ON THE ASYMPTOTIC EQUIVALENCE OF LINEAR DIFFERENTIAL EQUATIONS IN HILBERT SPACES

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Abstract. *In this paper we study conditions on the asymptotic equivalence of differential equations in Hilbert spaces. Besides, we discuss the relation between properties of solutions of differential equations of triangular form and those of truncated differential equations.*

I. Introduction

Let us consider in a given separable Hilbert space H differential equations of the form

$$\frac{dx}{dt} = f(t, x), \tag{1.1}$$

$$\frac{dy}{dt} = g(t, y), \tag{1.2}$$

where $f : R^+ \times H \to H$; $g : R^+ \times H \to H$ are operators such that $f(t, 0) = 0$, $g(t, 0) = 0$, $\forall t \in R^+$ which satisfy all conditions of global theorem on the existence and uniqueness of solutions (see e.g. [1, p. 187-189]). An interesting problem studied in the qualitative theory of solutions of ordinary differential equations is to find conditions such that (1) and (2) are asymptotically equivalent (see e.g. [3, 4, 5, 6, 8]).

Recall ([5], [3, p. 159]) that (1) and (2) are said to be *asymptotically equivalent* if there exists a bijection between the set of solutions $\{x(t)\}$ of (1) and the one of $\{y(t)\}$ of (2) such that

$$\lim_{t \to \infty} \|x(t) - y(t)\| = 0.$$

Let $\{e_i\}_1^\infty$ be a normalized orthogonal basis of the Hilbert space H and let $x = \sum\limits_{i=1}^{\infty} x_i e_i$ be an element of H. Then the operator $P_n : H \to H$ defined as follows:

$$P_n x = \sum_{i=1}^{n} x_i e_i$$

is a projection on H. We denote $H_n = \mathrm{Im} P_n$.

Suppose that $J = \{n_1, n_2, \ldots, n_j, \ldots\}$ is a strictly increasing sequence of natural numbers ($n_j \to \infty$ as $j \to +\infty$). Together with systems (1), (2) we consider the following systems of differential equations.

$$\begin{cases} \dfrac{dx}{dt} = P_m f(t, P_m x), \\ (I - P_m)x = 0, \quad m \in J, \end{cases} \tag{3}$$

$$\begin{cases} \dfrac{dy}{dt} = P_m g(t, P_m y), \\ (I - P_m)y = 0, \quad m \in J. \end{cases} \tag{4}$$

In this article, we study the asymtotic equivalence of a class of differential equations in the Hilbert space H. We will establish conditions for which the study of the asymptotic equivalence of (1) and (1) is reduced to the one of (3) and (4). There are some results on the stability of this class of differential equations (see [2, 7])

II. Main Results

We assume in this section the following conditions:

$$f(t, P_m x) \equiv P_m f(t, P_m x), \tag{5}$$

$$g(t, P_m x) \equiv P_m g(t, P_m x), \tag{6}$$

$$(\forall t \in R^+, \ \forall m \in J, \ \forall x \in H).$$

Definition 2.1. Differential equations (1) and (2) are said to be *asymptotically equivalent by part with respect to the set* J (or, simply, J - *asymptotically equivalent*) if systems (3) and (4) are asymptotically equivalent for all $m \in J$.

Using (5) we are going to prove the following.

Lemma 2.2. *For any solution* $x(t) = x(t, t_0, P_m x_0)$, $x_0 \in H$ *of equation* (1) *the following relation*

$$x(t, t_0, P_m x_0) = P_m x(t, t_0, P_m x_0)$$

holds for all $t \in R^+$, $m \in J$, $x_0 \in H$.

Proof. For given $m \in J$, let us consider the differential equation

$$\frac{du}{dt} = f(t, P_m u); \quad u \in H, \ t \in R^+. \tag{7}$$

For $\xi_0 \in P_m H$, the solution $u(t) = u(t; t_0, \xi_0)$ of (7) is also a solution of the equation

$$u(t) = \xi_0 + \int_{t_0}^{t} f(\tau, P_m u(\tau)) d\tau. \tag{8}$$

By (5) and $P_m \xi_0 = \xi_0$ we have

$$u(t) = P_m \xi_0 + P_m \int_{t_0}^{t} f(\tau, P_m u(\tau)) d\tau$$

or

$$u(t) = P_m\Big\{\xi_0 + \int_{t_0}^{t} f(\tau, P_m u(\tau))d\tau\Big\}.$$

Hence

$$u(t) = P_m u(t), \quad \forall t \in R^+.$$

Consequently we can rewrite (8) as follows

$$u(t) = \xi_0 + \int_{t_0}^{t} f(\tau, u(\tau))d\tau.$$

This shows that $u(t) = u(t, t_0, \xi_0)$ is a solution of (1), as well. Denoting by $x(t) = x(t, t_0, \xi_0)$ the solution of equation (1) satisfying the condition $x(t_0) = \xi_0$, by uniquness of solution we have:

$$x(t) = u(t).$$

Hence, for $x_0 \in H$, any solution $x(t) = x(t, t_0, P_m x_0)$, $m \in J$ of differential equation (1) will satisfy the relation:

$$x(t, t_0, P_m x_0) = P_m x(t, t_0, P_m x_0), \quad (\forall t \in R^+).$$

The Lemma is proved.

Remark. By Lemma 2.2, we can see that if the conditions (5) and (6) are satisfied, then all solutions of the equations (3), (4) are solutions of the equations (1), (2), respectively. Therefore, from the asymptotic equivalence of systems (1), (2), we can deduce their J - asymptotic equivalence.

Now we consider the following linear differential equations

$$\frac{dx}{dt} = Ax, \tag{9}$$

$$\frac{dy}{dt} = [A + B(t)]y, \tag{10}$$

where $A \in \mathcal{L}(H)$, $B(t) \in \mathcal{L}(H)$, $\forall t \in [0, \infty)$ and

$$\int_0^\infty \|B(\tau)\| d\tau < \infty. \tag{11}$$

We assume that, for them conditions (5), (6) are satisfied, that is,

$$(A - P_m A)P_m x = 0, \quad \forall m \in J, \ \forall x \in H \tag{12}$$

$$(B(t) - P_m B(t))P_m x = 0, \quad \forall m \in J, \ \forall x \in H. \tag{13}$$

Together with (9), (10) we consider also the sequences of truncated differential equations

$$\begin{cases} \dfrac{dx}{dt} = AP_m x, \\ (I - P_m)x = 0, \quad m \in J, \end{cases} \tag{14}$$

$$\begin{cases} \dfrac{dy}{dt} = [A + B(t)]P_m y, \\ (I - P_m)y = 0, \quad m \in J. \end{cases} \tag{15}$$

We denote by $X_m(t)$ the Cauchy operator of (14) satisfying $X_m(0) = E_m$ and by $Y_m(t)$ the Cauchy operator of (15) satisfying $Y_m(t_0) = E_m$, where E_m is the identity operator in H_m.

Lemma 2.3. *If all solutions of equation (14) are bounded, then*

1. *The Cauchy operator $X_m(t)$ of (14) can be written in the form*

$$X_m(t) = U_m(t) + V_m(t),$$

where $U_m(t)$ and $V_m(t) : H_m \to H_m$, so that there exist positive constants a_m, b_m, c_m satisfying

$$\|U_m(t)\| \le a_m e^{-b_m t}, \forall t \in R^+, \tag{16}$$

$$\|V_m(t)\| \le c_m, \forall t \in R; \tag{17}$$

2. *The operators $F_m : H \to H$ defined by*

$$F_m\xi = \int_{t_0}^{\infty} V_m(t_0 - \tau)B(\tau)Y_m(\tau)P_m\xi d\tau, \quad \xi \in H$$

are bounded and moreover the following inequality is valid

$$\|F_m\| \le \alpha_m < 1, \quad \forall t_0 \ge \Delta > 0.$$

Proof. By the assumption on the boundedness of all solutions of (14), we can see that $X_m(t)$ is bounded uniformly in t for every fixed m. Since $dim\ ImP_m < \infty$, the conclusion 1. of the Lemma can be proved by the similar method of proof as in [3, p. 160-161]

Denoting by $Y_m(t)$ the Cauchy operator of equation (15) satisfying $Y_m(t_0) = E$. We see that $Y_m(t)$ satisfies the equations

$$Y_m(t) = X_m(t - t_0) + \int_{t_0}^{t} X_m(t - \tau)B(\tau)Y_m(\tau)d\tau.$$

Thus,

$$\|Y_m(t)\| \le \|X_m(t-t_0)\| + \int_{t_0}^{t} \|X_m(t-\tau)\|\,\|B(\tau)\|\,\|Y_m(\tau)\|d\tau.$$

By (16), (17), we have

$$\|Y_m(t)\| \le a_1 + a_1 \int_{t_0}^{t} \|B(\tau)\|\,\|Y_m(\tau)\|d\tau,$$

where $a_1 = 2\max(a_m, c_m)$. From the Gronwall-Bellman inequality and (11), it follows that

$$\|Y_m(t)\| \le a_1 e^{\int_{t_0}^{t} \|B(\tau)\|d\tau} \le a_1 e^{\int_{0}^{\infty} \|B(\tau)\|d\tau}.$$

Hence, there exists a number K_m independent of t_0 so that

$$\|Y_m(t)\| \le K_m, \quad \forall t \in R^+. \tag{18}$$

Moreover, for any $\alpha_m < 1$, we can find a number $\Delta > 0$ so that

$$\int_{t_0}^{+\infty} \|B(\tau)\|d\tau \le \frac{\alpha_m}{c_m \cdot K_m}, \quad \forall t_0 > \Delta.$$

This implies that

$$\|F_m\| \le \int_{t_0}^{\infty} \|V_m(t_0-\tau)\|\,\|B(\tau)\|\,\|Y_m(\tau)\|d\tau$$

$$\le c_m K_m \int_{t_0}^{\infty} \|B(\tau)\|d\tau \le \alpha_m \le 1, \quad \forall t_0 > \Delta.$$

Theorem 2.4. *Assume that for any $m \in J$ the solutions of (14) are bounded. Then differential equations (9) and (10) are J - asymtotically equivalent.*

Proof For each $m \in J$ we put

$$Q_m x = (I + F_m)x, \quad x \in H_m.$$

By Lemma 2.3, the inequality $\|F_m\| < 1$ holds for $t_0 > \Delta$. Therefore, the operator $Q_m : H_m \to H_m$ is invertible.

Denoting $\eta_0 = Q_m^{-1}\xi_0$, $\xi_0 \in H_m$, $m \in J$, we consider the solutions $x(t) = x(t, t_0, \xi_0)$ of (14) and $y(t) = y(t, t_0, \eta_0)$ of (15). It is clear that

$$x(t) = X_m(t-t_0)\xi_0$$

and

$$y(t) = X_m(t-t_0)\eta_0 + \int_{t_0}^{t} X_m(t-\tau)B(\tau)y(\tau)d\tau.$$

As was shown in Lemma 2.3, by the boundedness of all solutions of (14) we have

$$X_m(t-t_0) = U_m(t-t_0) + V_m(t-t_0)$$
$$V_m(t-\tau) = X_m(t-t_0)V_m(t_0-\tau).$$

From the definition of Q_m, we have

$$\xi_0 = Q_m\eta_0 = \eta_0 + \int_{t_0}^{\infty} V_m(t_0-\tau)B(\tau)Y_m(\tau)\eta_0 d\tau.$$

Hence

$$x(t) = X_m(t-t_0)\eta_0 + X_m(t-t_0)\int_{t_0}^{\infty} V_m(t_0-\tau)B(\tau)Y_m(\tau)\eta_0 d\tau$$
$$= X_m(t-t_0)\eta_0 + \int_{t_0}^{\infty} V_m(t-\tau)B(\tau)Y_m(\tau)\eta_0 d\tau.$$

Consequently,

$$\|y(t)-x(t)\| =$$
$$= \Big\| - \int_{t_0}^{\infty} V_m(t-\tau)B(\tau)Y_m(\tau)\eta_0 d\tau + \int_{t_0}^{t} X_m(t-\tau)B(\tau)y(\tau)d\tau \Big\|$$
$$= \Big\| - \int_{t_0}^{\infty} V_m(t-\tau)B(\tau)Y_m(t)\eta_0 d\tau + \int_{t_0}^{t} U_m(t-\tau)B(\tau)y(\tau)d\tau +$$
$$+ \int_{t_0}^{t} V_m(t-\tau)B(\tau)y(\tau)d\tau \Big\|.$$

Since $y(t) = Y_m(t)\eta_0$, we have

$$\|y(t)-x(t)\| = \Big\| - \int_{t_0}^{\infty} V_m(t-\tau)B(\tau)y(t)d\tau + \int_{t_0}^{t} U_m(t-\tau)B(\tau)y(\tau)d\tau +$$
$$+ \int_{t_0}^{t} V_m(t-\tau)B(\tau)y(\tau)d\tau \Big\|$$
$$z = \Big\| - \int_{t}^{\infty} V_m(t-\tau)B(\tau)y(\tau)d\tau + \int_{t_0}^{t} U_m(t-\tau)B(\tau)y(\tau)d\tau \Big\|.$$

Using (16), (17), (18) and taking into account $y(t) = Y_m(t)\eta_0$, we have

$$\|y(t) - x(t)\| \leq a_m K_m \|\eta_0\| \int_{t_0}^{t} e^{-b_m(t-\tau)} \|B(\tau)\| d\tau + c_m K_m \|\eta_0\| \int_{t}^{\infty} \|B(\tau)\| d\tau$$

or

$$\|y(t) - x(t)\| \leq M_1 \int_{t_0}^{t} e^{-b_m(t-\tau)} \|B(\tau)\| d\tau + M_2 \int_{t}^{\infty} \|B(\tau)\| dt, \quad \forall t \geq t_0,$$

where $M_1 = a_m K_m \|\eta_0\|$, $M_2 = c_m K_m \|\eta_0\|$.

Thus, for every positive number $\varepsilon > 0$, there exists a sufficiently large number t and $t > 2t_0$ such that the following inequalities are valid

$$\int_{t_0}^{\frac{t}{2}} e^{-b_m(t-\tau)} \|B(\tau)\| d\tau \leq e^{-\frac{b_m t}{2}} \int_{t_0}^{\infty} \|B(\tau)\| d\tau < \frac{\varepsilon}{3M_1}$$

$$\int_{\frac{t}{2}}^{t} \|B(\tau)\| d\tau < \frac{\varepsilon}{3M_1}, \qquad \int_{t}^{\infty} \|B(\tau)\| d\tau < \frac{\varepsilon}{3M_2}.$$

Hence,

$$\|y(t) - x(t)\| \leq M_1 \Big(\int_{t_0}^{\frac{t}{2}} e^{-b_m(t-\tau)} \|B(\tau)\| d\tau + \int_{\frac{t}{2}}^{t} e^{-b_m(t-\tau)} \|B(\tau)\| d\tau \Big) +$$

$$+ M_2 \int_{t}^{\infty} \|B(\tau)\| d\tau < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

This means that

$$\lim_{t \to \infty} \|y(t) - x(t)\| = 0.$$

By the uniqueness of solutions of differential equations (14) and (15), the map Q_m is bijective between two set of solutions of equations (14) and (15).

Lemma 2.5. *If all solutions of the differential equations* (9) *are bounded, then*

1. *There exists a positive number* $\Delta = \Delta(\alpha)$ *such that*

$$\|F_m\| \leq \alpha < 1, \quad \forall t_0 \geq \Delta, \ \forall m \in J;$$

2. $\{F_m\}$ *and* $\{Q_m\}$ *are convergent sequences of operators as* $m \to \infty$.

Proof. By the boundedness of all solutions of (9), there is a number $\beta_1 > 0$ such that the Cauchy operator $X(t)$ of (9) satisfies relation

$$\|X(t)\| \leq \beta_1, \quad \forall t \in R^+.$$

Denoting by $Y(t)$ the Cauchy operator of (10) satisfying $Y(t_0) = E$ we see that $Y(t)$ satisfies the equation

$$Y(t) = X(t - t_0) + \int_{t_0}^{t} X(t - \tau)B(\tau)Y(\tau)d\tau.$$

Hence,

$$\|Y(t)\| \leq \|X(t - t_0)\| + \int_{t_0}^{t} \|X(t - \tau)\| \, \|B(\tau)\| \, \|Y(\tau)\|d\tau$$

$$\leq \beta_1 + \beta_1 \int_{t_0}^{t} \|B(\tau)\| \, \|Y(\tau\|d\tau.$$

By the Gronwall - Bellman inequality and (11) there exists a number β_2 independent of t_0 and m such that

$$\|Y(t)\| \leq \beta_2, \quad \forall t \in R^+.$$

Consequently,

$$\|X_m(t)\| \leq \beta_1, \quad \|Y_m(t)\| \leq \beta_2, \quad \forall t \in R^+, \ \forall m \in J.$$

On the other hand, for any $0 < \alpha < 1$ we can find a number $\Delta = \Delta(\alpha) > 0$ such that

$$\int_{t_0}^{\infty} \|B(t)\|d\tau \leq \frac{\alpha}{\beta_1 \cdot \beta_2} < +\infty, \quad \forall t_0 \geq \Delta.$$

Analogously, as in the proof of Lemma 2.3, we have

$$\|F_m\| \leq \int_{t_0}^{\infty} \|V_m(t_0 - \tau)\| \, \|B(\tau)\| \, \|Y_m(\tau)\|d\tau$$

$$\leq \beta_1 \cdot \beta_2 \int_{t_0}^{\infty} \|B(\tau)\|d\tau \leq \alpha < 1, \quad \forall m \in J, \ \forall t_0 \geq \Delta.$$

By definition,

$$F_m = \int_{t_0}^{+\infty} V_m(t_0 - \tau)B(\tau)Y_m(\tau)P_m\xi d\tau.$$

From (12) and (13) we can show that for all $m, m + p \in J, p > 0$,

$$X_{m+p}(t - t_0)P_m\xi = X_m(t - t_0)P_m\xi, \quad \forall \xi \in H$$
$$Y_{m+p}(t)P_m\xi = Y_m(t)P_m\xi, \quad \forall \xi \in H.$$

Hence,

$$F_{m+p}P_m\xi = F_mP_m\xi, \quad \forall m, m + p \in J, p > 0.$$

We now prove the convergence of $\{F_m\}$. In fact, for $\forall m, m+p \in J, p > 0$, we have

$$\begin{aligned}\|F_{m+p} - F_m\| &= \|F_{m+p}P_{m+p} - F_mP_m\| \\ &= \|F_{m+p}(P_{m+p} - P_m) + (F_{m+p} - F_m)P_m\| \\ &= \|F_{m+p}(P_{m+1} - P_m)\| \\ &\leq \|F_{m+p}\|\|P_{m+p} - P_m\|.\end{aligned}$$

By definition, $\lim_{m\to\infty} P_m = I$. Hence, by the boundedness of F_m the above yields that $\{F_m\}$ is a Cauchy sequence, so $\{F_m\}$ is convergent. This implies the convergence of $\{Q_m\}$.

Theorem 2.6. *If all solutions of the differential equation* (9) *are bounded then the equations* (9) *and* (10) *are asymptotically equivalent.*

Proof. By virtue of Lemma 2.5, we can put:

$$F = \lim_{m\to\infty} F_m \quad \text{and} \quad Q = \lim_{m\to\infty} Q_m.$$

Hence, $Q = I + F$. Since $\|F_m\| \leq \alpha < 1$, $\forall m \in J$, $\forall t_0 \geq \Delta$, we have

$$\|F\| \leq \alpha < 1, \quad \forall t_0 \geq \Delta.$$

Therefore, $Q : H \to H$ is an invertible operator. By the uniqueness of solutions of equations (9) and (10) we deduce that the map Q is also bijective between two sets of solutions $\{x(t)\}$ of (9) and $\{y(t)\}$ of (10). Let $y_0 = Q^{-1}x_0$ and $x(t) = X(t-t_0)x_0$, $y(t) = Y(t)y_0$. Since

$$\lim_{m\to\infty} P_my_0 = y_0, \quad \lim_{m\to\infty} Q_my_0 = Qy_0 = x_0,$$

we can see that for any given $\varepsilon > 0$, there exists sufficiently large $m_1 \in J$ such that for all $m \geq m_1$ we have for $\forall t \geq t_0$

$$\begin{aligned}\|y(t;t_0,y_0) - y(t;t_0,P_my_0)\| &< \frac{\varepsilon}{3}, \\ \|x(t;t_0,y_0) - x(t;t_0,Q_my_0)\| &< \frac{\varepsilon}{3}.\end{aligned}$$

By virtue of Theorem 2.4 and the boundedness of all solutions of (9), we deduce that differential equations (9) and (10) are J - asymptotically equivalent. Consequently, there exists $\tau_0 \in (t_0, \infty)$ such that for $\forall t \geq \tau_0$,

$$\|x(t;t_0,Q_{m_1}y_0) - y(t;t_0,P_{m_1}y_0)\| < \frac{\varepsilon}{3},$$

where t_0 is choosen sufficiently large such that

$$\|F_m\| \leq \alpha < 1, \quad \forall m \in J.$$

Therefore,

$$\begin{aligned}\|y(t;t_0,y_0)-x(t;t_0,x_0)\| &\le \|y(t;t_0,y_0)-y(t;t_0,P_{m_1}y_0)\|+\\ &+\|y(t,t_0,P_{m_1}y_0)-x(t,t_0,Q_{m_1}y_0)\|\\ &+\|x(t;t_0,Q_{m_1}y_0)-x(t;t_0,x_0)\|\\ &\le \frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon;\quad \forall t\ge\tau_0.\end{aligned}$$

This implies that

$$\lim_{t\to\infty}\|x(t;t_0,x_0)-y(t;t_0,y_0)\|=0.$$

By virtue of the Lemma 2.2 we get:

Corollary 2.7 Assume that all solutions of the differential equation (9) are bounded. Then, the differential equations (9) and (10) are asymptotically equivalent if and only if they are J - asymptotically equivalent.

Corollary 2.8 If all solutions of differential equations (14) are uniformly bounded for all $m\in J$, then differential equations (9) and (10) is asymptotically equivalent.

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VỀ SỰ TƯƠNG ĐƯƠNG TIỆM CẬN CỦA CÁC PHƯƠNG TRÌNH VI PHÂN TUYẾN TÍNH TRONG KHÔNG GIAN HILBERT

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Trong bài báo này chúng tôi nghiên cứu sự tương đương tiệm cận của các phương trình vi phân trong không gian Hilbert. Đồng thời chúng tôi cũng đã chỉ ra mối liên hệ giữa tính tương đương tiệm cận của các phương trình vi phân dạng tựa tam giác trong không gian Hilbert và tính tương đương tiệm cận của các phương trình vi phân được cắt ngắn tương ứng.