k=1}^{n} a_k$$

(where $a_k > 0, k = \overline{1,n}$ and $\beta > 0$ is given).

2. Sharpening mean-form inequality

We denote

$$B_n(a,p,q,\alpha) = \Big[\Big(\frac{1}{n}\sum_{k=1}^{n}(a_k^p + \alpha)^{\frac{p}{q}}\Big)^{\frac{q}{p}} - \alpha\Big]^{\frac{1}{q}},$$

where $a_k > 0 \quad (k = \overline{1,n}); p > 0; q > 0; \alpha \geqslant 0$.

We have

$$B_n(a,p,1,\alpha) = \Big(\frac{1}{n}\sum_{k=1}^{n}(a_k + \alpha)^p\Big)^{\frac{1}{p}} - \alpha$$

$$B_n(a,p,1,0) = \Big(\frac{1}{n}\sum_{k=1}^{n} a_k^p\Big)^{\frac{1}{p}}$$

$$B_n(a,1,1,0) = \frac{1}{n}\sum_{k=1}^{n} a_k.$$

Let us consider the inequality

$$B_n(a,r,1,0) \geqslant B_n(a,1,1,0), \text{ wherer } r > 1$$

and

$$B_n(a,r,1,0) \leqslant B_n(a,1,1,0) \text{where} 0 < r < 1.$$

Lemma 1.1. *$B_n(a,r,1,\alpha)$ is a non - increasing monotonic function (with variable α).*

Proof. We have

$$B_n'(a,r,1,\alpha) = \Big[\frac{1}{r}\Big(\frac{1}{n}\sum_{k=1}^{k=n}(a_k + \alpha)^r\Big)^{\frac{1}{r}-1}.\Big(\frac{1}{n}\sum_{k=1}^{k=n} r(a_k + \alpha)^{r-1}\Big)\Big] - 1.$$

Therefore

$$B_n'(a,r,1,\alpha) = \frac{\frac{1}{n}\sum_{k=1}^{k=n}(a_k + \alpha)^{r-1}}{\big(\frac{1}{n}\sum_{k=1}^{k=n}(a_k + \alpha)^r\big)^{\frac{r-1}{r}}} - 1.$$

Denote $A_k = (a_k + \alpha)^{r-1}$ and $q = \frac{r}{r-1.}$ We get

$$B'_n(a, r, 1, \alpha) = \frac{\frac{1}{n}\sum_{k=1}^{k=n} A_k}{\left(\frac{1}{n}\sum_{k=1}^{k=n} A_k^q\right)^{1/q}} - 1.$$

If $r > 1$, then $q > 1$ and if $0 < r < 1$, then $q < 0$. It yield

$$\left(\frac{1}{n}\sum_{k=1}^{k=n} A_k^q\right)^{1/q} \geqslant \frac{1}{n}\sum_{k=1}^{k=n} A_k$$

Therefore, $B'_n(a, r, 1, \alpha) \leqslant 0$ and $B_n(a, r, 1, \alpha)$ is non-increasing.

Lemma 1.2. *(i)* $B_n(a, r, 1, 0) \geqslant B_n(a, r, 1, \alpha) \geqslant B_n(a, 1, 1, 0)$ *where* $r > 1$
(ii) $B_n(a, r, 1, 0) \leqslant B_n(a, r, 1, \alpha) \leqslant B_n(a, 1, 1, 0)$ *where* $0 < r < 1$.

Proof. Let us consider $r > 1$. By inequality of Minkowski, we have

$$\begin{aligned}
&B_n(a, r, 1, 0) \geqslant B_n(a, r, 1, \alpha)\\
\Leftrightarrow &\left(\frac{1}{n}\sum_{k=1}^{k=n} a_k^r\right)^{1/r} \geqslant \left(\frac{1}{n}\sum_{k=1}^{k=n} (a_k + \alpha)^r\right)^{1/r} - \alpha\\
\Leftrightarrow &\left(\frac{1}{n}\sum_{k=1}^{k=n} a_k^r\right)^{1/r} + \left(\frac{1}{n}\sum_{k=1}^{k=n} \alpha^r\right)^{1/r} \geqslant \left(\frac{1}{n}\sum_{k=1}^{k=n} (a_k + \alpha)^r\right)^{1/r}.
\end{aligned}$$

It is learn that $B_n(a, r, 1, \alpha) \geqslant B_n(a, 1, 1, 0)$ is equivalent to

$$\begin{aligned}
\Leftrightarrow &\left(\frac{1}{n}\sum_{k=1}^{k=n} (a_k + \alpha)^r\right)^{1/r} - \alpha \geqslant \frac{1}{n}\sum_{k=1}^{k=n} a_k\\
\Leftrightarrow &\left(\frac{1}{n}\sum_{k=1}^{k=n} (a_k + \alpha)^r\right)^{1/r} - \alpha \geqslant \left(\frac{1}{n}\sum_{k=1}^{k=n} (a_k + \alpha)\right) - \alpha\\
\Leftrightarrow &\left(\frac{1}{n}\sum_{k=1}^{n} (a_k + \alpha)^r\right)^{\frac{1}{r}} \geqslant \frac{1}{n}\sum_{k=1}^{n} (a_k + \alpha)
\end{aligned}$$

For $0 < r < 1$, the proof is similar.

For $p > q > 1$, we have

$$B_n(a, p, 1, 0) \geqslant B_n(a, q, 1, 0).$$

Theorem 1.1. *Given* $p > q > 1, \alpha \geqslant 0, a_k > 0, (k = \overline{1, n})$. *Then*

(i) $B_n(a, p, 1, 0) \geqslant B_n(a, p, q, \alpha) \geqslant B_n(a, q, 1, 0)$.

(ii) $B_n(a, p, q, \alpha)$ *is a non-increasing monotonic function (in variable* α*).*

Proof.

(i) Denote $A_k = a_k^q$, the required inequality is equivalent to

$$\Big(\frac{1}{n}\sum_{k=1}^{k=n} A_k^{p/q}\Big)^{1/p} \geqslant \Big[\Big(\frac{1}{n}\sum_{k=1}^{k=n}(A_k+\alpha)^{p/q}\Big)\Big)^{q/p} - \alpha\Big]^{1/q} \geqslant \big(\frac{1}{n}\sum_{k=1}^{k=n} A_k\big)^{1/q}.$$

Taking the q power of both sides and set $r = p/q > 1$, we obtain

$$\big(\frac{1}{n}\sum_{k=1}^{k=n} A_k^r\big)^{1/r} \geqslant \big(\frac{1}{n}\sum_{k=1}^{k=n}(A_k+\alpha)^r\big)^{1/r} - \alpha \geqslant \frac{1}{n}\sum_{k=1}^{k=n} A_k.$$

Applying the lemma(1.2) deduces that the inequality is true (ii) Let $a_k^q = A_k, r = \frac{p}{q} > 1$, then the function

$$B_n(a,p,q,\alpha) = \Big[\Big(\frac{1}{n}\sum_{k=1}^{k=n}(A_k+\alpha)^r\Big)\Big)^{1/r} - \alpha\Big]^{1/q}$$

is a non-increasing monotonic (Lemma 1.1).

3. Sharpening some types of the cyclic inequalities

We denote

$$G_n(a,\alpha) = \sum_{k=1}^{k=n-1} \frac{(a_k+\alpha)^2}{a_{k+1}+\alpha} + \frac{(a_n+\alpha)^2}{a_1+\alpha} - n\alpha.$$

It follows

$$G_n(a,0) = \sum_{k=1}^{k=n-1} \frac{a_k^2}{a_{k+1}} + \frac{a_n^2}{a_1}.$$

We will strengthen a simpler inequality

$$G_n(a,0) \geqslant \sum_{k=1}^{k=n} a_k = S_n(a),$$

where $a_i > 0, i = \overline{1,n}$. We obtain the following result.

Theorem 1.2.

(i) $G_n(a,\alpha)$ is a non-increasing monotonic function.

(ii) $G_n(a,0) \geqslant G_n(a,\alpha) \geqslant S_n(a)$.

Proof.

We have

$$\frac{(a_k+\alpha)^2}{a_{k+1}+\alpha} = \frac{(\alpha+a_{k+1}+a_k-a_{k+1})^2}{a_{k+1}+\alpha}$$
$$= \alpha + a_{k+1} + 2(a_k - a_{k+1}) + \frac{(a_k-a_{k+1})^2}{a_{k+1}+\alpha}.$$

Therefore,

$$G_n(a,\alpha) = \sum_{k=1}^{k=n} a_k + \sum_{k=1}^{k=n-1} \frac{(a_k-a_{k+1})^2}{a_{k+1}+\alpha} + \frac{(a_n-a_1)^2}{a_1+\alpha}.$$

We have

$$G'_n(a,\alpha) = -\frac{(a_n-a_1)^2}{(\alpha+a_1)^2} - \sum_{k=1}^{n-1} \frac{(a_k-a_{k+1})^2}{(\alpha+a_{k+1})^2} \leqslant 0.$$

Hence $G_n(a,\alpha)$ is a non-increasing monotonic function.

Since $\alpha \geqslant 0$, we get

$$G_n(a,\alpha) \leqslant G_n(a,0).$$

We have

$$\sum_{k=1}^{k=n-1} \frac{(a_k+\alpha)^2}{a_{k+1}+\alpha} + \frac{(a_n+\alpha)^2}{a_1+\alpha} \geqslant \sum_{k=1}^{k=n}(a_k+\alpha).$$

Then

$$G_n(a,\alpha) \geqslant \sum_{k=1}^{k=n}(a_k+\alpha) - n\alpha = \sum_{k=1}^{k=n} a_k$$

(Theorem is proved).

We denote

$$D_n(a,\alpha) = \sum_{k=1}^{k=n-1} \frac{(a_k+\alpha)^2}{a_k+a_{k+1}+2\alpha} + \frac{(a_n+\alpha)^2}{a_n+a_1+2\alpha} - \frac{n\alpha}{2},$$

where $\alpha \geqslant 0$.

It follows that

$$D_n(a,0) = \sum_{k=1}^{k=n-1} \frac{a_k^2}{a_k+a_{k+1}} + \frac{a_n^2}{a_n+a_1}.$$

We will sharpen an inequality

$$D_n(a,0) \geqslant \frac{1}{2}\sum_{k=1}^{k=n} a_k = \frac{1}{2}S_n(a)$$

where $a_i > 0, i = \overline{1,n}$. We obtain the following result.

Theorem 1.3. *(i) $D_n(a,\alpha)$ is a non-increasing function.*
(ii) $D_n(a,0) \geqslant D_n(a,\alpha) \geqslant \frac{1}{2}S_n(a)$.

Proof. We have

$$\frac{(a_k+\alpha)^2}{a_k+a_{k+1}+2\alpha} = \frac{\alpha^2+2a_k\alpha+a_k^2}{2\alpha+a_k+a_{k+1}}$$
$$= \frac{\alpha}{2}+\frac{3a_k-a_{k+1}}{4}+\frac{(\frac{a_k-a_{k+1}}{2})^2}{2\alpha+a_k+a_{k+1}}.$$

It follows that

$$D_n(a,\alpha) = \frac{1}{2}\sum_{k=1}^{k=n} a_k + \frac{1}{4}\sum_{k=1}^{k=n-1}\frac{(a_k-a_{k+1})^2}{2\alpha+a_k+a_{k+1}} + \frac{1}{4}\cdot\frac{(a_n-a_1)^2}{2\alpha+a_n+a_1};$$

$$D'_n(a,\alpha) = -\frac{1}{2}\cdot\sum_{k=1}^{k=n-1}\frac{(a_k-a_{k+1})^2}{(2\alpha+a_k+a_{k+1})^2} - \frac{1}{2}\cdot\frac{(a_n-a_1)^2}{(2\alpha+a_n+a_1)^2} \leqslant 0$$

Hence $D_n(a,\alpha)$ is a non-increasing monotonic function. Since $\alpha \geqslant 0$, $D_n(a,\alpha)$ is a non-increasing monotonic function. Therefore

$$D_n(a,\alpha) \leqslant D_n(a,0)$$

To complete the proof., we will show that $D_n(a,\alpha) \geqslant \frac{1}{2}S_n(a)$.

We have

$$D(a,\alpha) = \sum_{k=1}^{k=n-1}\frac{(a_k+\alpha)^2}{a_k+a_{k+1}+2\alpha} + \frac{(a_n+\alpha)^2}{a_n+a_1+2\alpha} - \frac{n\alpha}{2} \geqslant$$
$$\geqslant \frac{1}{2}\sum_{k=1}^{k=n}(a_k+\alpha) - \frac{n\alpha}{2} = \frac{1}{2}\sum_{k=1}^{k=n} a_k.$$

Theorem is proved.

We denote

$$F_n(a,\alpha) = \sum_{k=1}^{n-2}\frac{(a_k+\alpha)^2}{a_{k+1}+\beta a_{k+2}+(1+\beta)\alpha} + \frac{(a_{n-1}+\alpha)^2}{a_n+\beta a_1+(1+\beta)\alpha}$$
$$+ \frac{(a_n+\alpha)^2}{a_1+\beta a_2+(1+\beta)\alpha} - \frac{n\alpha}{1+\beta}$$

(where variables $\alpha \geqslant 0$ and $\beta > 0$ is given)

It follows that

$$F_n(a,0) = \sum_{k=1}^{n-2}\frac{a_k^2}{a_{k+1}+\beta a_{k+2}} + \frac{a_{n-1}^2}{a_n+\beta a_1} + \frac{a_n^2}{a_1+\beta a_2}.$$

We will sharpen the inequality

$$F_n(a,0) \geqslant \frac{1}{1+\beta}S_n(a).$$

We obtain the following result

Theorem 1.4.

(i) $F_n(a,\alpha)$ *is a non-increasing function*

(ii) $F_n(a,0) \geqslant F_n(a,\alpha) \geqslant \dfrac{1}{1+\beta}S_n(a).$

Proof.

We have

$$\frac{(a_k+\alpha)^2}{a_{k+1}+\beta a_{k+2}+(1+\beta)\alpha} = \frac{\alpha^2+2a_k\alpha+a_k^2}{(1+\beta)\alpha+a_{k+1}+\beta a_{k+2}}$$

$$= \frac{\alpha}{1+\beta}+\frac{2a_k}{1+\beta}-\frac{a_{k+1}+\beta a_{k+2}}{(1+\beta)^2}+\frac{\dfrac{(a_{k+1}+\beta a_{k+2})^2}{(1+\beta)^2}-\dfrac{2a_k(a_{k+1}+\beta a_{k+2})}{(1+\beta)}+a_k^2}{(1+\beta)\alpha+a_{k+1}+\beta a_{k+2}}$$

$$= \frac{\alpha}{1+\beta}+\frac{2a_k}{1+\beta}-\frac{a_{k+1}+\beta a_{k+2}}{(1+\beta)^2}+\frac{\left(\dfrac{a_{k+1}+\beta a_{k+2}}{1+\beta}-a_k\right)^2}{(1+\beta)\alpha+a_{k+1}+\beta a_{k+2}}.$$

Similary, we have

$$\frac{(a_{n-1}+\alpha)^2}{a_n+\beta a_1+(1+\beta)\alpha} = \frac{\alpha}{1+\beta}+\frac{2a_{n-1}}{1+\beta}-\frac{a_n+\beta a_1}{(1+\beta)^2}+\frac{\left(\dfrac{a_n+\beta a_1}{1+\beta}-a_{n-1}\right)^2}{(1+\beta)\alpha+a_n+\beta a_1}$$

$$\frac{(a_n+\alpha)^2}{a_1+\beta a_2+(1+\beta)\alpha} = \frac{\alpha}{1+\beta}+\frac{2a_n}{1+\beta}-\frac{a_1+\beta a_2}{(1+\beta)^2}+\frac{\left(\dfrac{a_1+\beta a_2}{1+\beta}-a_n\right)^2}{(1+\beta)\alpha+a_1+\beta a_2}$$

It follows that

$$F_n(a,\alpha) = \frac{1}{1+\beta}\sum_{k=1}^{n}a_k+\frac{\left(\dfrac{a_n+\beta a_1}{1+\beta}-a_{n-1}\right)^2}{(1+\beta)\alpha+a_n+\beta a_1}+\frac{\left(\dfrac{a_1+\beta a_2}{1+\beta}-a_n\right)^2}{(1+\beta)\alpha+a_1+\beta a_2}$$
$$+\sum_{k=1}^{n-2}\frac{\left(\dfrac{a_{k+1}+\beta a_{k+2}}{1+\beta}-a_k\right)^2}{(1+\beta)\alpha+a_{k+1}+\beta a_{k+2}}$$

$$(1+\beta)F_n'(a,\alpha) = -\frac{\left(\dfrac{a_n+\beta a_1}{1+\beta}-a_{n-1}\right)^2}{[(1+\beta)\alpha+a_n+\beta a_1]^2}-\frac{\left(\dfrac{a_1+\beta a_2}{1+\beta}-a_n\right)^2}{[(1+\beta)\alpha+a_1+\beta a_2]^2}$$
$$-\sum_{k=1}^{n-2}\frac{\left(\dfrac{a_{k+1}+\beta a_{k+2}}{1+\beta}-a_k\right)^2}{[(1+\beta)\alpha+a_{k+1}+\beta a_{k+2}]^2}\leqslant 0$$

Hence, $F_n(a,\alpha)$ is a non-increasing monotonic function .

Since $\alpha \geqslant 0$, we have

$$F_n(a,\alpha) \leqslant F_n(a,0)$$

To complete the proof, we will show that

$$F_n(a,\alpha) \geqslant \frac{1}{1+\beta} S_n(a).$$

Indeed

$$F_n(a,\alpha) \geqslant \frac{1}{1+\beta} \cdot \sum_{k=1}^{n} (a_k + \alpha) - \frac{n\alpha}{1+\beta} = \frac{1}{1+\beta} \sum_{k=1}^{n} a_k$$

Theorem is proved.

References

1 . P.S Bullen, D.S.Metrinovic' and P.M Vasic', *Means and their inequalities,* Reidel Publishing co, Dordrecht - Boston 1988.

2. G.H.Hardy, J.E Littewood, G.Polya, *Inequalities* , Cambridge University Press. 1952 .

3. D.S.Metrinovic' (with P.M Vasic'), *Analytic inequalities,* Springer Verlag, Berlin - Heidelberg - New York 1970.

4. G.V.Milovanovic', *Recent progress in Inequalities,* Kluwer academic publishers. 1996.