ON THE LINEAR RADICAL OF LATTICES

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1. Introduction

The radical theory is an important tool for studying the structure and the classification of algebraic structures. It attracts large interest of many authors [1, 2, 3, 5, 7]. The concept of radical has been proposed and studied for rings, K-algebras and algebraic structures closely related to them. There, the radicals are defined based upon their particular substructures, namely their ideals.

This paper deals with the concept of the linear radical of lattices. For a lattice L, we consider a particular type of its congruencies, which we call linear congruencies. The intersection of these congruencies, denoted by $r(L)$, is called to be the linear radical of L. We will prove that this radical property satisfies some fundamental properties similar to those of radicals of rings. Moreover, we also show that the class of all distributive lattices is r-semi simple. By Theorem 3.1 we present an application of the radical to classifying modular lattices.

2. Linear radical of lattices and its properties

2.1. ***Notations.*** Let L be a lattice, $a, b \in L$ and ρ be a congruence on L:

(a) Symbol $a I b$ means that a is incomparable with b and $[a]_\rho$ is ρ- equivalence class of a.

(b) $[a]_\rho < [b]_\rho$ if and only if $[a]_\rho \neq [b]_\rho$ and $\exists\, x \in [a]_\rho, \exists\, y \in [b]_\rho, x < y$.

(c) The trivial and the largest congruencies on L are denoted by Δ and τ, respectively.

2.2. Definition.

a) Let L be a lattice. A congruence ρ on L is called to be linear if the quotient lattice L/ρ is linear.

b) The lattice L is called to be r-radical and r-semi simple if $r(L) = \tau$ and $r(L) = \Delta$ respectively. The class of all the r-semi simple lattices is called to be r-semi simple.

2.3. Proposition. *If D is a distributive lattice and $a, b \in D, a \neq b$, then there exists congruence ρ, which consists exactly of two classes: $[a]_\rho$ and $[b]_\rho$.*

For an arbitrary lattice L, there exists at least one linear congruence, for example t. Thus, the family of linear congruence on L is non-empty. Then we have:

2.4. Definition.. *Let L be a lattice. The intersection of all linear congruencies on L is called to be the linear radical of L and it is denote by $r(L)$.*

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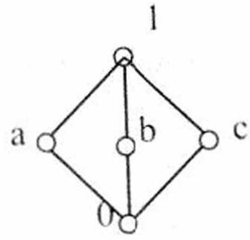
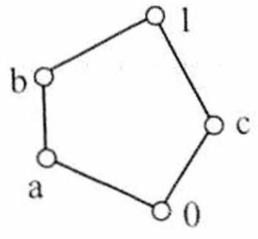
2.5. *Examples.* Consider lattices M_3 and N_5:



$\mathbf{M_3}$



$\mathbf{N_5}$

1) For M_3, there exists linear congruence τ only. Therefore $r(M_3) = \tau$.

2) For N_5, besides τ, there exist 2 linear congruencies. Namely, ρ_1 consists of the classes $\{a, b, 1\}, \{0, c\}$ and ρ_2 consists of the classes $\{0, a, b\}, \{c, 1\}$. Thus $\gamma(N_5) = \rho_1 \cap \rho_2$, which consists of classes $= \{a, b\}, \{c\}, \{0\}, \{1\}$.

2.6. Proposition.

(1) For an arbitrary lattice L, the quotient lattice $L/r(L)$ is distributive.

(2) Let D be a lattice. Then D is distributive if and only if $r(D) = \Delta$.

Proof. (1) We have $r(L) = \cap\{\rho_i | i \in I\}$, where $\rho_i, i \in I$, are linear congruencies on L. As well-known, $L/r(L)$ is the sub-direct product of quotient lattices $L/\rho_i, i \in I$. On the other hand, by definition, L/ρ_i is distributive for each $i \in I$. Thus, the quotient lattice $L/r(L)$ is distributive too.

(2) It follows directly from (2.3) and the part (1). □

2.7. Remark. *(a) Let ρ, σ be congruencies on L such that $\rho \subseteq \sigma$. The symbol ρ/σ denotes a congruence on L/ρ as follows:*

$$([x]_\rho, [y]_\rho) \in \sigma/\rho \Leftrightarrow (x, y) \in \sigma.$$

(b) Let $\varphi : L \to L'$ be a lattice homomorphism and σ be a congruence on L. Then on $\varphi(L)$ there exists a congruence $\varphi(\sigma)$:

$$(\varphi(x), \varphi(y)) \in \varphi(\sigma) \Leftrightarrow (x, y) \in \sigma.$$

Setting $\rho = Ker\,\varphi$ we have $\varphi(L) \cong L/\rho$ and $(\sigma \vee \rho)/\rho$ are a congruencies on L/ρ. It can be easily deduced that $(\varphi(x), \varphi(y)) \in \varphi(\sigma) \Leftrightarrow ([x]\rho, [y]\rho) \in (\sigma \vee \rho)/\rho$.

2.8.Proposition.. *Let L, L' be lattices and $\varphi : L \to L'$ be a homomorphism. then $\varphi(r(L)) \subseteq r(\varphi(L))$.*

Proof. Put $\rho = \mathrm{Ker}\varphi$. According to (2.7)(b), instead of $\varphi(L)$ and $\varphi(r(L))$ we consider L/ρ and $(r(L) \vee \rho)/\rho$, respectively.

According to the definition of radical, $r(L/\rho)$ is equal to the intersection of the linear congruence on L/ρ. By (2.7)(a) these congruencies are presented as σ/ρ, where σ is a congruence on L such that $\rho \subseteq \sigma \subseteq \tau$.

First, we prove: σ/ρ is linear on L/p if and only if σ is linear on L.

Necessity. Let σ/ρ be linear and $x, y \in L, (x, y) \notin \dot{\sigma}$. It is needed to prove that $[x]_\sigma < [y]_\sigma$ or $[y]_\sigma < [x]_\sigma$. According to (2.7)(a): $([x]_\sigma, [y]_\sigma) \notin \sigma/\rho$ and so we shall show: $[x]_\sigma < [y]_\sigma$. Since σ/ρ is linear, without loss of generality, we can assume that $[x]_\rho < [y]_\rho$.

By (2.1)(b), there exist $x' \in [x]_\rho$ and $y' \in [y]_\rho$ such that $x' < y'$. Since $\rho \subseteq \sigma$, it implies $x' \in [x]_\sigma, y' \in [y]_\sigma$, i.e., $[x]_\sigma < [y]_\sigma$.

Sufficiency. The proof is trivial.

Now, considering all the linear congruence $\sigma_i/\rho, i \in I$, on L/ρ, we have $r(L/\rho) = \cap\{\sigma_i/\rho|i \in I\} = \cap\{\sigma_i|i \in I\}/\rho$. May be $\{\sigma_i|i \in I\}$ does not contain all the linear congruence on L. Therefore $r(L) \subseteq \cap\{\sigma_i|i \in I\}$. Thus, $(r(L) \vee \rho)/\rho \subseteq \cap\{\sigma_i|i \in I\}/\rho$, i.e., $\varphi(r(L)) \subseteq r(\varphi(L))$.

The proof is completed. □

Summing up, we have proposed the concept of linear radical $r(L)$ of a lattice L and showed the important properties of $r(L)$ (see Propositions 2.6 and 2.8):

1) If $\varphi : L \to L'$ is a homomorphism then $\varphi(r(L)) \subseteq r(\varphi(L))$.

2) For an arbitrary lattice L, $r(L/r(L)) = \Delta$.

It is worth mentioning that these properties are formulated analogously to those of radicals of rings.

3. Application

In this section we present an application of the linear radical to classifying modular lattices. Here, for lattices, it is particular that P is r-semi simple class. Therefore, the classification problem is of interest itself.

For example, consider a modular lattice M. As in Example 2.5, we see that M_3 is r-radical lattice. On the other hand, the sublattices of M which are isomorphic to M_3 prevent M from being distributive. Since $M/r(M)$ is distributive, each sublattice isomorphic to M_3 belongs to one of equivalence classes of $r(M)$.

Based upon the above reasons we arrive at the study of the particular modular lattices formulated in the following theorem.

3.1. Theorem. *Let M be a modular lattice which is not distributive. Let A be a sublattice of M such that:*

1) A is convex.

2) A is r-radical.

3) A contains all sublattices of M which are isomorphic to M_3.

Then $r(M)$ has one class equal to A, the other classes (if they exist) are distributive.

Proof. We can assume that $M \neq\neq A$ and use the following lemma.

3.2.Lemma. *If $b \in M$ A then in M there exists a two-classes congruence, one class of which contains A , the other contains b.*

Proof For b and A, we have the following alternatives:

(I) $\exists a \in A, a < b$.

(II) Either $\forall a \in A, a||b$ or $\exists a \in A, a > b$.

For case (I), consider the principal filter generated by b : $F(b) = \{x \in M|x \geq b\}$. If $\exists c \in F(b) \cap A$ then $a < b < c$, it implies that $b \in A$ (due to convexity of A), but it contradicts the assumption. Thus $F(b) \cap A = \phi$.

We denote by $\mathcal{F}$ the family of all filters containing b but not any element of A. Consider $\mathcal{F}$ with relation $\subseteq$. Let $\{F_i|i \in I\}$ be a chain on $\mathcal{F}$, it is easy to deduce that

$\cap\{F_i/i \in I\} \in \mathcal{F}$. Due to Zorn's Lemma, there exists a maximal element of $\mathcal{F}$ which we denote by F.

We consider the ideal generated by A again $J(A) = \{x \in M/x \le a$ for some $a \in A\}$. Obviously, $J(b) \cap F = \phi$.

For case (II), we take the principal ideal generated by b : $J(A) = \{x \in M|x \le b\}$. Obviously, $J(b) \cap A = \phi$.

Similarly to case (I) we can deduce that there exist the maximal ideal J and the maximal filter F such that $b \in J, J \cap A = \phi$ and $A \subseteq F, J \cap F = \phi$.

Now, for both cases (I) and (II) we shall prove that $J \cup F = M$.

We suppose that $\exists c \in M, c \notin J \cup F$. First we prove the assertion:

(i) $\exists j \in J, j \vee c \in F$.

Indeed, if $\forall j \in J, j \vee c \notin F$, take the sublattice K generated by $J \cup \{c\}$. So K consists of elements as $c, j, j \vee c$ with $j \in J$. Denote $J(K) = \{x \in M|x \le k$ for some $k \in K\}$, it implies that $J \subseteq J(K)$ and $J(K) \cap F = \phi$. This contradicts the maximally of J. Thus (i) is proved.

By duality we have the similar assertion:

(ii) $\exists f \in F, f \wedge c \in J$

In the rest, we consider 3 elements j, f, c. We have:

$$j \wedge f, j \wedge c, f \wedge c \in J \text{ and } u = (j \wedge f) \vee (j \wedge c) \vee (f \wedge c) \in J,$$
$$j \vee f, j \vee c, f \vee c \in F \text{ and } v = (j \vee f) \wedge (j \vee c) \wedge (f \vee c) \in F.$$

It implies that $u \le v$ and since $J \cap F = \phi, u < v$.

Put x, y, z as follows:

$$x = (j \wedge v) \vee u = (j \vee u) \wedge v,$$
$$y = (f \wedge v) \vee u = (f \vee u) \wedge v,$$
$$z = (c \wedge v) \vee u = (c \vee u) \wedge v.$$

Furthermore, the equalities:

$$j \wedge v = j \wedge [(j \vee f) \wedge (j \vee c) \wedge (f \vee c)] = j \wedge (j \vee c) \quad (1)$$
$$f \vee u = f \vee [(j \wedge f) \vee (j \wedge c) \vee (f \wedge c)] = f \vee (j \wedge c) \quad (2)$$

hold.

Summing up, we have:

$$\begin{aligned}
x \wedge y &= [(j \wedge v) \vee u] \wedge [(f \vee u) \wedge v] \\
&= \{[(j \wedge v) \vee u] \wedge v\} \wedge (f \vee u) \\
&= \{[(j \wedge v) \wedge v] \vee u\} \wedge (f \vee u) && (u < v) \\
&= [(j \wedge v) \vee u] \wedge (f \vee u) \\
&= [(j \wedge v) \wedge (f \vee u)] \vee u && (u < f \vee u) \\
&= \{[j \wedge (f \vee c)] \wedge [f \vee (j \wedge c)]\} \vee u && (\text{see}(1),(2)) \\
&= \{(f \vee c) \wedge [j \wedge [f \vee (j \wedge c)]]\} \vee u \\
&= (f \vee c) \wedge [(j \wedge c) \vee (j \vee f)] \vee u && (j \wedge c < j) \\
&= [(j \wedge c) \vee (j \vee f)] \vee u = u
\end{aligned}$$

By duality we also obtain $x \vee y = v$.

Here, elements x, y, z play the similar role. Therefore, $x \wedge z = y \wedge z = u$ and $x \vee z = y \vee z = v$.

Thus, in M there exists sublattice $H = \{x, y, z, u, v\}$. Since H is isomorphic to M_3, it implies that either $H \subseteq J$ or $H \subseteq F$ according to the identification of J and F. But this contradicts the fact that $u \in J$ and $v \in F$.

In final, we see that $J \cup F = M$. This means that J and F form *a* two-class equivalence, where either $A \subseteq J, b \in F$ in case (I) or $b \in J, A \subseteq F$ in case (II). Because J is an ideal and F is a filter, it follows that this equivalence is a congruence. The Lemma is proved.

Now we finish the proof of Theorem 3.1. Since A is a r-radical lattice, every linear congruence on L has a class containing A. By virtue of Lemma (3.2) the equivalence class of $r(M)$ containing A is exactly equal to A. The other classes are sublattice of M. These sublattices contain no sublattice isomorphic to M_3, therefore, they are distributive.

The proof of theorem is completed. □

3.3.Remark. *It is worth remarking that lattices, in general, are not distributive because they may contain not only the sublattice M_3, but also N_5. Therefore, the classification problem is difficult (see Example 2.5).*

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VỀ CĂN TUYẾN TÍNH CỦA CÁC DÀN

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Bài báo này đề cập tới khái niệm của các dàn. Căn tuyến tính của dàn L được hiểu là giao của họ tất cả các tương đẳng tuyến tính trên nó và được ký hiệu là $r(L)$. Căn tuyến tính của dàn thoả mãn các tính chất cơ bản như các căn của vành và nhận lớp các dàn phaan phối làm lớp r-nửa đơn. Định lý 3.1 cho áp dụng hữu ích của căn tuyến tính trong việc phân loại các dàn Modular