# FUNCTION ALGEBRA ON A DISK

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**Abstract** *In this paper we prove the theorem on approximation of continous function algebra on a disk. This result is an extension of the Werner's one*

## I. Introduction

Let $D$ be small closed disk in the complex plane, centered at the origin and $f \in C(D)$. By $[z, f; D]$ we denote the function algebra consisting of uniform limits on $D$ of all polynomials in $z$ and $f$.

In 1964, J. Wermer [3] proved that if $f$ of class $C^1$ and $\frac{\partial f}{\partial \bar{z}}(0) \neq 0$ then $[z, f; D] = C(D)$. In 2001, P. J. de Paepe [4] show that if $f$ of class $C^1$, with $f(0) = 0$, $\frac{\partial f}{\partial z}(0) = 0$ and $\frac{\partial f}{\partial \bar{z}}(0) \neq 0$ then $[z^m, f^n; D] = C(D)$ with $D$ small enough and $m, n$ are coprime natural numbers. The proof of de Paepe does not work if $\frac{\partial f}{\partial z}(0) \neq 0$. In this paper, we give conditions such that $[z^m, f^n; D] = C(D)$ when $\frac{\partial f}{\partial z}(0) \neq 0$. The proofs are maked by the line of [2], the basis tool is Stout's version of Kallin's lemma.

## II. The main result

**Theorem.** *Let $f$ be a function of class $C^1$ defined in a neighbourhood of 0, with $f(0) = 0$, $\frac{\partial f}{\partial z}(0) = 1$ and $\frac{\partial f}{\partial \bar{z}}(0) = b \neq 0$. Suppose $m, n$ are coprime natural numbers with $m, n > 1$ and*

$$|b|^2 > 2\Big(1 + \max_{1 \le k, r \le m;\ 1 \le l,\ s \le n} \frac{1 + \left|\cos\left(2\pi\left(\frac{k-r}{m} + \frac{l-s}{n}\right)\right)\right|}{1 - \cos 2\pi\left(\frac{k-r}{m} + \frac{l-s}{n}\right)}\Big) \tag{*}$$

*for $k \neq r$ or $l \neq s$. Then $[z^m, f^n; D] = C(D)$ if $D$ is a sufficiently small disk around 0.*

**Lemma 1.** *Let $X$ be a compact subset of $\mathbb{C}^2$, and let $\pi : \mathbb{C}^2 \to \mathbb{C}^2$ be defined by $\pi(z, w) = (z^m, w^n)$. Let $\pi^{-1}(X) = X_{11} \cup ... \cup X_{kl} \cup ... \cup X_{mn}$ with $X_{mn}$ compact, and $X_{kl} = \{(\rho^k z, \tau^l w) : (z, w) \in X_{mn}\}$ for $1 \le k \le m$, $1 \le l \le n$, where $\rho = \exp\left(\frac{2\pi i}{m}\right)$ and $\tau = \exp\left(\frac{2\pi i}{n}\right)$. If $P(\pi^{-1}(X)) = C(\pi^{-1}(X))$, then $P(X) = C(X)$.*

*Proof.* Let $f \in C(X)$. Then $f \circ \pi \in C(\pi^{-1}(X))$, so there is a polynomial $Q$ in two variables with $f \circ \pi \sim Q$ on $\pi^{-1}(X)$. In particular, this is true on $X_{kl}$, so

$$f(z^m, w^n) \sim Q(\rho^k z, \tau^l w) := Q_{kl}(z, w) \text{ on } X_{mn}.$$

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It follows that

$$f(z^m, w^n) \sim \frac{Q_{11}(z,w) + \cdots + Q_{mn}(z,w)}{mn} \quad \text{on } X_{mn}.$$

Now, if $Q(z,w) = \sum a_{p,q} z^p w^q$, the right hand side above equals $\sum a_{pm,qn} z^{pm} w^{qn}$ (all other terms drop out), so equals $P(z^m, w^n)$, where $P$ is polynomial in two variables. So $f(z^m, w^n) \sim P(z^m, w^n)$ on $X_{mn}$, that is, $f \sim P$ on $X$. So $P(X) = C(X)$.

**Lemma 2.** (Stout's version of Eva Kallin's lemma) [4]. *Suppose that:*

*(1) $X_1$ and $X_2$ are compact subsets of $\mathbb{C}^n$ with $P(X_1) = C(X_1)$ and $P(X_2) = C(X_2)$;*

*(2) $Y_1$ and $Y_2$ are polynomially convex subsets of $\mathbb{C}$ such that $0$ is boundary point of both $Y_1$ and $Y_2$, and $Y_1 \cap Y_2 = \{0\}$;*

*(3) $p$ is polynomial such that $p(X_1) \subset Y_1$ and $p(X_2) \subset Y_2$;*

*(4) $p^{-1}(0) \cap (X_1 \cup X_2) = X_1 \cap X_2$.*

*Then $P(X_1 \cup X_2) = C(X_1 \cup X_2)$.*

*Proof of Theorem.* The conditions on $f$ imply that $f(z) = z + b\overline{z} + h(z)$, with $h(z)$ of class $C^1$ and $h(z) = o(|z|)$.

First, we show that $z^m$ and $f^n$ separate points near 0. Indeed, first we see that points $u$ and $v$ with $v \neq u\exp\left(\frac{2\pi i k}{m}\right)$ for all $1 \leq k \leq m$ are separated by $z^m$. Now, suppose that $(f(z))^n$ take the same value at $u\exp\left(\frac{2\pi ik}{m}\right)$ and $u\exp\left(\frac{2\pi il}{m}\right)$ for $k \neq l$ and $u \neq 0$. Then, there is $1 \leq r \leq n$ such that $f\left(u\exp\left(\frac{2\pi ik}{m}\right)\right) = \exp\left(\frac{2\pi ir}{n}\right) f\left(u\exp\left(\frac{2\pi il}{m}\right)\right)$. It implies that

$$\begin{aligned} &b|u|\exp(-i\varphi)\left(\exp\left(\frac{-2\pi ik}{m}\right) - \exp\left(\frac{-2\pi il}{m} + \frac{2\pi ir}{n}\right)\right) \\ &= -|u|\exp(i\varphi)\left(\exp\left(\frac{2\pi ik}{m}\right) - \exp\left(\frac{2\pi il}{m} + \frac{2\pi ir}{n}\right)\right) \\ &\quad - h\left(u\exp\left(\frac{2\pi ik}{m}\right)\right) + \exp\left(\frac{2\pi ir}{n}\right) h\left(u\exp\left(\frac{2\pi il}{m}\right)\right), \end{aligned}$$

where $u = |u|\exp(i\varphi)$. It follows that

$$\begin{aligned} |b|^2\,|u|^2\left(1 - \cos 2\pi\left(\frac{k-l}{m} + \frac{r}{n}\right)\right) &\leq 2|u|^2\left(1 - \cos 2\pi\left(\frac{k-l}{m} - \frac{r}{n}\right)\right) \\ &+ 4\left(\left|h\left(u\exp\left(\frac{2\pi k}{m}\right)\right)\right|^2 + \left|h\left(u\exp\left(\frac{2\pi il}{n}\right)\right)\right|^2\right) \end{aligned}$$

If $m, n$ are coprime, then $\frac{k-l}{m} + \frac{r}{n}$ is not integer with $1 \leq k \neq l \leq m$ and $1 \leq r \leq n$, so $\cos 2\pi\left(\frac{k-l}{m} + \frac{r}{n}\right) \neq 1$. Therefore

$$|b|^2 \leq 2\left(\frac{1 - \cos 2\pi\left(\frac{k-l}{m} - \frac{r}{n}\right)}{1 - \cos 2\pi\left(\frac{k-l}{m} + \frac{r}{n}\right)}\right) + 4\left(\frac{\left|h\left(u\exp\left(\frac{2\pi ik}{m}\right)\right)\right|^2 + \left|h\left(u\exp\left(\frac{2\pi il}{m}\right)\right)\right|^2}{|u|^2\left(1 - \cos 2\pi\left(\frac{k-l}{m} + \frac{r}{n}\right)\right)}\right).$$

Since $h(z) = o(|z|)$ for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|h(z)| < \varepsilon|z|$ for all $z \in B(0,\delta) := \{z \in \mathbb{C} : |z| \le \delta\}$. So, for $D$ is small enough, we have

$$|b|^2 \le 2\left(\frac{1-\cos 2\pi\left(\frac{k-l}{m}-\frac{r}{n}\right)}{1-\cos 2\pi\left(\frac{k-l}{m}+\frac{r}{n}\right)}\right) + \frac{8\varepsilon^2}{1-\cos 2\pi\left(\frac{k-l}{m}+\frac{r}{n}\right)}$$

It follows that

$$|b|^2 \le 2\left(\frac{1-\cos 2\pi\left(\frac{k-l}{m}-\frac{r}{n}\right)}{1-\cos 2\pi\left(\frac{k-l}{m}+\frac{r}{n}\right)}\right),$$

because $\varepsilon$ is arbitrary. This contradicts to $(*)$. So $z^m$ and $f^n$ separate points near 0. Now, let $X = \{(z^m, f^n) : z \in D\}$. Furthermore, let $\pi$ be as in Lemma 1, and

$$\pi^{-1}(X) = X_{11} \cup \ldots \cup X_{mn} \quad \text{with} \quad X_{mn} = \{(z, f(z)) : z \in D\}.$$

By Wermer's theorem $P(X_{kl}) = C(X_{kl})$ for $1 \le k \le m$, $1 \le l \le n$ . Next we consider polynomial $p(z,w) = zw$ and put

$$Y_{kl} := p(X_{kl}) = \{\rho^k\tau^l(z^2 + b|z|^2 + zh(z)) : z \in D\}.$$

We show that $Y_{kl} \cap Y_{rs} = \{0\}$ for all $1 \le k,r \le m$; $1 \le l,s \le n$ with $k \ne r$ or $l \ne s$. Indeed, it is easy to see that $0 \in Y_{kl}$ for all $1 \le k \le m$ and $1 \le l \le n$. Suppose, there is $0 \ne y \in Y_{kl} \cap Y_{rs}$ with $k \ne r$ or $l \ne s$. Then, there are $z_1 = a\exp(i\alpha)$ and $z_2 = c\exp(i\beta)$, where $a, c$ are real numbers with $ac \ne 0$ such that

$$\begin{aligned}&\rho^k\tau^l\big(ba^2 + a^2\exp(2i\alpha) + a\exp(i\alpha)h(a\exp(i\alpha))\big)\\ &= \rho^r\tau^s\big(bc^2 + c^2\exp(2i\beta) + c\exp(i\beta)h(c\exp(i\beta))\big).\end{aligned}$$

It implies that

$$\begin{aligned}b(\rho^r\tau^s c^2 - \rho^k\tau^l a^2) &= \rho^k\tau^l a^2\exp(2i\alpha) - \rho^r\tau^s c^2\exp(2i\beta)\\ &+ a\exp(2i\alpha)h(a\exp(2i\alpha)) - c\exp(2i\beta)h(c\exp(2i\beta)).\end{aligned}$$

Put $A = \cos 2\left((\alpha-\beta) + \pi\left(\frac{k-r}{m} + \frac{l-s}{n}\right)\right)$ and $B = \cos 2\pi\left(\frac{k-r}{m} + \frac{l-s}{n}\right)$. By the coprimeness of $m, n$ we see that $\frac{k-r}{m} + \frac{l-s}{n}$ is not integer for all $1 \le k,r \le m$ and $1 \le l,s \le n$ with $k \ne r$ or $l \ne s$, so $B \ne 1$. We obtain

$$|b|^2 \le 2\left(\frac{a^4+c^4-2a^2c^2A}{a^4+c^4-2a^2c^2B}\right) + 4\left(\frac{a^2|h(a\exp(2i\alpha))|^2 + c^2\,|h(c\exp(2i\beta))|^2}{a^4+c^4-2a^2c^2B}\right)$$

Since $h(z) = o(|z|)$, for every $\varepsilon > 0$, there exists $D$ is sufficiently small disk such that

$$\frac{a^2|h(a\exp(2i\alpha)|^2 + c^2|h(c\exp(2i\beta)|^2}{a^4+c^4-2a^2c^2B} \le \frac{4(a^4+c^4)\varepsilon}{a^4+c^4-2a^2c^2B} \le M\varepsilon,$$

where $M = 4$ if $B \le 0$ and $M = \frac{4}{1-B}$ if $B > 0$. So, for $D$ small enough, we obtain

$$|b|^2 \le 2\left(\frac{a^4+c^4-2a^2c^2A}{a^4+c^4-2a^2c^2B}\right) + M\varepsilon.$$

We have

$$|b|^2 \leq 2\Big(\frac{t^2-2At+1}{t^2-2Bt+1}\Big)+M\varepsilon = 2\Big(1+\frac{2(B-A)t}{t^2-2Bt+1}\Big)+M\varepsilon, \quad \text{where} \quad t=\frac{a^2}{c^2}>0.$$

Considering the real function $g(t)=\dfrac{t}{t^2-2Bt+1}$, we obtain $\max\limits_{(0,+\infty)} g(t)=\dfrac{1}{2(1-B)}$. It implies that

$$|b|^2 \leq 2\Big(1+\frac{|B-A|}{1-B}\Big)+M\varepsilon \leq 2\Big(1+\frac{|A|+|B|}{1-B}\Big)M\varepsilon \leq 2\Big(1+\frac{1+|B|}{1-B}\Big)+M\varepsilon.$$

Since $\varepsilon$ is arbitrary, we conclude that

$$|b|^2 \leq 2\Big(1+\frac{1+|B|}{1-B}\Big)$$

This inequality contradicts to $(*)$.

Furthermore, it is easy to see that $\mathbb{C}\setminus Y_{kl}$ is connected set if $D$ small enough, so $Y_{kl}$ is polynomially convex [Ga] and $p^{-1}(0)\cap X_{kl}=(0,0)$ for $1\leq k\leq m$ and $1\leq l\leq n$. Therefore $p^{-1}(0)\cap(X_{kl}\cup X_{rs})=X_{kl}\cap X_{rs}$ for all $1\leq k,r\leq m$ and $1\leq l,s\leq n$ with $k\neq r$ or $r\neq s$. Now apply Stout's version of Kallin's lemma repeatedly, to obtain

$$P(\pi^{-1}(X))=C(\pi^{-1}(X))$$

By Lemma 1, it follows that $P(X)=C(X)$, or equivalently $[z^m,f^n;D]=C(D)$. The theorem is proved.

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