ON FD-CAP SETS IN CONVEX GROWTH HYPERSPACES OF CONVEX N-CELLS

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Abstract *If X is a convex n-cell, $n \geq 2$, then every non-trivial convex growth polyhedron hyperspace G is an fd-cap set in the closure $\overline{G}$ of G in $CC(X)$.*

1. Introduction

Let X be a compact convex set lying in a Banach space. We write $CC(X)$ for the hyperspace of all non-empty convex sets in X topologized by Hausdorff metric:

$$d(A,B) = \max\left\{\max_{a\in A}\min_{b\in B}\|a-b\|, \max_{b\in B}\min_{a\in A}\|a-b\|\right\}$$

for $A, B \in CC(X)$.

By $P(X)$ we denote the family of all *convex polyhedrons* in X. A family $G \subset CC(X)$ (*resp.* $G \subset P(X)$) is *a convex growth hyperspace* (*resp. convex growth polyhedron hyperspace*) provided it satisfies the condition: If $A \subset G$ and $B \in CC(X)$ (*resp.* $B \in P(X)$) such that $A \subset B$, then $B \subset G$.

2. The results

Proposition 2.1. *If G is a convex growth polyhedron hyperspace then closure $\overline{G}$ of G in $CC(X)$ is a closed convex growth hyperspace.*

Curtis [2] has shown that if G is a non-trivial closed convex growth hyperspace of convex n-cell, $n \geq 2$, then G is homeomorphic to the Hilbebrt cub Q iff $G \setminus \{X\}$ is contractible. In this note we prove the following theorem strengthening the theorem of Curtis.

Theorem 2.2. *If X is a convex n-cell, $n \geq 2$, then every non-trivial convex growth polyhedron hyperspace G is an fd-cap set in the closure $\overline{G}$ of G in $CC(X)$.*

Here we say that a subset M of a metric space X is *an fd-cap set* in X iff M is a countable union of finite dimensional compact z-sets and the following condition hold

(*Cap.*) There is an increasing sequence of finite dimensional compact z-sets $\{M_n\}$ with $\bigcup_{n\in N} M_n$ is dense in M such that given a finite dimensional compact set $K \subset X, \forall\varepsilon > 0, n \in N$, there is an embedding $h : K \to M_m$ for some $m > n$ such that $h|_{K\cap M_n} = id$ and $d(h(x), x) < \varepsilon$ for each $x \in K$.

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Definition 2.3. *We say that M is a cap-set in X iff M is a countable union of compact z-sets and the above condition is satisfied for every finite dimensional compact set $K \subset X$.*

Combining the theorem with the result of Curtis [2] we obtain the following fact.

Corollary 2.4. *Let G be a non-trivial convex growth polyhedron hyperspace of a convex n-cell X, $n \geq 2$ and let $\overline{G}$ denote the closure of G in $CC(X)$. If $\overline{G} \setminus \{X\}$ is contractible, then $(\overline{G}, G) \cong (Q, Q^f)$, where $Q^f = \{x = (x_i) \in Q : x_i = 0 \; for \; almost \; i\}$.*

3. Proof of the theorem

For each $n \in N$, put

$$G_n = \{A \in G : \vee A \leq n\}$$
$$F_n = \{A \in G : A \subset IntX \quad \text{and} \quad \vee A \leq n\},$$

where $\vee A$ denotes the number of vertices of A. Obviously, G_n, F_n are z-sets in $\overline{G}$ for each $n \in N$ and $G \supset F \overset{df}{=} \bigcup\limits_{n\in N} F_n$, $G = \bigcup\limits_{n\in N} G_n$ and $\overline{G} = \overline{F} (= CC(X))$.

The proof of the theorem is divided into two steps.

Step 1. Given $\varepsilon > 0$, $n \in N$ and a finite dimensional compact set $K \subset \overline{G}$, there is a map $g : K \to F_p$, for some $p > n$ such that

$$g|_{K \cap F_n} = id \text{ and } d[g(x), x] < \varepsilon/2$$

for $x \in K$.

Proof. Take an $m > n$ such that F_m is an $\frac{1}{4}\varepsilon - net$ for K. Let $\{U_j, C_j\}_{j\in J}$ be a Dugundji system for $K \setminus F_m$ (see [1]) and let $U = \{U_j\}_{j \in J}$, $k = \dim K$.

By $N(U)$ we denote the nerve of U and let $N_0(U)$ be the 0-skeleton of $N(U)$. Since $\dim K = k$, we may assume that every simplex σ of $N(U)$ has at most $k+1$ vertices. We define

$$f : N_0(U) \to F_m$$

by the formula $f(U_j) = a_j$ for every $j \in J$, and extend f over the 1-skeleton $N_1(U)$ of $N(U)$ as follows:

Let C be edge of $N(U)$ with endpoints U_i, U_j and midpoint C^*. We define f on $C = [U, C^*] \cup [C^*, U_j]$ by the formula

$$f[(1-t)U_i + tC^*] = \text{Conv}\{a_i, (1-t)a_i + ta_j\},$$
$$f[(1-t)U_j + tC^*] = \text{Conv}\{a_j, (1-t)a_j + ta_i\},$$

for $t \in [0,1]$. It is easy to see that $f(x) \in F_{2m^2}$ for each $x \in C = [U_i, U_j]$.

We now extend f over $N(U)$.

Let C denote the hyperspace of subcontinua of the 1-skeleton of $N(U)$. Take a map $\varphi : N(U) \to C$ such that $\varphi(x) = \{x\}$ for each $x \in N_1(U)$, and if σ is the carrier of point x, then $\varphi(x) \subset \sigma^{(1)}$.

We define $f : N(U) \to F$ by the formula

$$f(x) = Conv\{f(p) : \ p \in \varphi(x)\} \text{ for } x \in N(U).$$

It is easy to see that f is continuous (i.e. f/σ is continuous for every simplex σ of $N(U)$) and $f(x) \in F_{2km^2}$ for every $x \in K$.

Let $p = 2km^2$, we define $g : K \to F_p$ by the formula

$$g(x) = \begin{cases} x & \text{if } x \in K \cap F_m \\ f\left[\sum_{i \in J} \lambda_i(x) U_j\right] & \text{if } x \in K \setminus F_m, \end{cases}$$

where $\{\lambda_j\}_{j \in J}$ is a locally finite partition of unity inscribed into $U = \{U_j\}_{j \in J}$. Since $m > n$, we have

$$g|_{K \cap F_n} = id.$$

For each $x \in K \setminus F_m$, let

$$E(x) = \{j \in J, \ \lambda_j(x) > 0\}.$$

Then $\text{card}E(x) \leq k + 1$ and

$$\begin{aligned} d[g(x), x] = d\left[f\left(\sum_{j \in E(x)} \lambda_j(x) U_j\right), x\right] \leq \\ \leq d\left[\text{Conv}\left\{a_j : j \in E(x)\right\}, x\right] \leq \\ \leq \sup\left\{d(a_j, x) : j \in E(x)\right\} \leq 2d(F_m, x). \end{aligned}$$

This shows that g is continuous.

Since F_m is an $\frac{1}{4}\varepsilon$ – net for K, we infer that $d(g(x), x) < \frac{\varepsilon}{2}$ for $x \in K$.

Step 2. There is an embedding $h : K \to F_m$ for some $m > p > n$ such that

$$h|_{K \cap F_n} = g|_{K \cap F_n} = \text{id}$$

and

$$d(h(x), g(x)) \leq \frac{1}{2}\varepsilon$$

for each $x \in K$.

Proof. Without loss of generality we may assume that $X \subset R^2$. Let us put

$$\widetilde{K} = \bigcup \{g(x) : x \in K\} \subset \text{IntX}.$$

Since $\widetilde{K}$ is compact, $\text{dist}(\widetilde{K}, \partial X) > 0$, where ∂X denotes the boundary of X.

Let h be an embedding of K into I^K. For some $k \in N$, let $h_i, \ i = 1, \dots, k$ be the it's coordinate functions h.

For each $x \in K$, put

$$S(x) = \text{Conv}\left\{ e^{\dfrac{2\pi \widehat{x}_j i}{3p(k+1)}},\ j = 0, \dots, 3p(k+1) - 1 \right\} \tag{1}$$

$$i^2 = -1 \tag{2}$$

$$\widehat{x}_j = \begin{cases} j & \text{if } j = r(k+1) \text{ for } r = 0, \dots, 3p(k+1) - 1 \\ j + \frac{1}{2} h_q(x) & \text{if } j = r(k+1) + q \text{ for } r = 0, \dots, 3p(k+1) - 1, \\ & q = 1, \dots, k. \end{cases} \tag{3}$$

We define $h : K \to F$ by the formula

$$h(x) = g(x) + \delta d(x, B) S(x) \qquad \text{for each } x \in K, \tag{4}$$

where

$$B = F_n \cap K$$

and

$$\delta = \frac{1}{2} \min\left\{\varepsilon,\ \text{dist}(\widetilde{K}, \partial X)\right\}.$$

It is easy to see that

$$h(K) \subset F_n$$

for $m = 3p(k+1)$, $h|_B = g|_B = \text{id}$ and $d[h(x), g(x)] \leq \dfrac{\varepsilon}{2}$ for each $x \in K$.

Let us show that h is an embedding. Given $x, y \in K$ with $x \neq y$. Consider three case.

Case 1. $x, y \in B = K \cap F_n$. Then we have

$$h(x) = g(x) = x \neq y = g(y) = h(y).$$

Case 2. $x \in B$ and $y \in K \setminus B$. Then we have

$$\vee h(x) = \vee g(x) \leq p < 3p(k+1) = \vee h(y).$$

Thus $h(x) \neq h(y)$.

Case 3. $x, y \in K \setminus B$. Let V be a vertex of $g(x)$ such that

$$\widehat{V} \leq \frac{(\vee g(x) - 2)2\pi}{\vee g(x)},$$

where $\widehat{V}$ denotes the angle of $g(x)$ at V.

Let $\widehat{V^*}$ denote the angle pictured as in figure 1.

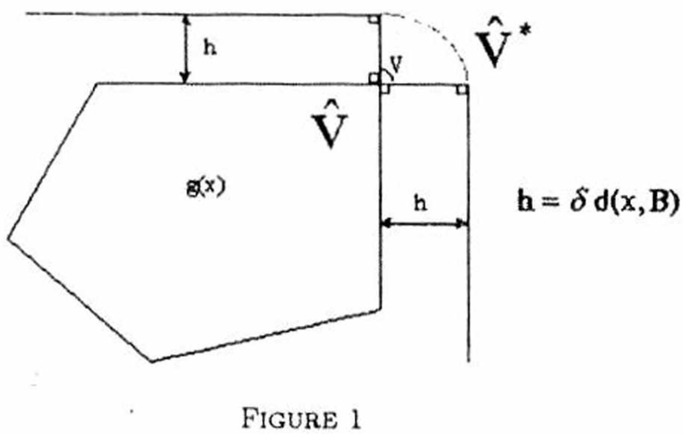

FIGURE 1

Then we have

$$\widehat{V^*} = 2\pi - \widehat{V} \geq \frac{4\pi}{Vg(x)} \geq \frac{4\pi}{p}.$$

In this case $h(x)$ has at least

$$\frac{4\pi}{p} \cdot \frac{3p(k+1)}{2\pi} = 6(k+1) \text{ vertices of the form}$$

$$\left\{ V + \delta d(x,B) \cdot e^{\frac{2\pi \widehat{x_j} i}{3p(k+1)}},\ j = q, \dots, q + 6(k+1) \right\}. \tag{5}$$

Consider two cases.

Case 3a. V is a vertex of $g(y)$. Since $x \neq y$, from (1), (2), (3), (4) follows that there are at least nine points of the form (5), which are not vertices of $h(y)$. This shows that $h(x) \neq h(y)$.

Case 3b. V is not a vertex of $g(y)$, whence $h(y)$ has at most two vertices of the form (5), thus $h(x) \neq h(y)$.

Thus h is one-to-one. Since K is compact, it follows that f is an embedding.

This completes the proof of the theorem.

References

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