EXTENDING PROPERTY OF INFINITE DIRECT SUMS OF UNIFORM MODULES

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Introduction

Let R be any ring and $M = \oplus_{i\in I} Mi$ be a direct sum of uniform right R- submodules $M_i, i \in I$. We are interested in the question, when this module M is extendible. If the index set I is finite, this question has been studied by Harmanci and Smith [6]. In deed, Harmanci and Smith have shown that if $M = \oplus_{i=1}^{n} M_i$, then M is extendible if and only if every direct summand of M with uniform dimension 2 is an extending module, where each M_i is uniform, [6, Theorem 3].

In the first part of this paper we give some conditions for M to be extendible, where the index set I is not necessarily finite. We show that a module over an arbitrary ring is extendible if it has $(1 - C_1)$ and every local direct summand is a direct summand. Moreover, properties of extending modules have been obtained.

In the last part of the paper the results have been applied to characterize quasi-Frobenius rings and rings whose projective right modules are extendible.

2. Preliminaries

Throughout this paper all rings R are associative rings with identity and all R-modules are unitary right R- modules.

We consider the following conditions on a module M:

(C_1) Every submodule of M is essential in a direct summand of M.

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand of M.

$(1 - C_1)$ Every uniform submodule of M is essential in a direct summand of M.

A module M is called *continuous* if it satisfies conditions (C_1) and (C_2), *U-continuous* if it satisfies (C_2) and $(1 - C_1)$.

A module M is said to be an *extending module* if it satisfies condition (C_1), and M is said to have the *extending property of uniform submodule* if it satisfies condition $(1 - C_1)$.

A submodule N of a module M is closed in M, if it has no proper essential extensions in M. It is easy to check that M is an extending module if and only if every closed submodule of M is a direct summand of M.

A module M is said to have finite uniform dimension if M does not contain an infinite direct sum non-zero submodule.

A ring R is called Quasi - Frobenius (briefly QF) if R is right artinian and right self - injective. It is know that a ring R is QF if every projective R - module is injective if every injective R - module is projective (See [4, Theorem 24.20]).

A ring R is called a *right* $H-ring$ [8, Theorem I], if it satisfies one of the following equivalent conditions:

1) Every injective R- module is a lifting module.

2) R is right artinian in which every non- small R- module contains a non-zero injective submodule.

3) R is right perfect and the family of all injective R- modules is closed under taking small covers, i.e for any exact sequence $P \xrightarrow{f} E \to 0$ where E is injective and kerf is small in P, P is injective.

4) Every R- module is expressed as a direct sum of an injective module and a small module.

Dually a ring R is called a *right* $co-H$ *ring* [8, Theorem II] if it satisfies one of the following equivalent conditions:

1) Every projective R-module is an extending module.

2) R satisfies acc on right annihilator ideals and every non-cosmall R-module contains a non-zero projective direct summand.

3) The family of all projective R- modules is closed under taking essential extensions.

4) Every R- module is expressed as a direct sum of a projective module and a singular module.

For a submodule X of M, $X \subseteq^e M$ means that X in an essential submodule of M. The injective hull of a module M will be denoted by $E(M)$.

Let $M = \oplus_{i \in I} M_i$ be a direct sum of modules $M_i, i \in I$ and let J be a subset of I, then we put $M(J) := \oplus_{j \in J} M_j$.

3. Direct sums of uniform modules

Lemma 1. *([2, Proposition 2.2] or [6, Lemma 1). Let M be any module and $K \subseteq L$ be submodules of M such that K is closed in L and L is closed in M. Then K is a closed submodule of M.*

The following result is obvious.

Lemma 2. *Let M have $(1 - C_1)$. Then every direct summand of M also has $(1 - C_1)$.*

Lemma 3. *Let $M = \oplus_{i \in I} M_i$ be a direct sum of uniform modules $M_i, i \in I$. Then any non-zero sunmodule of M contains a uniform submodule.*

Proof. Let A be a non-zero submodule of M. Then there exists a subset J of I which is maximal with respect to $A \cap M(J) = 0$.

Consider $k \in I \setminus J$ and let Π_k be the projection

$$\Pi_k : M_k \oplus M(J) \to M_k.$$

Let $A_k = A \cap (M_k \oplus M(J))$. By the choice of J, A_k is non- zero. Since $A_k \cap M(J) = 0$, we have

$$A_k \cong \Pi_k(A_k) \subseteq M_k.$$

Therefore A_k is a uniform submodule of A.

Lemma 4. *Let M be a module with $(1 - C_1)$ and $X \oplus U$ be a closed submodule of M, where X is a direct summand of M and U is a uniform submodule of M. Then $X \oplus U$ is a direct summand of M.*

Proof. Let $M = X \oplus M_1$ for some submodule M_1 of M and $\Pi : M \to M_1$ be the projection. Now let U be a uniform submodule of M with $U \cap X = 0$. Let V be a maximal essential extension of $\Pi(U)$ in M_1. Since $U \cong \Pi(U), V$ is uniform. By Lemma 2, M_1 has $(1 - C_1)$. Consequently V is a direct summand of M_1. Thus V is a direct summand of M. Moreover, it is easy to check that

$$X \oplus U \subseteq \Pi^{-1}(V) \subseteq X \oplus V.$$

Since V is uniform, we obtain $X \oplus U \subseteq^e X \oplus V$. It follows that $X \oplus V$ and so $X \oplus U$ is a direct summand of M. □

Corollary 5. *Let M be a module with $(1 - C_1)$. If M has finite uniform dimension, then M is a finite direct sum of uniform submodules.*

Proposition 6. *Let M be a module with $(1 - C_1)$ and A be a closed submodule of M. If A has finite uniform dimension then A is a direct summand of M.*

Proof. By Lemma 1, A has $(1 - C_1)$. Hence, by Corollary 5, A has a direct sum decomposition

$$A = A_1 \oplus \cdots \oplus A_n,$$

where each A_i is uniform. By induction on n and by using Lemma 4, we can show that $A = (A_1 \oplus \cdots \oplus A_{n-1}) \oplus A_n$ is a direct summand of M.

Corollary 7. *(see [7, Proposition 3]). Let M be a module with $(1 - C_1)$, then every closed submodule of the form $\oplus_{i=1}^n A_i$, with all A_i uniform, is a direct summand of M.*

Corollary 8. *([7, Proposition 6]). Let $M = \oplus_{i \in I} M_i$ with all M_i uniform. If M has $(1 - C_1)$, then every non- zero closed submodule of M contains a uniform direct summand of M.*

Proof. Let A be a closed submodule of M. By Lemma 3, there exists a uniform closed submodule V of A. By Lemma 1, V is closed in M. Since M has $(1 - C_1), V$ is a direct summand of M □.

It is clear that the extending modules have $(1 - C_1)$, but the following example shows that the converse is not true.

Example 9. *There exists a Z - module with $(1-C_1)$, such that it is not an extending module.*

Proof. Let F be an infinitely generated free abelian group. Then $F = \oplus_{i\in I} U_i$, where I is an infinite set and $U_i \cong Z$, for all $i \in I$. Since F has infinite rank, F is not an extending module ([8, p.19] or [6, p.3]).

Now let A be a uniform closed submodule of F. Since F is a hereditary Z-module and A is uniform we can show that A is a finitely generated Z - module. Then there exists a finite direct sum $X := U_1 \oplus \cdots \oplus U_n$, where $\{1, \cdots, n\} \subset I$, such that $A \subseteq X$. Since X is an extending module by [8, p.19], A is a direct summand of X, and hence also of F. Thus F has $(1-C_1)$.

A family $\{M_i : i \in I\}$ of submodules of a module M is called a *local direct summand* of M, if $\sum_{i\in I} M_i$ is direct and $\sum_{i\in F} M_i$ is a direct summand of M for every finite subset F of I.

The following result gives a condition when a direct sum of uniform modules is extendible. □

Theorem 10. *Let M be an R-module such that M is a direct sum of uniform modules $M_i, i \in I$ and assume that every local direct summand of M is a direct summand. Then M is an extending module if and only if M has $(1-C_1)$.*

Proof. Let M be an extending module. Obviously, M has $(1-C_1)$.

Conversely, let $M = \sum_{i\in I} M_i$, where each M_i is uniform and assume that any local direct summand of M is a direct summand. Let A be a closet submodule of M. By Corollary 8, A contains a uniform direct summand X of M. Hence we can define a non-empty set $\mathcal{P}$ of direct sums of uniform modules in M as follows: $\mathcal{P} = \{\oplus_{\alpha\in\Lambda} A_\alpha : A_\alpha \subseteq A, A_\alpha$ is uniform and $\oplus_{\alpha\in\Lambda} A_\alpha$ is a local direct summand of $M\}$.

By Zorn's Lemma, we can find a maximal member $\oplus_{k\in K} A_k$ in $\mathcal{P}$.

Hence $A' = \oplus_{k\in K} A_k$ is a direct summand of M, i. e. $M = A' \oplus M_1$ for some submodule M_1 of M. From this, $A = A' \oplus B$, where $B = A \cap M_1$. By Lemma 1, B is closed in M. If $B \neq 0$, B contains a uniform direct summand B_β of M. Then $A' \oplus B_\beta$ is member of $\mathcal{P}$, a contradiction to the maximality of A'. Hence $B = 0$, i. e. $A = A'$ is a direct summand of M. Thus M is an extending module. □

Corollary 11. *Let M be a module with finite uniform dimension.*

a *If M has $(1-C_1)$, then M is an extending module.*

b *If M is U - continuous, then M is continuous.*

Theorem 12. *Let $M = \oplus_{i\in I} M_i$ with all M_i uniform and assume that this decomposition of M complements direct summands. Suppose further that for all $i, j \in I, i \neq j$, M_i cannot be properly embedded in M_j . Then the following statements are equivalent:*

(i) M is an extending module.

(ii) M has $(1-C_1)$.

(iii) $M(J)$ is $M(K)$ - injective, for any subsets J and K of I such that $K \cap J = \oslash$.

Proof. $(i) \Rightarrow (ii)$ is trivial.

$(ii) \Rightarrow (iii)$. By [8, Proposition 1.5], it suffices to prove that for each $k \in K, M(J)$ is M_k- injective. For this purpose, let U be a sunmodule of M_k and α be a homomorphism of U in $M(J)$. We show that α is extended to one in $\text{Hom}_R(M_k, M(J))$. Since $M(J) \oplus M_k$ has $(1 - C_1)$, there is a direct summand X^* of M such that

$$\{x - \alpha(x) : x \in U\} \subseteq^e X^*.$$

Since as a direct summand of $M, M(J) \oplus M_k$ has a decomposition that complements direct summands. We consider two cases:

a) $M(J) \oplus M_k = X^* \oplus M(J')$, where J' is a subset of J. Then $M(J) \oplus M_k = X^* \oplus M(J) \subseteq X^* \oplus M(J) \subseteq M(J) \oplus M_k$. Hence $X^* \oplus M(J') = X^* \oplus M(J)$. It follows that $J' = J$. Therefore $\Pi|_{M_k}$ extends α, where $\Pi : X^* \oplus M(J) \to M(J)$ is the projection.

b) $M(J) \oplus M_k = X^* \oplus M(J_1) \oplus M_k$, where J_1 is a subset of J. Let $\Pi_k : X^* \oplus M(J_1) \oplus M_k \to M_k$ be the projection and let $A = (X^* \oplus M(J_1)) \cap M(J)$.

If $A \neq 0$ and suppose that $A \cap M_j - \neq 0$ for each $j \neq J$, then by [3, Proposition 3.6], A is essential in $M(J)$. Then it is easy to check that $X^* \oplus A$ is essential in $M_k \oplus M(J)$. Hence $M_k \cap (X^* \oplus M(J)) \neq 0$, a contradiction. Consequently there exists $j \in J$ such that $M_j \cap A = 0$. Hence $M_j \cap \ker \Pi_k = 0$ and thus $M_j \cong \Pi_k(M_j)$. By hypothesis we have

$$\Pi_k(M_j) = M_k.$$

Therefore we have

$$X^* \oplus M(J_1) \oplus M_k = X^* \oplus M(J_1) \oplus M_j = X^* \oplus M(J_1)$$

where $J_2 = J1 \cup \{j\}$. Hence we may use a) to show that α is extended to one in $\text{hom}_R(M_k, M(J))$.

Now assume that $A = 0$. Then $M(J_1) = 0$. It implies that $M(J) \oplus M_k = X^* \oplus M_k$. From this, it is easy to see that $M(J)$ is uniform and

$$M(J) \oplus M_k = M_j \oplus M_k = X^* \oplus M_k,$$

where $J = \{j\}$ is a set of only one element. Hence we have $\Pi_k(M_j) = M_k$ and $M_j \oplus M_k = X^* \oplus M_j$. There fore $\Pi|_{M_k}$ extends α, where $\Pi : X^* \oplus M_j \to M_j$ is the projection, proving (iii).

$(iii) \Rightarrow (i)$. Let A be a closed submodule of M, and J be a subset of I which is maximal with respect to $A \cap M(J) = 0$. Then it is easy to see that $A \oplus M(J)$ is essential in M. Let $K = I \setminus J$ and Π_K, Π_J be the projections of M onto $M(K)$ and $M(J)$, respectively. Then $\Pi_K|_A$ is a monomorphism. Let $\alpha = \Pi_J(\Pi_K|_A)^{-1} : \Pi_K(A) \to M(J)$. It is easy to check that $A = \{x + \alpha(x) : x \in \Pi_K(A)\}$. Since $M(J)$ is $M(K)$-injective, there exists an extension $\hat{\alpha} : M(K) \to M(J)$ of α. Put

$$A' = \{y + \hat{\alpha}(y) : y \in M(K)\}.$$

Since $A \oplus M(J)$ is essential in $M, \Pi_K(A)$ is essential in $M(K)$. Hence A is essential in A' and therefore $A = A'$. It follows that $\Pi_K(A) = M(K)$. From this we have $M = A \oplus M(J)$. Thus M is an extending module. □

Proposition 13. *Let M be a finite direct sum of uniform modules $U_i (1 \leq i \leq n)$ such that $U_i \oplus U_j$ is an extending module for all $i, j \in 1, 2, ..., n$. If M is U- continuous, then M is a quasi-injective module.*

Proof. By Corollary 11 and [8, Corollary 1.19, Proposition 2.10].

4. Application

First we consider co- H rings.

Theorem 14. *Let R be a right perfect ring with finite right uniform dimension. Then R is a right $co - H$ ring if and only if the projective cover of every semisimple R-module has $(1 - C_1)$.*

Proof. Let R be a right $co - H$ ring. By [9, Theorem II], the projective cover of every semisimple R-module has $(1 - C_1)$.

Conversely, let R is the projective cover of a semisimple R-module. Since R is right perfect

$$R_R - e_1 R \oplus \cdots \oplus e_n R,$$

where $\{e_i\}_{i=1}^n$ is a set of orthogonal primitive idempotents of R. By [1, Theorem 27.11], $P = \oplus_{i \in I} P_i$, for some index set I and each P_i is isomorphic to some $e_i R$ in $\{e_1 R, \cdot, e_n R\}$. Since each P_i has finite uniform dimension, P_i is an extending module by Corollary 11. Consequently, each P_i is uniform. By [1, Theorem 28.14], $P = \oplus_{i \in I} P_i$ has a decomposition that complements direct summands. Then by [1, Theorem 27.12] and [8, Theorem 2.15] every local direct summand of P is direct summand. Now by Theorem 10, P is an extending module and hence by [12, Proposition 2.10], R is a right co-H ring.

Corollary 15. *The following statements are equivalent for a given ring R:*

(i) *R is right $co - H$.*
(ii) *R is right perfect, right extending and every projective right R-module has $(1 - C_1)$.*
(iii) *R is (right and left) perfect and every projective right R-module has $(1 - C_1)$.*

Proof. $(i) \Rightarrow (ii)$ and $(i) \Rightarrow (ii)$ are clear.

$(ii) \Rightarrow (i)$ by Theorem 14.

$(iii) \Rightarrow (i)$. Since R is left perfect, each $e_i R$ contains a uniform submodule, where $\{e_i\}_{i=1}^n$ is a set of orthogonal primitive idempotents of R. Since each $e_i R$ has $(1 - C_1), e_i R$ is uniform for all $1 \leq i \leq n$. Then every projective right R-module P has a direct sum decomposition

$$P = \oplus_{i \in I} P_i,$$

where each P_i is uniform. As in the proof of Theorem 14 we see that P is an extending module. By [9, Theorem II], R is a right co-H ring.

For QF- rings we prove the following.

Theorem 16. *For a ring R, the following statements are equivalent:*

(i) *R is QF.*
(ii) *R is right perfect, right continuous and the projective cover of every semisimple R-module has $(1 - C_1)$.*

(iii) R is right co – H and right continuous.
(iv) R is right co – H and right U-continuous.
(v) R is a right H-ring and right continuous.
(vi) R is a right H-ring and right U-continuous.

Proof. Since R is right continuous, $Z(R_R) = J(R)$, by [11, Lemma 4.1]. Hence, by [9, Theorem 4.3] we have $(i) \leftrightarrow (iii)$ and $(i) \leftrightarrow (v)$.

$(i) \Rightarrow (ii), (iii) \Rightarrow (iv)$ and $(v) \Rightarrow (vi)$ are clear.

$(ii) \Rightarrow (iii)$. Since R is right perfect and right continuous, R_R has finite uniform dimension. By Theorem 14, R is a right $co - H$ ring.

$(iv) \Rightarrow (iii)$. By Corollary 15, R is right perfect. Hence R is right continuous, by Corollary 11.

$(vi) \Rightarrow (v)$. By [9, Theorem 2.11], R is right artinian. Hence, R is right continuous by Corollary 11.

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TÍNH CHẤT MỞ RỘNG CỦA TỔNG TRỰC TIẾP VÔ HẠN CỦA CÁC MÔĐUN ĐỀU

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Cho $M = \oplus_{i\in I} M_i$, trong đó M_i là các môđun con đều và I là tập vô hạn bất kỳ, câu hỏi đặt ra là khi nào M là CS - môđun.

Trong bài báo này chúng tôi đưa ra một số điều kiện để môđun M là CS thông qua lớp $(1 - C_1)$-môđun. Các kết quả thu được là mở rộng một số kết quả của A. Kamal - J.Müller [7] và Phan Dan [10].