STABILITY RADII FOR DIFFERENTIAL ALGEBRAIC EQUATIONS

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Abstract. *In this article, we deal with the problem of computing stability radii for systems described by differential algebraic equations of the form $AX'(t)+BX(t)=0$, where A, B are constant matrices. A computable formula for complex radii is given and the key difference between ODEs and DAEs cases is pointed out. A special case where the real stability radii and complex one are equal is considered.*

Key Words and Phrases: Differential algebraic equation, index of matrices pencil, stability radii.

Introduction.

In the last decade, a large amount of works has been devoted to robustness measures among them there is a powerful tool, namely the stability radius, which was introduced by Hinrichsen and Pritchart (see [2]). It is defined as the smallest value ρ of the norm of real or complex perturbations destabilizing the system. If complex perturbations are allowed, ρ is called the complex stability radius. If only real perturbations are considered, the real radius is obtained. A detailed analysis of the stability radius for ordinary differential equations can be found in [2,3,4].

In this article, we deal with the computation of stability radii of systems described by a differential algebraic equation

$$AX'(t) - BX(t) = 0, \tag{1.1}$$

with constant matrices A and B. This problem has been well investigated for the case of nonsingular matrix A, when (1.1) turns into an explicit system of ordinary differential equations (ODEs for short) $X'(t) = MX(t)$, where the matrix $M = A^{-1}B$. According to the works in [2], [3]... the stability radii can be characterized by the matrix M and it is computed in principle. If the matrix A is singular, then the investigation of the index of the pencil $\{A, B\}$ is necessary but the situation becomes more complicated.

It is known that in ODEs case, if the original equation (1.1) is stable, then by continuity of spectrum, the stability radius is positive. However, this property is no longer

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX valid in the case of differential algebraic equations (DAEs for short). The main reason is that the structure of solutions of differential algebraic equations depends strongly on the index of the pencil $\{A, B\}$ and the solutions of (1.1) have some fixed components. Under the perturbations, the index of the perturbed systems may be changed, that implies the changing of the dimension of these fixed one and some of eigenvalues may be "disappeared" which causes the stability radius of (1.1) perhaps to be equal to 0. Moreover, it is different to ODEs case, in which we always are able to find a disturbance whose norm equals to stability radius ρ and under which our system is unstable, such a matrix in DAEs case may not exist.

Therefore, to study stability radii of algebraic differential equations, one must pay attention on the index of equation or the disturbances must having some special forms that we call structured perturbations in order to exclude "violent factors".

The article is organized as follows: In the next section, we study some basic properties of differential algebraic equations. Section 3 deals with a formula for computing the stability radius of (1.1) where structured disturbances are considered. Section 4 is concerned with a special class of the pencil of matrices $\{A, B\}$ for which the complex and real stability radii are equal.

2. Preliminary.

Consider the equation

$$AX'(t) - BX(t) = 0, \tag{2.1}$$

where $X \in R^m$, A and B are constant matrices in $K^{m\times m}$, ($K = C$ or $K = R$), $\det A = 0$; the pencil of matrices $\{A, B\}$ is supposed to be regular, index $\{A, B\} = k \geq 1$. It is known that there exists a pair of nonsingular matrices W, T such that

$$A = W\begin{pmatrix} I_r & 0 \\ 0 & U \end{pmatrix} T^{-1}, \qquad B = W\begin{pmatrix} B_1 & 0 \\ 0 & I_{m-r} \end{pmatrix} T^{-1}, \tag{2.2}$$

where I_s is the unit matrix in $K^{s\times s}$. Further $B_1 \in K^{r\times r}$, U is a $k-$ nilpotent matrix having the Jordan box form, i.e., $U = diag(J_1, J_2, ..., J_l)$ with

$$J_i = \begin{pmatrix} 0 & 1... & 0 \\ 0 & 0... & 1 \\ 0 & 0... & 0 \end{pmatrix} \in K^{p_i\times p_i}, (i = 1, 2, ..., l) \tag{2.3}$$

such that $\max_{1\leq i\leq l} p_i = k$ (see [5]). Multiplying both sides of (2.1) by W^{-1} we obtain

$$Y'(t) - B_1Y(t) = 0, \tag{2.4}$$

$$UZ'(t) - Z(t) = 0, \tag{2.5}$$

where $T^{-1}X(t) = \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix}$, $Y(t) \in K^r, Z(t) \in K^{m-r}$. Since U is a $k-$ nilpotent matrix, it is easy to see that the equation $UZ'(t) - Z(t) = 0$ has only a unique solution $Z = 0$. Thus, the above system is reduced to

$$Y'(t) - B_1 Y(t) = 0,$$
$$Z(t) = 0,$$

where $Y(t) \in K^r, Z(t) \in K^{m-r}$.

The trivial solution $X \equiv 0$ of (2.1) is said to be asymptotically stable if there are a certain projection $P \in \mathcal{L}(K^m)$ and positive constants α, c, such that the solution of the initial value problem

$$AX'(t) - BX(t) = 0,$$
$$P(X(0) - X_0) = 0$$

is unique and the estimate $\|X(t)\| \le c.\|\Pi X_0\| e^{-\alpha t}, t \ge 0$ holds. In fact, if the index of $\{A, B\} = 1$ we choose $P = I - Q$ where Q is the projection on $ker\, A$ along $S = \{z \in C : Bz \in imA\}$.

We denote by $\sigma(C, D)$ the spectrum of the pencil $\{C, D\}$, i.e., the set of all solutions of the equation $\det(\lambda C - D) = 0$. In case $C = I$, we write simply $\sigma(D)$ for $\sigma(I, D)$. It is known that system (2.1) is asymptotically stable iff all finite eigenvalues of the pencil $\{A, B\}$ lie within the half left hand side of complex plan (see[5]). If $\sigma(A, B) = \emptyset$ then (2.1) has only a unique solution $X(t) = 0$. Indeed, $\sigma(A, B) = \emptyset$ implies that for any s, $\det(sA - B) = \det W \det(sI_{d-k} - B_1)\det(sU - I)\det T^{-1} =$ nonzero constant. Thus $d - k = 0$, i.e., the equation (2.4) must be absent. Hence, (2.1) is equivalent to (2.5) which has only a trivial solution $X(t) = 0$. In this case we also consider (2.1) is asymptotically stable by choosing $P = 0$.

3. Structured disturbances.

As is done in ODE's case, one fixes a pencil of matrices $\{A, B\}$ to be stable; a pair of matrices $E \in K^{m \times p}, F \in K^{q \times m}$, and consider the disturbed system

$$AX'(t) - (B + E\Delta F)X(t) = 0, \tag{3.1}$$

where $\Delta \in K^{p \times q}$. The matrix $E\Delta F$ is called structured disturbance. Denote by

$$\mathcal{V}_K = \{\Delta \in K^{p \times q} : \text{ (3.1) is either irregular or unstable } \}$$

i.e., $\mathcal{V}_K$ is the set of "bad" disturbance. Let $d_K = \inf\{\|\Delta\| : \Delta \in \mathcal{V}_K\}$. We call d_K the structured stability radius of the quadruple $\{A, B, E, F\}$. If $K = C$, we have complex stability radius and if $K = R$ we have real stability radius.

First, we investigate the complex stability radius of (2.1), i.e., $K = C$. Similar as in ODEs, put $G(s) = F(sA - B)^{-1}E$ and we shall prove that

$$d_C = [\sup_{s \in C^+} \|G(s)\|]^{-1}.$$

We point out $d_C \ge [\sup_{s \in C^+} \|G(s)\|]^{-1}$. Taking $\Delta \in \mathcal{V}_C$, there are two cases:

a) The pencil of matrices $\{A, B + E\Delta F\}$ is regular. Then, we take a value $s \in \sigma(A, B + E\Delta F)$ (it notes that $\sigma(A, B + E\Delta F) \neq \emptyset$ since $\Delta \in \mathcal{V}_C$). Suppose that $x \neq 0$

is its corresponding eigenvector, that is $sAx - (B + E\Delta F)x = 0$, or equivalently, $x = (sA - B)^{-1}E\Delta Fx$, which follows that

$$Fx = F(sA - B)^{-1}E\Delta Fx = G(s)\Delta Fx$$

Hence,

$$\|\Delta\| \geq \|G(s)\|\|^{-1} \geq [\sup_{s\in C^+} \|G(s)\|\|^{-1}$$

for all $\Delta \in \mathcal{V}_C$ which implies that $d_C \geq [\sup_{s\in C^+} \|G(s)\|\|^{-1}$.

b) The pencil of matrices $\{A, B + E\Delta F\}$ is irregular, then for any $s \in C^+$ it exists a vector $x \neq 0$ such that $sAx - (B + E\Delta F)x = 0$. By using a similar procedure we can prove $d_C \geq [\sup_{s\in C^+} \|G(s)\|\|^{-1}$.

We now prove the inverse relation $d_C \leq [\sup_{s\in C^+} \|G(s)\|\|^{-1}$. For any $\varepsilon > 0$, we find $s_0 \in C^+$ such that $\|G(s_0)\|^{-1} \leq [\sup_{s\in C^+} \|G(s)\|\|^{-1} + \varepsilon$. Suppose that $u \in C^q$ such that $\|u\| = 1$ and $\|G(s_0)u\| = \|G(s_0)\|$. A corollary of Haln-Banach theorem follows that there is a linear function y^* defined on C^p such that $\|y^*\| = 1$ and $y^*G(s_0)u = \|G(s_0)u\| = \|G(s_0)\|$. Put $\Delta = \|G(s_0)\|^{-1}uy^* \in C^{p\times q}$. It is clear that

$$\Delta G(s_0)u = \|G(s_0)\|^{-1}uy^*G(s_0)u = \|G(s_0)\|^{-1}u\|G(s_0)\| = u.$$

Hence, $\|\Delta\| \geq \|G(s_0)\|^{-1}$. On the other hand, from $\Delta = \|G(s_0)\|^{-1}uy^*$ we have $\|\Delta\| \leq \|G(s_0)\|^{-1}$. Therefore, $\|\Delta\| = \|G(s_0)\|^{-1}$. Further, since $\Delta G(s_0)u = u$, we obtain $E\Delta G(s_0)u = Eu \neq 0$. Let $x := (s_0A - B)^{-1}Eu$ then $(s_0A - B)x = Eu$ which follows $E\Delta Fx = (s_0A - B)x$ or $(s_0A - B - E\Delta F)x = 0$, i.e., $s_0 \in \sigma(A, B + E\Delta F)$. This means that the system

$$AX'(t) - (B + E\Delta F)X(t) = 0$$

is unstable. Therefore, $\Delta \in \mathcal{V}_C$. Further,

$$d_C \leq \|\Delta\| = \|G_(s_0)^{-1} \leq [\sup_{s\in C^+} \|G(s)\|\|^{-1} + \varepsilon$$

Because ε is arbitrary then $d_C \leq [\sup_{s\in C^+} \|G(s)\|\|^{-1}$. Thus,

$$d_C = [\sup_{s\in C^+} \|G(s)\|\|^{-1}.$$

We note that the function $G(s)$ is analytic on the half plan C^+ then by maximum principle, it only attains maximum at $s = \infty$ or on iR. Thus

$$d_C = [\sup_{s\in iR} \|G(s)\|\|^{-1}.$$

Following the above argument, we see that if there exists $s_0 \in C^+$ such that $\|G(s_0)\| = [\sup_{s\in C^+} \|G(s)\|]$ then

$$d_C = \|G(s_0)\|^{-1} = [\max_{s\in C^+} \|G(s)\|\|^{-1}.$$

Moreover, if the matrix Δ is given by

$$\Delta = \|F(s_0A - B)^{-1}E\|^{-1}uy^*$$

then Δ is "bad" matrix with $\|\Delta\| = d_C$.

However, the above argument does not allow us to compute a "bad" matrix Δ whose norm equals to d_C as in ODEs case even we take the limit as $s \to \infty$. We now show that if $\|G(s)\|$ does not attain its maximum over half plane C^+ then there is no matrix Δ such that $d_C = \Delta$ and the system $AX' - (B + E\Delta F)X = 0$ is unstable. Suppose, in contrary, there is such a matrix Δ. Let $s_0 \in \sigma(A, B + E\Delta F) \cap C^+$ and x is its eigenvector, i.e., $s_0Ax - (B + E\Delta F)x = 0$ which implies that $\|\Delta\| \geq \|G(s_0)\|^{-1} > [\sup_{s \in C^+} \|G(s_0)\|]^{-1} = d_C$. This is contradiction.

Moreover, for any sequence (s_n) in C^+ which maximizes $\|G(s)\|$ at ∞ and Δ_n associated to s_n is constructed as above (we can suppose that there exists $\lim_{n\to\infty} \Delta_n = \Delta_0$, if not we take a subsequence), then the system $(AX' - (B + E\Delta_0F)X = 0$ is stable. Since the set of matrices Δ such that the pencil of matrices $\{A, B + E\Delta F\}$ has the index 1 is open then the index of $\{A, B + E\Delta_0F\}$ must bigger then 1.

We consider a special case where $E = F = I$ (unstructured disturbances). As is seen, the stability radius with unstructured disturbances is

$$d_C = [\sup_{s \in iR} \|G(s)\|]^{-1},$$

where $G(s) = (sA - B)^{-1}$. We prove that if $\operatorname{ind}(A, B) = k > 1$ then the matrix function $G(s)$ is unbounded on iR. Indeed,

$$\begin{aligned} G(s) = (sA - B)^{-1} &= T\left(\begin{pmatrix} sI_r & 0 \\ 0 & sU \end{pmatrix} - \begin{pmatrix} B_1 & 0 \\ 0 & I_{m-r} \end{pmatrix}\right)^{-1} W^{-1} \\ &= T\begin{pmatrix} (sI_r - B_1)^{-1} & 0 \\ 0 & (sU - I)^{-1} \end{pmatrix} W^{-1} \\ &= T\begin{pmatrix} (sI_r - B_1)^{-1} & 0 \\ 0 & -\sum_{i=0}^{k-1}(sU)^i \end{pmatrix} W^{-1} \to \infty \end{aligned}$$

as $s \to \infty$. Therefore, in this case, $d_C = 0$. This means that under a very small disturbance, the DAEs with the index greater then 2 is no longer stable.

If $\operatorname{ind}(A, B) = 1$, it is easy to prove that $\|G(s)\|$ is bounded on C^+, i.e., $d_C > 0$ but perhaps it does not exist any "bad" matrix Δ such that $\|\Delta\| = d_C$.

Summing up we obtain

Theorem 3.1. *a) The complex stability radius of System (2.1) is given by*

$$d_C = [\sup_{s \in iR} \|G(s)\|]^{-1},$$

where, $G(s) = F(sA - B)^{-1}E$.

b) There exists a "bad" matrix Δ such that $\|\Delta\| = d_C$ if and only if $G(s)$ attains its maximum over iR.

c) In the case $E = F = I$, $d_C > 0$ if and only if $ind(A, B) = 1$.

A question rises here: whenever the function $\|G(s)\|$ attains its maximum at a finite value s_0. We firstly remark that the answer depends strongly on the chosen norm of C^m since $\|G(s)\|$ has maximum values in one norm but has not in another one. To simplify the situation, we solve the problem with $A, B \in R^m$ and with a Euclid norm in the set of $m \times m-$ matrices, that is if $M = (m_{ij})$ is a $m \times m-$ matrix then $\|M\|^2 = \sum |m_{ij}|^2$. We deal with the way to obtain the decomposition (2.2). First we decompose $(A-B)^{-1}A$ into Jordan form by a nonsingular matrix S, that is $(A - B)^{-1}A = Sdiag(M, V)S^{-1}$, where V is a nilpotent matrix of the form (2.3) and M is nonsingular. The matrix W and T in (2.2) is given by

$$W = (A - B)Sdiag(M, I); \qquad T = Sdiag(I, (V - I)^{-1}); \qquad U = V(V - I)^{-1}. \quad (3.2)$$

If $G(s)$ is unbounded on C_+ then $d_C = 0$ and there is no thing to say. The assumption $G(s)$ to be bounded implies that $FTdiag(0, U^j)W^{-1}E = 0 \ \forall j > 1$. Thus

$$G(s) := FTdiag((sI - B_1)^{-1}, (U - I)^{-1})W^{-1}E = FTdiag((sI - B_1)^{-1}, -I)W^{-1}E.$$

Let $f(s) = \|G(1/s)\|^2$ if $s \neq 0$ and $f(0) = \lim_{s\to\infty} \|G(1/s)\|^2$ (we remark that this limit always exists). It is easy to see that $f(s) = \|FTdiag(s(I - sB_1)^{-1}, -I)W^{-1}E\|^2$.

Since all entries of the matrix $G(s)$ are only rational functions which are analytic then by the maximum principle, the maximum of $G(s)$ takes place only at $s = \infty$ or $s \in iR$. Therefore, $G(s)$ attains its maximum at $s = \infty$ iff $f(s)$ has the maximum value at $s = 0$ (of course we consider only s in C^+). Thus, taking a ray $t \to t \cdot e$, $t \geq 0$ where $e = (\cos\alpha, \sin\alpha)$, $\frac{-\pi}{2} \leq \alpha \leq \frac{\pi}{2}$, the attainment of maximum value at 0 of $f(s)$ implies that $f'_e(0) \leq 0$, for every e. It is easy to see that

$$f'_e(0) = 2\cos\alpha\big[FTdiag(0, -I)W^{-1}E\big] * \big[FTdiag(I, 0)W^{-1}E\big] = 2\cos\alpha C * D,$$

where $C = \big[FTdiag(0, -I)W^{-1}E\big]$; $D = \big[FTdiag(I, 0)W^{-1}E\big]$ and $C * D$ denotes the Frobenius inner product of two matrices C, D.

In using the expressions of W and T in (3.2) we obtain

$$\begin{aligned} C := &\ FTdiag(0, -I)W^{-1}E \\ = &\ FTdiag(0, U(U - I)^{-1} - I)W^{-1}E = FTdiag(0, V - I)W^{-1}E \\ = &\ FSdiag(I, (V - I)^{-1}) \cdot diag(0, V - I)diag(M^{-1}, I)S^{-1}(A - B)^{-1}E \\ = &\ FSdiag(0, I)S^{-1}(A - B)^{-1}E \end{aligned}$$

and

$$\begin{aligned} D := &\ FTdiag(I, 0)W^{-1}E = FTW^{-1}E + C \\ = &\ FSdiag(I, (V - I)^{-1})diag(M^{-1}, I)S^{-1}(A - B)^{-1}E + C \\ = &\ F[Sdiag(M, V - I)S^{-1}]^{-1}(A - B)^{-1}E + C \\ = &\ F[(A - B)^{-1}A - Sdiag(0, I)S^{-1}]^{-1}(A - B)^{-1}E + C \\ = &\ F[A - (A - B)Sdiag(0, I)S^{-1}]^{-1}E + C \end{aligned}$$

Summing up, we have: if $C * D > 0$ then $G(s)$ has maximum at a finite value s. In the case $C * D = 0$ we can compute higher derivatives of f to obtain the answer but the formula is complicated and we do not realize here.

Example 1. Let us calculate stability radius of the structured perturbed equation $AX'(t) - (B + E\Delta F)X(t) = 0$ where Δ is disturbance and

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} -2 & -1 & 0 \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{pmatrix} \quad E = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It is seen that ind $(A, B) = 2$ and $\sigma(A, B) = -\frac{1}{3}$. Therefore the pencil $\{A, B\}$ is asymptotically stable. By a direct computation we obtain

$$G(s) = F(sA - B)^{-1}E = \begin{pmatrix} \frac{s}{3s+1} & \frac{s}{3s+1} & \frac{s}{3s+1} \\ \frac{s+1}{3s+1} & \frac{s+1}{3s+1} & \frac{s+1}{3s+1}, \\ -\frac{s}{3s+1} & -\frac{s}{3s+1} & -\frac{s}{3s+1} \end{pmatrix}.$$

Thus, $\|G(s)\| = 3\max\{|\frac{s+1}{3s+1}|, |\frac{s}{3s+1}|\}$ which attains its maximum at $s_0 = 0$ and $\|G(0)\| = 3$. Hence, $d_C = 1/3$. Choose $u = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then $\|G(0)u\| = G(0) = 3$. Let $y^* = (0 \ \ 1 \ \ 0)$, we have $\Delta = \|G(0)\|^{-1}uy^* = \begin{pmatrix} 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \\ 0 & 1/3 & 0 \end{pmatrix}$. Moreover, $\det(sA - B - E\Delta F) = 2s = 0$ for $s = 0$.

Example 2. Let us consider the equation $AX'(t) - BX(t) = 0$ where $A = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -2 \\ -2 & 0 \end{pmatrix}$. It is seen that ind $(A, B) = 1$; $\sigma(A, B) = -1$. and $G(s) = (sA - B)^{-1} = \begin{pmatrix} s/(s+1) & 1/2 \\ 1/2 & 1/4 \end{pmatrix}$. Hence, $\|G(s)\| = \max\{3/4, 1/2 + |s/(s+1)|\}$ which doesn't attain its maximum on C^+. Further, $\lim_{s\to\infty} \|G(s)\| = 3/2$, i.e., $d_C = 2/3$. If we choose $u = \begin{pmatrix} (t-i)/\sqrt{t^2+1} \\ 1 \end{pmatrix}$, it is clear that $\|u\| = 1$ and $\|G(s)u\| = \|G(s)\|$ when $\Re s$ is large. Thus, with $y^* = (1 \quad 0)$, we have

$$\Delta = \|G(s)\|^{-1}uy^* = \quad \text{converges to} \quad \begin{pmatrix} 2/3 & 0 \\ 2/3 & 0 \end{pmatrix} \quad \text{as} \quad s \to \infty.$$

It is easy to verify $det(sA - B - \Delta) = -8/3$ for all s, i.e., $\sigma(A, -(B + \Delta)) = \emptyset$ and the equation $AX'(t) - (B + \Delta)X(t) = 0$ i.e., the system

$$-x_1' + 2x_2' - \frac{5}{3}x_1 + 2x_2 = 0$$

$$2x_1' - 4x_2' + \frac{4}{3}x_1 = 0$$

has a unique solution $x_1 = 0; x_2 = 0$ which is asymptotically stable.

4. The equality of real and complex stability radiis of DAEs

In this section, we are concerned with a special case where the complex stability radius is equal to real stability radius. For DAEs, this is a difficult question because under the action of the pencil of matrices $\{A, B\}$, the positive cone R_+ is no longer invariant even both A and B are positive. We are able to solve problem under a very strict hypothesis. Suppose that $A, B \in R^{m\times m}$.

A matrix $H = (a_{i,j}) \in R^{m\times m}$ is said to be positive if $a_{ij} \geq 0$ for any i, j. Denote the absolute of the matrix $M = (m_{i,j})$ by $|M| = (|m_{i,j}|)$ and of the vector x by $|x| = (|x_1|, |x_2|, \ldots, |x_m|)$. We define a partial order relation in $R^{m\times m}$ by

$$M \leq N \Leftrightarrow M - N \leq 0.$$

Let $\mu(A, B)$ be the abscissa spectrum of the pencil $\{A, B\}$, i.e., $\mu(A, B) := \max\{\Re\lambda : \lambda \in \sigma(A, B)\}$.

We consider the equation

$$A\dot{X}(t) - BX(t) = 0, \tag{4.1}$$

where A, B are constant matrices in $R^{m\times m}$, the pencil $\{A, B\}$ is regular. If $ind(A, B) > 1$ then there is nothing to say because $d_C = d_R = 0$. So we suppose that $ind(A, B) = 1$ and the following conditions are satisfied:

i) $A \geq 0$ (4.2)

ii) There exists a sequence (t_n): $t_n > 0$; $t_n \to \infty$ such that $(t_n A - B)^{-1} \geq 0$ for all n. (4.3)

iii) The equation (4.1) is asymptotically stable.

We remark that the above conditions ensure a positive system on ODEs case.

Let us choose the monotonous norm in R^m. That is, if $|x| \leq |y|$ then $\|x\| \leq \|y\|$.

Lemma 4.1. *Let the system (4.1) satisfies above conditions, then for all λ such that $\Re\lambda > \mu(A, B)$, we have $|(\lambda A - B)^{-1}x| \leq (\Re\lambda A - B)^{-1}|x|$ for any $x \in R^m$.*

Proof. Let us take an $t, t_n \in R$ such that $t > \mu(A, B)$.and $t_n - t > \mu(A, B)$. Suppose that $\lambda = t + i\omega$, we have to prove that $|[(t + i\omega)A - B]^{-1}x| \leq (tA - B)^{-1}|x|$ for all $x \in R^m$. By simple calculation we have

$$((t + i\omega)A - B)^{-1} = (t_n A - B)^{-1}[I - (t_n - t - i\omega)A(t_n A - B)^{-1}]^{-1}.$$

Putting $G(t_n) = (t_n A - B)^{-1}$, we obtain

$$\begin{aligned} [(t + i\omega)A - B)]^{-1} &= G(t_n)[I - (t_n - t - i\omega)AG(t_n)]^{-1} \\ &= G(t_n)\sum_{n=0}^{\infty}(t_n - t - i\omega)^n(AG(t_n))^n. \end{aligned} \tag{4.3}$$

The above series absolutely converges if we can prove that $\|(t_n - t - i\omega)r(AG(t_n))\| < 1$ where $r(M)$ denotes the spectrum radius of M.

First, it is easy to see that $\lim_{t_n\to\infty}(t_n - |t_n - t - i\omega|) = t$. Therefore, for $\varepsilon = t - \mu(A,B) > 0$ we have $t_n - |t_n - t - i\omega| > t - \varepsilon = \mu(A,B)$ for t_n sufficiently large, i.e., $t_n - \mu(A,B) > |t_n - t - i\omega|$. On the other hand, by hypotheses i) and ii), $AG(t_n)$ is positive matrix, then by Perron-Frobenius theorem : $r(AG(t_n)) = \mu(AG(t_n)) \in \sigma(AG(t_n))$. This means that $\det[r(AG(t_n))I - AG(t_n)] = 0$. Hence,

$$\begin{aligned}\det[r(AG(t_n))I - AG(t_n)] = 0 &\Leftrightarrow \det[(t_nA - B) - A/r(AG(t_n))] = 0\\ &\Leftrightarrow \det[(t_n - 1/r(AG(t_n)))A - B] = 0.\end{aligned}$$

Thus, $t_n - \frac{1}{r(AG(t_n))} \in \sigma(A,B)$. Therefore, $\frac{1}{r(AG(t_n))} \geq t_n - \mu(A,B)$, which implies $|t_n - t - i\omega| r(AG(t_n)) < 1$. Hence, by (4.3)

$$\begin{aligned}|((t+i\omega)A - B)^{-1}x| &\leq G(t_n)\sum_{n=0}^{\infty} |t_n - t - i\omega)|^n (AG(t_n))^n |x|\\ &= G(t_n)[I - |t_n - t - i\omega|(AG(t_n)]^{-1}|x| = [(t_n - |t_n - t - i\omega|)A - B]^{-1}|x|.\end{aligned}$$

Let $t_n \to \infty$ we obtain

$$|[(t+i\omega)A - B]^{-1}x| \leq (tA - B)^{-1}|x|.$$

Lemma 4.1 is proved. ◇

Lemma 4.2. *$G(t) = (tA - B)^{-1} \geq 0$ for any $t > \mu(A,B)$. Moreover, $G(t)$ is decreasing on $(\mu(A,B),\infty)$.*

Proof. Let $t_0 > \mu(A,B)$. By using Lemma 4.2 we see that $|G(t_0)| \leq G(\Re t_0) = G(t_0)$ then $G(t_0) \geq 0$.

The decreasing of $G(t)$ on $(\mu(A,B),\infty)$ follows from the first part of the lemma and the fact that for $s,t \geq \mu(A,B)$ one has $G(s) - G(t) = (t-s)G(s)AG(t)$. ◇

Because the function $G(s)$ is analytic on half plane C^+ then it only attains maximum at $s = \infty$ or $s \in iR$. Furthermore, monotonous norm is chosen then by Lemma 4.1 $G(s)$ attains the maximum on $[0,\infty)$. On the other hand, by Lemma 4.2, it follows that $G(s)$ has its maximum at $t = 0$, i.e., $\|G(0)\| = \max\{\|G(\lambda)\| : \Re\lambda \geq 0\} = \|B^{-1}\|$.

From Perron - Frobenius theorem, there exists $u > 0$, $\|u\| = 1$ such that $\|G(0)u\| = \|G(0)\|$. By using once more Haln-Banach theorem for positive system there exists positive linear functional y^* satisfying $y^*(G(0)u) = \|G(0)u\| = \|G(0)\|$ and $\|y^*\| = 1$. Let $\Delta = \|G(0)\|^{-1}uy^* > 0$. Following the way as above we can prove that Δ is "bad" matrix and $\|\Delta\| = d_C$. Therefore,

Theorem 4.3. *Suppose that the system (4.1) satisfies hypotheses i); ii) and iii) and a monotonous norm in R^m is chosen, then the complex stability radius d_C and the real stability radius d_R are equal and $d_R = (\|B^{-1})\|^{-1}$.*

As is mentioned above, assuming the positivity of $G(t_n)$ for a sequence (t_n) is strong and it is difficult to verify. We know give a sufficient condition to ensure the above hypotheses. It is said that the system (2.1) is positive if for any $x_0 \in R_+^m = \{y = (y_1, y_2, \dots, y_m) : y_i \geq 0 \forall i = 1, 2, \dots, m\}$, the solution X_t with $X_0 = x_0$, if it exists, satisfies the condition $X_t \geq 0$ for all $t > 0$. Let Q be a projection on $KerA$ then it is known that (2.1) is equivalent to

$$X' - \widehat{B}X = 0; \qquad \widehat{Q}X = 0, \tag{4.4}$$

where, $\widehat{Q} = -Q(A - BQ)^{-1}B$; $\widehat{P} = 1 - \widehat{Q}$ and $\widehat{B} = \widehat{P}(A - BQ)^{-1}B$. We note that $\widehat{Q}$ does not depend on the choice of the projection Q and $\widehat{P}(A - B\widehat{Q})^{-1} = \widehat{P}(A - BQ)^{-1}$ which implies that $\widehat{B}$ is independent of the choice of Q. So it is seen that (2.1) is positive if and only if $\widehat{P} \geq 0$ and $\widehat{B}$ is a $\widehat{P}-$ Metlez matrix, i.e., all entries of $\widehat{B}$ are positive except for entries $\widehat{b}_{ij}$ with $\widehat{p}_{ij} > 0$, where $\widehat{P} = (\widehat{p}_{ij})$. Indeed, from (4.4), the general solution of (2.1) is $X_t = \exp(t\widehat{B})\widehat{P}X_0$. Thus, the positivity condition implies that $\widehat{P} \geq 0$ (with $t = 0$). On the other hand, for t is small we have

$$0 \leq \exp(t\widehat{B})\widehat{P} = \widehat{P} + \sum_{n=1}^{\infty}(t\widehat{B})^n = \widehat{P} + t\widehat{B} + o(t), \qquad \text{as} \qquad t \to \infty$$

which follows that if $\widehat{p}_{ij} = 0$ then $\widehat{b}_{ij} \geq 0$. Conversely, if $\widehat{B}$ is a $\widehat{P}-$ Metlez matrix then in noting that $\widehat{P}$ is a projection which commute with $\widehat{B}$ then for an α such that $\alpha\widehat{P} + \widehat{B} \geq 0$ we have

$$\begin{aligned}\exp(t\widehat{B})\widehat{P} &= \exp(-\alpha t\widehat{P} + t(\widehat{B} + \alpha\widehat{P}))\widehat{P} = \exp(-\alpha t\widehat{P})\exp(t(\widehat{B} + \alpha\widehat{P}))\widehat{P} \\ &= \exp(-\alpha t)\exp(t(\widehat{B} + \alpha\widehat{P}))\widehat{P} \geq 0.\end{aligned}$$

We now suppose that the system (2.1) is positive. In adding conditions that $\widehat{P}(A - B\widehat{Q})^{-1} \geq 0$ and $\widehat{Q}(A - B\widehat{Q})^{-1} \geq 0$ we can prove that $G(t) = (tA - B)^{-1} \geq 0$ for any $t > 0$ and t large. To verify this attestation we have only to remark that

$$\begin{aligned}(A - B\widehat{Q})^{-1}(tA - B) &= t\widehat{P} - (A - B\widehat{Q})^{-1}B = t\widehat{P} + \widehat{Q} - \widehat{B} \\ &= (\widehat{P} + \widehat{Q}/t)\left[tI - (\widehat{P} + t\widehat{Q})\widehat{B}\right] = (\widehat{P} + \widehat{Q}/t)(tI - \widehat{B})\end{aligned}$$

Thus,

$$\begin{aligned}G(t) &= \left[(A - B\widehat{Q})^{-1}(tA - B)\right]^{-1}(A - B\widehat{Q})^{-1} = \left[(\widehat{P} + \widehat{Q}/t)(tI - \widehat{B})\right]^{-1}(A - B\widehat{Q})^{-1} \\ &= (tI - \widehat{B})^{-1}\widehat{P}(A - B\widehat{Q})^{-1} + t(tI - \widehat{B})^{-1}\widehat{Q}(A - B\widehat{Q})^{-1} \\ &= (t - \widehat{B})^{-1}\widehat{P}(A - B\widehat{Q})^{-1} + \widehat{Q}(A - B\widehat{Q})^{-1}.\end{aligned} \tag{4.5}$$

Since $\widehat{B}$ is a $\widehat{P}-$ Metlez then there is a t_0 such that $(t - \widehat{B})^{-1}\widehat{P} \geq 0$ for any $t > t_0$. Thus, under the hypotheses $\widehat{P}(A - B\widehat{Q})^{-1} \geq 0$ and $\widehat{Q}(A - B\widehat{Q})^{-1} \geq 0$, the relation (4.5) tells that $G(t) \geq 0$ for $t > t_0$.

However, it is easy to give an example where the system is positive but the resolvent $(tA - B)^{-1}$ is not positive. So far we do not know if the positive condition ensures the equality of d_C and d_R. An answer of this problem is welcome.

We consider the case where $G(s)$ attains maximum at $s = \infty$. From the relation (4.5) we see that

$$\lim_{s\in C^+;\, s\to\infty} \|G(s)\| = \lim_{t\in R_+;\, t\to\infty} \|G(t)\| = \|\widehat{Q}(A - B\widehat{Q})^{-1}\| = \max_{s\in C^+} \|G(s)\|.$$

where $\widehat{Q}$; $\widehat{B}$ are given as in (4.4). Let (t_n) be a sequence in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \infty$. For any n we choose $u_n \in R^m$ and y_n^* with $(y_n^*)^T \in R^m$ such that $\|u_n\| = 1$; $\|y_n^*\| = 1$; $\|G(t_n)u_n\| = \|G(t_n)\|$ and $y^*(G(t_n)u_n) = \|G(t_n)u_n\|$ as in §3. Let $\Delta_n = \|G(t_n)\|^{-1}u_n.y_n^*$. It is clear that t_n is an eigenvalue of the pencil $\{A, B + \Delta_n\}$ with the corresponding eigenvector $x_n = (t_nA - B)^{-1}u_n$. This means that $\Delta_n \in \mathcal{V}_R$. It is easy to see that $\lim_{n\to\infty} \|\Delta_n\| = \lim_{n\to\infty} \|G(t_n)\|^{-1} = d_C$, i.e., $d_C = d_R$. Thus,

Theorem 4.4. *If the resolvent $G(s)$ has not maximum value on the right-hand half of complex plan then $d_C = D_R$.*

Example 1. Compute the stability radius of the system $AX'(t) - BX(t) = 0$ with

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix}.$$

It is seen that $\operatorname{ind}(A, B) = 1$; $\sigma(A, B) = \{-3/2 - \sqrt{5}/2; -3/2 + \sqrt{5}/2\}$ and

$$\widehat{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}; \quad \widehat{B} = \begin{pmatrix} -2 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix};$$

Thus $\widehat{B}$ is $\widehat{P}-$ Metlez. Moreover,

$$\widehat{Q}(A - B\widehat{Q})^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \geq 0; \quad \widehat{P}(A - B\widehat{Q})^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \geq 0$$

Then $G(s) \geq 0$ for any $t > 0$ and it attains the maximum at $s_0 = 0$ with

$$G(0) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 0 & 0 & 1 \end{pmatrix}; \quad \text{and} \quad \|G(0)\| = 5.$$

Therefore, $d_R = d_C = 1/5 = \|\Delta\|$ where

$$\Delta = \begin{pmatrix} 0 & 1/5 & 0 \\ 0 & 1/5 & 0 \\ 0 & 1/5 & 0 \end{pmatrix}$$

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BÁN KÍNH ỔN ĐỊNH ĐỐI VỚI CÁC PHƯƠNG TRÌNH VI PHÂN ĐẠI SỐ

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Trong bài báo này chúng tôi đề cập đến việc tính bán kính ổn định cho hệ được mô tả bởi phương trình vi phân đại số có dạng $AX'(t) + BX(t) = 0$, trong đó a, b là các ma trận hằng số. Một công thức bán kính ổn định phức đã được đưa ra và sự khác biệt giữa các trường hợp phương trình vi phân thường và phương trình vi phân đại số cũng được chỉ ra. Chúng tôi cũng nghiên cứu trường hợp đặc biệt mà ở đó bán kính ổn định thực và và phức bằng nhau