VNU. JOURNAL OF SCIENCE, Mathematics - Physics. t.XVIII, n^{0}1 - 2002

ON THE RADICAL CHARACTERISTIC OF REGULARITIES

Tran Trong Hue
Faculty of Mathematics
College of Natural Sciences, Vietnam National University, Hanoi

I. Introduction

In this paper we shall work in the variety W of algebra's over an associative and commutative ring K which unity element. For a given subclass R of the variety W each algebra $A \in R$ is called R - algebra, and an ideal I of an algebra A is called R- ideal if I is an R-algebra.

Radical classes are meant in the sense of Kurosh [11] and Amitsur [1] and for details of radical theory we refer to [18] and [19]. It is well known that a non-empty subclass R is a radical class in W if and only if the following conditions are satisfied.

(i) R is homomorphically closed.

(ii) The sum $R(A)$ of all R-ideals of an algebra A is an R-ideal.

(iii) R is closed under extensions, that is, if both I and A/I are R- algebra's, then A is also R- algebra.

In ring theory much so - called regularity appear. The oldest one seems to be the von Neumann regularity. In 1936 Von Neumann [14] defined a ring A (with identity) to be regular if and only if for any element a of A there exists an element x of A such that $a = axa$. In 1950 Brown and Mccoy [6] generalized to rings without identity, and they succeeded in proving that it is a radical property. In the meantime (1942) Perlis [15] had introduced the concept of quasi-regularity for algebra with identity. He defined an algebra A with identity to be quasi-regular if and only if for any element x of A there exist an element y of A such that $x + y + xy = 0$. In 1945 Jacobson [10] generalized this concept to arbitrary ring without identity, and he showed that it is a radical property, called later on the Jacobson radical. In 1948 Brown and Mccoy [5] attempted for define a general concept of regularity for rings. That is the Brown-McCoy radical. At that time their theory was general enough. All regularities introduced up to then were regularities in the sense of it. However, in 1971 a wide class of regularities was introduced by Goulding and Ortiz [8], McKnight and Musser [12], Musser [13], namely the so-called (p, q) - regularities. They had showed that the (p, q) - regularities are radical properties. In 1975 Roos [16] gave a general definition of regularity for rings is terminology of the class of mappings $\{F_A\}$, where F_A mapps each ring A into the set of all subgroups of the additive group (A, t) of the ring A. The regularity in this sense satisfies the conditions of radical property. In 1981 Hue and Szasa [9] gave the definition of regularity of associative rings in the common terminology of polynomials and formal power series and showed the radical characteristic of regularities in this sense.

The aim of this paper is to give the general definition of regularity for arbitrary algebra, which includes all regularities known up till now, and to show the radical characteristic of regularities in this sense. We hope that we are going to get a diagram to define correct radical classes of W.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

II. S-regularities

1. *Definitions*

S - regularity

Let there be assigned to each an algebra A belonging to W a mapping S_A which mapps the direct sum $A^\infty = A_\oplus A_\oplus \ldots$ into the algebra A. The class S consisting of all mappings S_A will be called a regularity for algebra of W if the following condition is satisfied.

For every $A, B \in W$ and $f \in \mathrm{Hom}_K(A, B)$, we have the commutative diagram

$$\begin{tikzcd} A^\infty \arrow[r, "S_A"] \arrow[d, "f^\infty"'] & A \\ B^\infty & B \arrow[l, "S_B"] \arrow[u, "f"'] \end{tikzcd} \tag{a}$$

where $f^\infty = (f, f, \ldots.)$.

S - regular algebra

An element a of the algebra A is called S-regular if $a \in \Im - S_A$. An algebra A is said to be S-regular if $\Im S_A = A$. An ideal I of an algebra A is called S-regular if I is an S-regular algebra.

2. *Properties*

We obviously have the following.

Proposition 1. *If the class S is a regularity then $S_A((0, 0, \ldots)) = 0$ for every algebra A of the class W.*

Proposition 2. *The class of the S-regular algebras is homomorphically closed.*

Proof. Let B be an image of an S-regular algebra A under the homomorphism f. Now let b be an arbitrary element of B. Then there exists an element a of A such that $b = f(a)$. Since A is an S-regular algebra there is an element x of A^∞ such that $S\_A(x) = a$. By the commutative diagram (a) we have:

$$b = f(a) = f(S_A(x)) = S_B(f^\infty(x))\Im S)\_B - .$$

Therefore the algebra B is S-regular.

Theorem 3. *If the class $S = \{S_A : A^\infty \to A\}_{A \in W}$ is a regularity then the class of all S-regular algebra is a radical class in W if and only if the following condition is satisfied.*

If I is an S-regular ideal of the algebra A and for every element a of A there exists an element x of A^∞ such that $S_A(x) - a \equiv 0 \mod I$, then A is a regular algebra.

Proof. Assume that the class R of all S-regular algebras is a radical class. Now suppose that I is an S-regular ideal of an algebra and for every a of A there exists an element x of A^∞ such that $S_A(x) - a \equiv 0 \mod I$. We have to show that the algebra A is S-regular. Let us consider the factor algebra A/I. Take any element $\overline{a}$ of A/I. By hypothesis there exists an element x of A^∞ such that $S\_A(x) - a \equiv 0 \mod I$. So in the factor algebra A/I the equality $\overline{a} = \overline{S_A(x)}$ holds. For the natural homomorphism $p : A \to A/I$ we have the commutative diagram

$$\begin{tikzcd} A^{\infty} \arrow[r, "S_A"] \arrow[d, "p^{\infty}"'] & A \\ (A/I)^{\infty} & A/I \arrow[u, "p"'] \arrow[l, "S_{A/I}"] \end{tikzcd} \tag{b}$$

We have $\overline{a} = \overline{S_A(x)} = p(S_A(x)) = S_{A/I}(p^{\infty}(x))$. Therefore the element $\overline{a}$ is S-regular. This implies the S-regularity of the algebra A/I. Since radical classes are closed under extensions, the algebra A is S-regular.

Conversely, assume that the S-regularity satisfies the condition of the theorem. We shall show that the class R of all S-regular algebras is a radical class. Clearly, the class R is not empty.

By proposition 2 the class R is homomorphically closed. The condition (i) of the radical property is satisfied.

Now suppose that for an ideal of an algebra A, both I and A/I are R-algebra. Since the algebra A/I is S-regular therefore for every element a of A there exists an element $\overline{x}$ of $(A/I)^{\infty}$ such that $S_{A/I}(\overline{x}) = \overline{a}$. By the commutative diagram (b) we have $\overline{a} = S_{A/I}(\overline{x}) = S_{A/I}(p^{\infty}(x)) = p(S_A(x)) = \overline{S_A(x)}$. This implies $S_A(x) - a \equiv 0 \mod I$. By the condition of theorem the algebra A is S-regular. Hence the class R is closed under extensions. The condition (iii) of the radical property is satisfied.

By the proposition 1 the zero ideal of an algebra A is an S-regular ideal. Hence the set of the R-ideals of algebra A is not empty. Suppose both I_1 and I_2 be R-ideals of the algebra A. By the second isomorphism theorem we have.

$$\frac{I_1 + I_2}{I_2} \cong \frac{I_1}{I_1 \cap I_2}$$

Since the class R is homomorphically closed and closed under extensions, the above isomorphism implies that $I_1 + I_2$ is an R-algebra. By a simple induction we can prove that the sum of any finite number of R-ideals of the algebra A is again an R-ideal. Finally, we have to show that the sum $R(A)$ of all R-ideals of the algebra A is an S-regular ideal. Take any element a of $R(A)$. Then there are the R_n-ideals $I_1, ..., I_n$ such that the ideal $J = \sum_{k=1}^{n} I_k$ contains the element a. Sine J is an S-regular ideal there is an element x of J^{∞} such that $S_J(x) = a$. For the embedding $i_J : J \to R(A)$ we have the commutative diagram.

$$\begin{tikzcd} J^{\infty} \arrow[r, "S_J"] \arrow[d, "i_J^{\infty}"'] & J \\ R(A)^{\infty} & R(A) \arrow[u, "i_J"'] \arrow[l, "S_{R(A)}"] \end{tikzcd} \tag{c}$$

We have

$$a = i_{-}J - (a) = i_J(S_J(x)) = S_{R(A)}(i_{-}J^{\infty}(x))$$

Therefor the ideal $R(A)$ is S-regular. This completes the proof of the theorem.

As the radical criterions of S-regularities we have the following assertions.

Proposition 4. *The class of all S-regular algebras is a radical class if the following condition is satisfied.*

For arbitrary elements a in an algebra A, and x of A^∞ if the element $S_A(x) - a$ is S-regular then the element a is also S-regular.

Proof. We are going to show that the condition of theorem 3 is valid. Let I be can S-regular ideal of an algebra A with the following property: For every element a of A there exists an element x of A^∞ such that $S_A(x) - a \equiv 0 \mod I$. Therefore the element $S_A(x) - a$ belongs to the ideal I. Since I is an S-regular ideal so the element $S_A(x) - a$ is S-regular. By hypothesis the element a is S-regular. Thus the algebra A is S-regular. The condition of Theorem 3 is satisfied. The proposition is proved.

Proposition 5. *The class of all S-regular algebras is a radical class if the following condition class if the following condition is satisfied.*

Let I be can S-regular ideal of an algebra A. If the element $\overline{a}$ of the factor algebra A/I is S-regular, then the element a is S-regular in the algebra A.

Proof. Assume that I is an S-regular ideal of an algebra A, and for every element a of A there exists an element x of A^∞ such that $S_A(x) - a \equiv 0 \mod I$. Hence in the factor algebra A/I we have $\overline{S_A(x)} = \overline{a}$. By the commutative diagram (b) we have $\overline{a} = \overline{S_A(x)} = p(SA(x)) = S_{A/I}(p^\infty(x))$.

Therefore $\overline{a}$ is an S-regular element of the algebra A/I. By hypothesis the element a is S-regular in the algebra A. Thus the algebra A is S-regular. The condition of Theorem 3 is valid. The proposition is proved.

III. Examples of S-regularities

In this section we shall list the S-regularities which are known to us. One had proved that these regularities are the radical properties in the sense of Kurosh and Amitsur, i.e. these well-known regularities are the S-regularities satisfying the condition of Theorem 3. Suppose that W_0 is the variety of associative algebra. For arbitrary subsets X and Y of an algebra A we denote $XY = \left\{\sum_{i=1}^n x_i y_i : x_i \in X, y_i \in Y\right\}$. Let us consider some following regularities $S^K = \{S_A^k : A^\infty \to A\}_{A \in W_0}$.

1. $S_A^1((a_1, a_2, \ldots)) = a_1 a_2 a_1$.

An element a of an algebra A is said to be regular in the sense of Neumann [14] if $a \in aAa$. Clearly, S^1-regular coincides with the regularity in the sense of Neumann.

2. $S_A^2((a_1, a_2, \ldots)) = -(a_1 + a_1 a_2)$.

The right quasi-regularity had been defined by Pelis [15] and later studied by Baer [3] and Jacobson [10]. An element a of an algebra A is said to be right quasi-regular if $a + b + ab = 0$ for some element b of A. Hence S^2-regularity is right - quasi regularity.

3. $S_A^3((a_1, a_2, \ldots)) = a_2 + a_2 a_1 + \sum i = 1^\infty a_{2i+1} a_{2(i+1)} + a_{2i+1} a a_{2(i+1)}$

Brown and Mccoy [5] have introduced the notion of G-regularity. An element a of an algebra A is said to be G-regular if the element a is in $G(a)$, where

$$G(a) = A(1 + a) + A(1 + a)A$$

It is clear to see that S^3-regularity is G-regularity.

4. $S_A^4((a_1, a_2, ...)) = a_1^2 a_2$

The notion of strongly regular algebra had been introduced by Arens and Kaplansky [2] and was later studied by others. An algebra A is strongly regular if $a \in a^2A$ for every $a \in A$. It is clear that S^4-regularity is the same as strong regularity.

5. $S_A^5((a_1, a_2, ...)) = \sum_{i=1}^{\infty} a_{2i}a_1a_{2i+1}$

De La Rose [17] has introduced the notion of λ-regularity. An element a of an algebra A is λ-regular if $a \in AaA$. Clearly, S^5-regularity is λ-regularity.

6. $S_A^6((a_1, a_2, ...)) = -a_2(a_1 + a_1^2)$

Divinsky [7] has introduced left pseudo-regularity. An element a of an algebra A is left pseudo-regular if $a + ba + ba^2 = 0$ for some element b of A. It is easy to see that S^6-regularity coincides with left pseudo-regularity.

7. $S_A^7((a_1, a_2, ...)) = \sum_{i=1}^{\infty} a_{3i-1}a_1a_{3i}a_1a_{3i+1}$

Blair [4] introduced the notion of f-regularity, which was later studied by others. An element a of an algebra A is said to be f-regular if $a \in (a^2)$, where (a) denotes the principal ideal of A generated by a. Blair has shown that an element a in an algebra A is f-regular if and only if there exist elements $u_- - i, v_i$ and w_i in A such as $a = \sum_{i=1}^{n} u_iav_iaw_i$. Hence, S^7-regularity is the same as f-regularity in the sense of Blair.

8. $S_A^7((a_1, a_2, ...)) = p(a_-1)a_2q(a_1)$, where $p(x)$ and $q(x)$ are in the polynomial ring $K[x]$.

The (p, q)-regularity was introduced by Mcknight and Musser [12]. An algebra A is (p, q) - regular if the inclusion $a \in p(a)Aq(a)$ holds for every element a of A, where $p(x), q(x)$ are in $K[x]$. It is easy to see that S^8-regularity is (p, q)- regularity in the sense of Kcknight and Musser.

The open problems

Problem 1. *Find a necessary and sufficient condition for the S-regularity is hereditary.*

Problem 2. *Establish some diagrams to define concret radical classes by S-regularities.*

References

1. S.A. Amitsur. A general theory of radicals, *i. Amer. J. Math.*, **74**(1952), 774-786.
2. R.F. Arens and I. Kaplansky. Topological representation of algebras, *Trans. Amer. Math. Soc.*, **63**(1948), 457-481.
3. R. Baer. Radical ideals, *Amer. J. Math.*, **65**(1943), 537 - 568.
4. R.L. Blair. A note on f-regularity in rings. *Proc. Amer. Math. Soc.*, **6** (1955), 511 - 515.
5. B.Brown and N.H. Mccoy. The radical of a ring, *Duke Math. J.*, **15**(1948), 495 - 499.
6. B. Brown and N.H. Mccoy. The maximal regular ideal of a ring, *Proc. Amer. Math. Soc.*, **1**(1950), 165 - 171.
7. N. Divinsky. Pseudo-regularity, *Canad. J. Math.*, **7** (1958), 401 - 410.

8. T. L. Goulding and A. H. Ortiz. Structure of semiprime (p, q) - radicals. *Pac. J. Math.*, **37**(1971), 97 - 99.
9. T. T. Hue and F. Szasz. On the radical classes determined by regularities. Acta. Sci. Math., **43**(1981), 131 - 139.
10. N. Jacobson. Some remarks on one-sided inverses. *Proc. Amer. Math. Soc.*, **1**(1950), 352 - 355.
11. A. G. Kurosh. Radicals of rings and algebras. *Mat. Sb.*, **33**(75) (1953), 13 - 26.
12. J. D. Kcknight and G. L. Musser. Special (p,q) - radicals. *Canad. J. Math.*, **24**(1972), 38 - 44.
13 G. L. Musser. Linear semiprime (p,q) -radical. *Pac. J. Math.*, **37**(1971), 749 - 757.
14. Von Neumann. On regular rings. *Proc. Math. Acad. Sci.*, **22**(1936), 767 - 783.
15. S. Perlis. A characterization of the radical of an algebra. *Bull. Amer. Math. Soc.*, **48**(1942), 128 - 132.
16. C. Roos. *Regularities of rings*. Dissertation Delft. 1975.
17. B. De La Rose. *Ideals and radicals*. Dissertation Delft. 1970.
18. F. A. Szasz. *Radiacals of rings*. Akadmiai Kiad, Budapest 1981.
19. R. Wiegandt. Radical and semisimple classes of rings. *Queen's Papers in Pure and Appl. Math.*, Vol. **37** (Kingston, 1974).

TAP CHÍ KHOA HOC ĐHQGHN, Toán - Lý, t.XVIII, n^{0}1 - 2002

VỀ ĐẶC TRƯNG CĂN CỦA CÁC TÍNH CHẤT CHÍNH QUY

Trần Trọng Huệ

Khoa Toán - Cơ - Tin học

Đại học Khoa học Tự nhiên - ĐHQG Hà Nội

Xét W là đa tạp các đại số (không nhất thiết kết hợp) trên vành K giao hoán có đơn vị. Với mỗi đại số A thuộc W ký hiệu A^∞ là tổng trực tiếp $A_o plus A_\oplus$.... Mỗi lớp các ánh xạ $S = \{SA : A^\infty \to A\}_{A\in W}$ gọi là một tính chất S - chính quy nếu điều kiện sau được thoả mãn:

Đối với mọi A, B thuộc W và f thuộc $\hom_K(A, B)$ ta có hệ thức giao hoán $f.S_A = S_B.f^\infty$, trong đó $f^\infty = (f, f, ...)$. Phần tử a của đại số A gọi là S- chính quy nếu $a \in \Im S_A$. Đại số A gọi là S-chính quy nếu $\Im S_A = A$. Iđêan I của đại số A gọi là S-chính quy nếu I là một đại số S- chính quy.

Trong bài báo này chúng tôi đã chứng minh được rằng tính chất S - chính quy là một tính chất căn theo nghĩa Kurosh và Amitsur khi và chỉ khi điều kiện sau được thoả mãn:

Nếu I là một iđêan S - chính quy của đại số A và đối với mọi $a \in A$ tồn tại phần tử $x \in A^\infty$ sao cho $S_A(x) - a \equiv 0 \mod I$ thì A là một đại số S- chính quy.

Từ đặc trưng này ta chứng minh được hai điều kiện đủ để một tính chất S - chính quy là tính chất căn.

Trong trường hợp W là đa tạp các đại số kết hợp thì khái niệm S- chính quy và các kết quả của bài báo này là sự tổng quát hoá các tính chất chính quy của các tác giả Von Neumann, Perlis (căn Jacobson) Brown - McCoy, Kaplansky, De la Rose, Divinsky, Blair, Maknigh - Musser, v.v...