SOME REMARKS ON THE FINITE-TIME BEHAVIOR OF WIENER PATHS

Dang Phuoc Huy
Department of Mathematics, University of Da Lat

Abstract *We establish some properties of the finite-time behavior of Wiener paths. Some applications of these results are also given.*
Keywords: Wiener's measure, stopping-time, Wiener scaling invariance.

1. Introduction

Throughout this note, by $\mathfrak{B}(\mathbb{R}^N)$ we shall denote the Polish space of all continuous paths $\Psi : [0,\infty) \longrightarrow \mathbb{R}^N$, and let $M_1(\mathfrak{B}(\mathbb{R}^N))$ be the space of Borel probability measures on $\mathfrak{B}(\mathbb{R}^N)$(see, for example, [1, Section 1]). Define, for each $\mathbf{x} \in \mathbb{R}^N$, the transformation $T_{\mathbf{x}} : \mathfrak{B}(\mathbb{R}^N) \longrightarrow \mathfrak{B}(\mathbb{R}^N)$ by

$$\Big[T_{\mathbf{x}}\Psi\Big](t) = \mathbf{x} + \Psi(t),\ t \in [0,\infty), \tag{1.1}$$

and let $\mathcal{W}^{(N)}_{\mathbf{x}} = T_{\mathbf{x}} * \mathcal{W}^{(N)}$ be the distribution of $T_{\mathbf{x}}$ under $\mathcal{W}^{(N)}$, where $\mathcal{W}^{(N)}$ is Wiener's measure for $\mathbb{R}^N$- valued paths. As usual, we use $\mathfrak{B}_E$ to denote the Borel field over the topological space E, and set

$$\gamma^{(N)}_t(d\mathbf{y}) = \gamma^{(N)}_t(\mathbf{y})d\mathbf{y},$$

where $\gamma^{(N)}_t(\mathbf{y})$ is the Gauss kernel on $\mathbb{R}^N$.

We mainly refer the reader to [2, Section 3.3] for all questions about the existence, the uniqueness and the independence of the coordinates of $\mathcal{W}^{(N)}$, under which the above probability distribution was also satisfied. Namely, we have the following properties.

(a) $\mathcal{W}^{(N)}_{\mathbf{x}}$ *is a unique probability measure on* $M_1(\mathfrak{B}(\mathbb{R}^N))$ *with the properties that* $\Psi(0) = \mathbf{x}$ *for* $\mathcal{W}^{(N)}_{\mathbf{x}}$*-almost all* $\Psi \in \mathfrak{B}(\mathbb{R}^N)$ *and*

$$\mathcal{W}^{(N)}_{\mathbf{x}}\left(\left\{\Psi : \Psi(t_1) - \Psi(t_0) \in B_1, ..., \Psi(t_k) - \Psi(t_{k-1}) \in B_k\right\}\right)$$
$$= \gamma^{(N)}_{t_1}(B_1) \times \gamma^{(N)}_{t_2-t_1}(B_2) \times \cdots \times \gamma^{(N)}_{t_k-t_{k-1}}(B_k), \tag{1.2}$$

for all $k \in Z^+, 0 = t_0 < t_1 < \cdots < t_k$, *and* $B_1, \cdots, B_k \in \mathfrak{B}_{\mathbb{R}^N}$.

(b)

$$\mathcal{W}^{(N)}_{\mathbf{x}} = \mathcal{W}_{x_1} \times \cdots \times \mathcal{W}_{x_N} = \prod_{i=1}^{N} \mathcal{W}_{x_i}. \tag{1.3}$$

Here we use $\mathcal{W}_{x_i}$ *in place of* $\mathcal{W}^{(1)}_{x_i}$, *and* $\mathcal{W}_{x_1} \times \cdots \times \mathcal{W}_{x_N}$ *is the product measure, for any* $\mathbf{x} = (x_1, \cdots, x_N)^T \in \mathbb{R}^N$.

Our aim here is to investigate some of the basic facts about the finite-time behavior of Wiener paths. Thus, in Section 2, we shall present the invariance properties of Wiener's measure and, as a consequence of Theorem 4.3.8 in [2], some results are obtained. Finally, applications to the properties of Wiener paths are given in Section 3.

2. Some properties

We begin this section with the invariance properties of the probability distribution $\mathcal{W}_{\mathbf{x}}^{(N)}$. Firstly, we recall two families of transformations on $\mathfrak{B}(\mathbb{R}^N)$. The first of these is the family $\{S_\alpha : \alpha \in (0,\infty)\}$ of Scaling maps given by

$$\left[S_\alpha \Psi\right](t) = \alpha^{-\frac{1}{2}}\Psi(\alpha t),\ t \in [0,\infty), \tag{2.1}$$

and the second family of transformations which we will want are the rotation $\mathcal{R}$ relative to $\mathbf{R}$, given by

$$\left[\mathcal{R}\Psi\right](t) = \mathbf{R}\Psi(t),\ t \in [0,\infty), \tag{2.2}$$

where $\mathbf{R}$ is a orthogonal matrix of order N.

From the invariance properties were introduced in [2] (see [p. 182 and Exercise 3.3.28]), we immediately obtain the following result.

Proposition 2.1.
(a) *(Translation invariant)*

$$\mathcal{W}_{\mathbf{x}}^{(N)} = T_{\mathbf{x}-\mathbf{y}} * \mathcal{W}_{\mathbf{y}}^{(N)} = T_{\mathbf{y}} * \mathcal{W}_{\mathbf{x}-\mathbf{y}}^{(N)}, \tag{2.3}$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$.
(b) *(Scaling invariance)*

$$\mathcal{W}_{\mathbf{x}}^{(N)} = S_\alpha * \mathcal{W}_{\alpha^{\frac{1}{2}}\mathbf{x}}^{(N)}, \tag{2.4}$$

for each $\alpha \in (0,\infty)$ *and* $\mathbf{x} \in \mathbb{R}^N$.
(c) *(Rotation invariant)*

$$\mathcal{W}_{\mathbf{x}}^{(N)} = \mathcal{R} * \mathcal{W}_{\mathbf{R}^T\mathbf{x}}^{(N)}, \tag{2.5}$$

(where $\mathbf{R}^T$ *is the transposed matrix of* $\mathbf{R}$*), for each* $\mathbf{x} \in \mathbb{R}^N$.

Proof. We begin by noting that, for each $\Psi \in \mathfrak{B}(\mathbb{R}^N)$,

$$\left[T_{\mathbf{x}} S_\alpha \Psi\right](t) = \mathbf{x} + \alpha^{-\frac{1}{2}}\Psi(\alpha t) = \left[S_\alpha T_{\alpha^{\frac{1}{2}}\mathbf{x}}\Psi\right](t),\ t \in [0,\infty)$$

and

$$\left[T_{\mathbf{x}}\mathcal{R}\Psi\right](t) = \mathbf{x} + R\Psi(t) = \left[\mathcal{R}T_{\mathbf{R}^T\mathbf{x}}\Psi\right](t),\ t \in [0,\infty).$$

Hence,

$$T_{\mathbf{x}} \circ S_\alpha = S_\alpha \circ T_{\alpha^{\frac{1}{2}}\mathbf{x}} \tag{2.6}$$

and

$$T_{\mathbf{x}} \circ \mathcal{R} = \mathcal{R} \circ T_{\mathbf{R}^T\mathbf{x}}. \tag{2.7}$$

So, by Wiener Scaling invariance (see [2, p. 182]) [resp. Rotation invariant (see [2, Exercise 3.3.28])] together with (2.6) [resp. (2.7)] implies (2.4) [resp. (2.5)].

Now, in order to prove the first assertion, we see that

$$T_{\mathbf{x}} \circ T_{\mathbf{y}} = T_{\mathbf{y}} \circ T_{\mathbf{x}} = T_{\mathbf{x}+\mathbf{y}},$$

for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^N$. Hence,

$$\begin{aligned}\mathcal{W}^{(N)}_{\mathbf{x}+\mathbf{y}} = T_{\mathbf{x}+\mathbf{y}} * \mathcal{W}^{(N)} &= T_{\mathbf{x}} * \left(T_{\mathbf{y}} * \mathcal{W}^{(N)}\right) = T_{\mathbf{x}} * \mathcal{W}^{(N)}_{\mathbf{y}} \\ &= T_{\mathbf{y}} * \left(T_{\mathbf{x}} * \mathcal{W}^{(N)}\right) = T_{\mathbf{y}} * \mathcal{W}^{(N)}_{\mathbf{x}},\end{aligned}$$

which proves (2.3). Proposition 2.1 is thus established. □

In the next lemma, by $B(\mathbb{R}^N; R)$ we denote the space of bounded $\mathfrak{B}_{\mathbb{R}^N}$-measurable functions from $\mathbb{R}^N$ into R.

Lemma 2.2. *Let $f \in B(\mathbb{R}^N; R)$ be a given function. Then, for each $t \in [0, \infty)$ and any $\mathbf{x}, \mathbf{z} \in \mathbb{R}^N$,*

$$\int_{\mathfrak{B}(\mathbb{R}^N)} f(\Phi(t) + \mathbf{z}) \mathcal{W}^{(N)}_{\mathbf{x}}(d\Phi) = \int_{\mathbb{R}^N} f(\mathbf{y}) \gamma^{(N)}_t(\mathbf{y} - \mathbf{x} - \mathbf{z}) d\mathbf{y}, \ t \in [0, \infty).$$

Proof. Indeed, from (1.2) it implies that,

$$\mathcal{W}^{(N)}_{\mathbf{x}+\mathbf{z}}\Big(\{\Psi : \Psi(t) \in B\}\Big) = \int_B \gamma^{(N)}_t(\mathbf{y} - \mathbf{x} - \mathbf{z}) d\mathbf{y},$$

for every $B \in \mathfrak{B}_{\mathbb{R}^N}$. Hence, by (2.3) and the fact that $\mathcal{W}^{(N)}_{\mathbf{x}}$ is the Lesbesgue measure for $\mathfrak{B}(\mathbb{R}^N)$, we have

$$\begin{aligned}\int_{\mathfrak{B}(\mathbb{R}^N)} f(\Phi(t) + \mathbf{z}) \mathcal{W}^{(N)}_{\mathbf{x}}(d\Phi) &= \int_{\mathfrak{B}(\mathbb{R}^N)} f\Big(\big[T_{\mathbf{z}}\Phi\big](t)\Big) \mathcal{W}^{(N)}_{\mathbf{x}}(d\Phi) \\ &= \int_{\mathfrak{B}(\mathbb{R}^N)} f(\Psi(t)) \mathcal{W}^{(N)}_{\mathbf{x}+\mathbf{z}}(d\Psi) \\ &= \int_{\mathbb{R}^N} f(\mathbf{y}) \gamma^{(N)}_t(\mathbf{y} - \mathbf{x} - \mathbf{z}) d\mathbf{y},\end{aligned}$$

this completes the proof. □

We need the following well known notions.

Define, for each $t \in [0, \infty)$, the coordinate projections $\pi_t : \mathfrak{B}(\mathbb{R}^N) \longrightarrow \mathbb{R}^N$ by

$$\pi_t \Psi = \Psi(t), \tag{2.8}$$

and let

$$\mathfrak{B}_t^N = \sigma\left(\pi_s : s \in [0,t]\right),\ t \in [0,\infty) \tag{2.9}$$

be the σ-algebra over $\mathfrak{B}(\mathbb{R}^N)$ generated by all maps $\pi_s, s \in [0,t]$. Given a $\{\mathfrak{B}_t^N : t \in [0,\infty)\}$-stopping time τ, we will use the notation

$$\mathfrak{B}_\tau^N = \{A \subseteq \mathfrak{B}(\mathbb{R}^N) : A \cap \{\tau \le t\} \in \mathfrak{B}_t^N \ for\ all\ t \in [0,\infty)\}.$$

Then it is well known that, $\mathfrak{B}_\tau^N$ is always a sub σ-algebra of $\mathfrak{B}_{\mathfrak{B}(\mathbb{R}^N)}$ and τ itself is a $\mathfrak{B}_\tau^N$-measurable function (concerning this subject, see, for example, [2, Section 4.3] and [3, Chapter 2, Sections 4,5] for more information).

The following theorem extends a particular case of Theorem 4.3.8 in [2].

Theorem 2.3. *Let τ be a $\{\mathfrak{B}_t^N : t \in [0,\infty)\}$-stopping time and $F : \mathfrak{B}(\mathbb{R}^N) \longrightarrow R$ a bounded $\mathfrak{B}_\tau^N$-measurable function. Suppose that $\eta : (\tau < \infty) \longrightarrow [0,+\infty]$ is a $\mathfrak{B}_\tau^N$-measurable function. Then, for each $f \in B(\mathbb{R}^N;R)$, $\mathbf{x} \in \mathbb{R}^N$ and $h \in C(\mathbb{R}^N;\mathbb{R}^N)$, we have*

$$\begin{aligned}\int_{(\eta<\infty)} & F(\Psi) f\Big(\Psi(\tau+\eta) + h\big(\Psi(\tau)\big)\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi)\\ &= \int_{(\eta<\infty)} F(\Psi)\left(\int_{\mathbb{R}^N} f\Big(\mathbf{y} + \Psi(\tau) + h\big(\Psi(\tau)\big)\Big)\gamma_{\eta(\Psi)}^N(d\mathbf{y})\right)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi).\end{aligned} \tag{2.10}$$

(Here we use $\Psi(\tau)$ in place of $\Psi\big(\tau(\Psi)\big)$.)

Proof. Define, from the above assumptions, the function $H : \mathfrak{B}(\mathbb{R}^N) \times \mathfrak{B}(\mathbb{R}^N) \longrightarrow R$ by

$$H(\Phi,\Psi) = \mathcal{I}_{[0,\infty)}\big(\tau(\Phi)\big) \cdot \mathcal{I}_{[0,\infty)}\big(\eta(\Phi)\big) \cdot F(\Phi) f\Big(\Psi\big(\eta(\Phi)\big) + h\Big(\Phi\big(\tau(\Phi)\big)\Big)\Big),$$

where $\mathcal{I}_A$ denotes the characteristic function of a set A .

Then H is $\mathfrak{B}_\tau^N \times \mathfrak{B}_{\mathfrak{B}(\mathbb{R}^N)}$-measurable. Note that $(\eta < \infty) = (\tau < \infty) \cap (\eta < \infty)$, applying Theorem 4.3.8 in [2] and Lemma 2.2, we have

$$\begin{aligned}\int_{(\eta<\infty)} & F(\Psi) f\Big(\Psi\big(\tau(\Psi)+\eta(\Psi)\big) + h\big(\Psi(\tau)\big)\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi)\\ &= \int_{(\eta<\infty)} F(\Phi)\left(\int_{\mathfrak{B}(\mathbb{R}^N)} f\Big(\Psi\big(\eta(\Phi)\big) + h\big(\Phi(\tau)\big)\Big)\mathcal{W}_{\Phi(\tau)}^{(N)}(d\Psi)\right)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Phi)\\ &= \int_{(\eta<\infty)} F(\Phi)\left(\int_{\mathbb{R}^N} f(\mathbf{y})\gamma_{\eta(\Phi)}^N\Big(\mathbf{y} - \Phi(\tau) - h\big(\Phi(\tau)\big)\Big)d\mathbf{y}\right)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Phi),\end{aligned}$$

by change of variables of the integral in parentheses, we get (2.10) and the theorem is established. □

From the above theorem we obtain the following corollaries.

Corollary 2.4. *Let τ be a $\{\mathfrak{B}_t^N : t \in [0,\infty)\}$-stopping time and $\eta : (\tau < \infty) \longrightarrow [0,+\infty]$ is a $\mathfrak{B}_\tau^N$-measurable function. Then, for any $A \in \mathfrak{B}_\tau^N$ and $A \subset (\eta < \infty)$, $\mathbf{x} \in \mathbb{R}^N$ and* $B \in \mathfrak{B}_{\mathbb{R}^N}$, we have

$$\mathcal{W}_{\mathbf{x}}^{(N)}\Big(A \cap \{\Psi : \Psi(\tau+\eta) \in B\}\Big) = \int_A \mathcal{W}_{\Psi(\tau)}^{(N)}\Big(\{\Phi : \Phi(\eta(\Psi)) \in B\}\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi). \quad (2.11)$$

Proof. Setting $F(\Psi) = \mathcal{I}_A(\Psi), f(y) = \mathcal{I}_B(y)$ and $h \equiv 0$, by applying (2.10) and Lemma 2.2, we get

$$\begin{aligned}
&\mathcal{W}_{\mathbf{x}}^{(N)}\Big(A \cap \{\Psi : \Psi(\tau+\eta) \in B\}\Big) \\
&\quad = \int_{(\eta<\infty)} \mathcal{I}_A(\Psi)\Big(\int_{\mathbb{R}^N} \mathcal{I}_B\big(\mathbf{y}+\Psi(\tau)\big)\gamma_{\eta(\Psi)}^N(d\mathbf{y})\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi) \\
&\quad = \int_A \Big(\int_{\mathbb{R}^N} \mathcal{I}_B(\mathbf{y})\gamma_{\eta(\Psi)}^N\big(\mathbf{y}-\Psi(\tau)\big)d\mathbf{y}\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi) \\
&\quad = \int_A \Big(\int_{\mathfrak{B}(\mathbb{R}^N)} \mathcal{I}_B\Big(\Phi\big(\eta(\Psi)\big)\Big)\mathcal{W}_{\Psi(\tau)}^{(N)}(d\Phi)\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi).
\end{aligned}$$

Corollary 2.4 is thus completely proved. □

Corollary 2.5. *Let $\mathbf{R}$ be any orthogonal matrix of order N, define the transformation $h : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ by $h(\mathbf{y}) = \mathbf{R}^T(\mathbf{y} - \mathbf{R}\mathbf{y})$. Under the assumptions of Theorem 2.3, we have*

$$\begin{aligned}
&\int_{(\eta<\infty)} F(\Psi) f\Big(\Psi(\tau+\eta) + \mathbf{R}^T\Psi(\tau) - \Psi(\tau)\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi) \\
&\quad = \int_{(\eta<\infty)} F(\Psi)\Big(\int_{\mathbb{R}^N} f\big(\mathbf{y} + \mathbf{R}^T\Psi(\tau)\big)\gamma_{\eta(\Psi)}^N(d\mathbf{y})\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi),
\end{aligned} \quad (2.12)$$

for each $\mathbf{x} \in \mathbb{R}^N$.

Proof. It follows immediately from Theorem 2.3 and the linearity of the transformation h. □

3. Three applications

The above results can be used to study properties of the behavior of Wiener paths. As our first application about these, we give the following computation.

Example 3.1. The following notations will be used from now on:

$$\begin{aligned}
B_N(\mathbf{a}; r) &= \{\mathbf{y} \in \mathbb{R}^N : |\, \mathbf{y} - \mathbf{a} \,| < r\} \quad (3.1)\\
\overline{B}_N(\mathbf{a}; r) &= \{\mathbf{y} \in \mathbb{R}^N : |\, \mathbf{y} - \mathbf{a} \,| \le r\},
\end{aligned}$$

where $|\, \mathbf{z} \,|$ denotes the Euclidean norm of $\mathbf{z} \in \mathbb{R}^N$

for any $\mathbf{a} \in \mathbb{R}^N$ and $r > 0$. It is easy to see (see (2.8)) that $\pi_t^{-1}(\overline{B}_N(\mathbf{a}; r)) = \{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |\Psi(t) - \mathbf{a}| \le r\} \in \mathfrak{B}_{\mathfrak{B}(\mathbb{R}^N)}$. Identify $\Psi \in \mathfrak{B}(\mathbb{R}^N) \longleftrightarrow (\Psi_1, \cdots, \Psi_N) \in (\mathfrak{B}(R))^N$ by

$$\Psi(s) = (\Psi_1(s), \cdots, \Psi_N(s))^T, \quad s \in [0, \infty),$$

(see, for example, [1, p.16], [2, p.179]). Then, for each $\mathbf{x} = (x_1, \cdots, x_N)^T \in \mathbb{R}^N$, satisfying condition $\prod_{i=1}^N x_i \ne 0$, putting $r = \sqrt{N}\epsilon$ for any $\epsilon > 0$, by the independence of the coordinates under $\mathcal{W}_{\mathbf{x}}^{(N)}$ (see [2, Exercise 3.3.28]) we have, for any fixed $t \in [0, \infty)$,

$$\begin{aligned}
&\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |\Psi(t)| \le \sqrt{N}\epsilon\}\Big) \\
&\qquad \ge \mathcal{W}_{\mathbf{x}}^{(N)}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |\Psi_j(t)| \le \epsilon;\ for\ 1 \le j \le N\}\Big) \\
&\qquad = \prod_{i=1}^N \mathcal{W}_{x_i}\Big(\{\phi \in \mathfrak{B}(R) : |\phi(t)| \le \epsilon\}\Big).
\end{aligned}$$

Next, taking $\alpha_i = x_i^{-2}$, $1 \le i \le N$, use (2.4) and (2.5) to see that

$$\begin{aligned}
\prod_{i=1}^N \mathcal{W}_{x_i}\Big(\{\phi \in \mathfrak{B}(R) : |\phi(t)| \le \epsilon\}\Big) &= \prod_{i=1}^N \mathcal{W}_{sgn\, x_i}\Big(\{\phi \in \mathfrak{B}(R) : |\phi(x_i^{-2}t)| \le \epsilon |x_i|^{-1}\}\Big) \\
&= \prod_{i=1}^N \mathcal{W}_1\Big(\{\phi \in \mathfrak{B}(R) : |x_i\,\phi(x_i^{-2}t)| \le \epsilon\}\Big).
\end{aligned}$$

Hence, we obtain the following inequality

$$\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |\Psi(t)| \le \sqrt{N}\epsilon\}\Big) \ge \prod_{i=1}^N \mathcal{W}_1\Big(\{\phi \in \mathfrak{B}(R) : |x_i\,\phi(x_i^{-2}t)| \le \epsilon\}\Big),$$

for any $\epsilon > 0$, $t \in [0, \infty)$, and $\mathbf{x}$ is given in the above.

Remark. If putting $A_i = \{\phi \in \mathfrak{B}(R) : |x_i\,\phi(x_i^{-2}t)| \le \epsilon\}$, $1 \le i \le N$, then (see (1.3))

$$\begin{aligned}
\prod_{i=1}^N \mathcal{W}_1(A_i) &= \mathcal{W}_{(1,\cdots,1)^T}^{(N)}\Big(\prod_{i=1}^N A_i\Big) \\
&= \mathcal{W}_{(1,\cdots,1)^T}^{(N)}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |x_i\,\Psi_i(x_i^{-2}t)| \le \epsilon;\ for\ 1 \le i \le N\}\Big).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |\Psi(t)| \le \sqrt{N}\epsilon\}\Big) \\
&\qquad \ge \mathcal{W}_{(1,\cdots,1)^T}^{(N)}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : |\Psi_i(x_i^{-2}t)| \le \frac{\epsilon}{|x_i|};\ for\ 1 \le i \le N\}\Big).
\end{aligned}$$

The next applications deal with the behavior of Wiener paths over finite time intervals.

Example 3.2. For any $r \in (0,\infty)$, we consider a function $\tau^{(N)}_{\geq r}$ from $\mathfrak{B}(\mathbb{R}^N)$ into $[0, +\infty]$ which given by

$$\tau^{(N)}_{\geq r}(\Psi) = inf\ \{t \geq 0 : \mid \Psi(t) \mid \geq r\},\ \Psi \in \mathfrak{B}(\mathbb{R}^N). \tag{3.2}$$

Then $\tau^{(N)}_{\geq r}$ is a $\{\mathfrak{B}^N_t : t \in [0,\infty)\}$-stopping time. In this example, we will show that the Wiener paths satisfy the following properties.

(a) For each $x \in B_1(o;r)$ (see the notation (3.1)) and $T > 0$, then

$$\int_{(\tau^{(1)}_{\geq r} > T)} \cos\Big(\pi(\frac{\Psi(T)}{r} - \frac{1}{2}) + k\pi\Big)\mathcal{W}_x(d\Psi) = e^{-\frac{\pi^2}{2}\cdot\frac{T}{r^2}}\cos\Big(\pi(\frac{x}{r} - \frac{1}{2}) + k\pi\Big) \tag{3.3}$$

and

$$\int_{(\tau^{(1)}_{\geq r} > T)} \cos\Big(\frac{\pi}{2}\cdot\frac{\Psi(T)}{r} + k\pi\Big)\mathcal{W}_x(d\Psi) = e^{-\frac{\pi^2}{8}\cdot\frac{T}{r^2}}\cos\Big(\frac{\pi}{2}\cdot\frac{x}{r} + k\pi\Big), \tag{3.4}$$

for every $k \in Z$. Further,
(b) for any $\epsilon > 0$ and $\mathbf{x} \in B_N(0; \frac{\epsilon}{\sqrt{N}})$, then

$$\mathcal{W}^{(N)}_{\mathbf{x}}\Big(\tau^{(N)}_{\geq\epsilon} > T\Big) \geq e^{-\frac{\pi^2}{2}\cdot\frac{N^2T}{\epsilon^2}}\cos^N\Big(\pi\Big(\frac{\mid \mathbf{x} \mid}{\epsilon} - \frac{1}{2}\Big)\Big) \tag{3.5}$$

and

$$\mathcal{W}^{(N)}_{\mathbf{x}}\Big(\tau^{(N)}_{\geq\epsilon} > T\Big) \geq e^{-\frac{\pi^2}{8}\cdot\frac{N^2T}{\epsilon^2}}\cos^N\Big(\frac{\pi}{2}\cdot\frac{\mid \mathbf{x} \mid}{\epsilon}\Big). \tag{3.6}$$

In order to prove these assertions,we proceed in several steps.

Step 1. Using The same techniques of the proof of Theorem 7.2.4 in [2] with respect to the function

$$f(t,x) = e^{\frac{\pi^2}{2}\cdot\frac{t}{r^2}}\cos\Big(\pi\Big(\frac{x}{r} - \frac{1}{2}\Big) + k\pi\Big),\ (t,x) \in [0,\infty) \times R$$

$$\Big[resp.\ \ g(t,x) = e^{\frac{\pi^2}{8}\cdot\frac{t}{r^2}}\cos\Big(\frac{\pi}{2}\frac{x}{r} + k\pi\Big),\ (t,x) \in [0,\infty) \times R\Big]$$

we see that, the assertion (3.3) [resp. (3.4)] is obtained from the fact that $f\big(t\wedge\tau^{(1)}_{\geq r}, \pi_{t\wedge\tau^{(1)}_{\geq r}}\big)$ $\Big[resp.\ \ g\big(t\wedge\tau^{(1)}_{\geq r}, \pi_{t\wedge\tau^{(1)}_{\geq r}}\big)\Big]$ is $\mathcal{W}_x$-martingale relative to $\{\mathfrak{B}^1_t : t \in [0,\infty)\}$ (see the notations (2.8), (2.9)). Furthermore, by the independence of coordinates of Wiener's measure, we also have

$$\mathcal{W}^{(N)}_{\mathbf{x}}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : \sup_{t\in[0,T]} \mid \Psi(t) \mid < \epsilon\}\Big)$$
$$\geq \mathcal{W}^{(N)}_{\mathbf{x}}\Big(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : \sup_{t\in[0,T]} \mid \Psi_j(t) \mid < \frac{\epsilon}{\sqrt{N}};\ for\ 1 \leq j \leq N\}\Big)$$
$$= \prod_{i=1}^{N}\mathcal{W}_{x_i}\Big(\{\phi \in \mathfrak{B}(R) : \sup_{t\in[0,T]} \mid \phi(t) \mid < \frac{\epsilon}{\sqrt{N}}\}\Big), \tag{3.7}$$

for each $\mathbf{x} = (x_1, \cdots, x_N)^T \in \mathbb{R}^N$ and $\epsilon > 0$.

Now, choosing $\alpha = \epsilon^{-2} \cdot r^2 \cdot N$, again by (2.4),

$$\mathcal{W}_{x_i}\left(\{\phi \in \mathfrak{B}(R) : \sup_{t\in[0,T]} \mid \phi(t) \mid < \frac{\epsilon}{\sqrt{N}}\}\right)$$
$$= \mathcal{W}_{\frac{r\sqrt{N}}{\epsilon}\cdot x_i}\left(\{\phi \in \mathfrak{B}(R) : \sup_{t\in[0,T]} \mid \phi(\epsilon^{-2}\cdot r^2 \cdot N\,t) \mid < r\}\right)$$
$$= \mathcal{W}_{\frac{r\sqrt{N}}{\epsilon}\cdot x_i}\left(\{\phi \in \mathfrak{B}(R) : \sup_{t\in[0,\epsilon^{-2}\cdot r^2\cdot NT]} \mid \phi(t) \mid < r\}\right),$$

for each $\mathcal{W}_{x_i}$, $1 \le i \le N$. Hence, it shows from (3.7) that

$$\mathcal{W}^{(N)}_{\mathbf{x}}\left(\{\Psi \in \mathfrak{B}(\mathbb{R}^N) : \sup_{t\in[0,T]} \mid \Psi(t) \mid < \epsilon\}\right)$$
$$\ge \prod_{i=1}^{N} \mathcal{W}_{\frac{r\sqrt{N}}{\epsilon}\cdot x_i}\left(\{\phi \in \mathfrak{B}(R) : \sup_{t\in[0,\epsilon^{-2}\cdot r^2\cdot NT]} \mid \phi(t) \mid < r\}\right). \tag{3.8}$$

Step 2. Denote by R_0^+ the set of the non-negative real numbers. From (3.3) (with $k = 0$), we see that

$$\mathcal{W}_x\left(\tau^{(1)}_{\ge r} > T\right) \ge e^{-\frac{\pi^2}{2}\cdot\frac{T}{r^2}} \cos\left(\pi\left(\frac{x}{r} - \frac{1}{2}\right)\right) \ge 0, \tag{3.9}$$

for $x \in [0, r)$.

Taking $\mathbf{x} \in (R_0^+)^N \cap B_N(0; \frac{\epsilon}{\sqrt{N}})$, then $\frac{r\sqrt{N}}{\epsilon} \cdot x_i \in [0, r)$ (for each $i(1 \le i \le N)$), and therefore, applying (3.9) (with T replaced by $\epsilon^{-2} \cdot r^2 \cdot NT$), we get

$$\prod_{i=1}^{N} \mathcal{W}_{\frac{r\sqrt{N}}{\epsilon}\cdot x_i}\left(\tau^{(1)}_{\ge r} > \epsilon^{-2}\cdot r^2 \cdot NT\right) \ge e^{-\frac{\pi^2}{2}\cdot\frac{N^2T}{\epsilon^2}} \prod_{i=1}^{N} \cos\left(\pi\left(\frac{\sqrt{N}}{\epsilon}\cdot x_i - \frac{1}{2}\right)\right). \tag{3.10}$$

Hence, since $\mathcal{W}_{\frac{r\sqrt{N}}{\epsilon}\cdot x_i}\left(\{\phi(0) = \frac{r\sqrt{N}}{\epsilon}\cdot x_i\}\right) = 1$ and $\mathcal{W}^{(N)}_{\mathbf{x}}(\{\Psi(0) = \mathbf{x}\}) = 1$ (see (1.2)), (3.8)(3.10) plus the definition of the stopping-time (3.2) leads to

$$\mathcal{W}^{(N)}_{\mathbf{x}}\left(\tau^{(N)}_{\ge \epsilon} > T\right) \ge e^{-\frac{\pi^2}{2}\cdot\frac{N^2T}{\epsilon^2}} \prod_{i=1}^{N} \cos\left(\pi\left(\frac{\sqrt{N}}{\epsilon}\cdot x_i - \frac{1}{2}\right)\right). \tag{3.11}$$

Step 3. Next, for any fixed $\mathbf{x} \in B_N(0; \frac{\epsilon}{\sqrt{N}})$, putting $\mathbf{y} = (\frac{|\mathbf{x}|}{\sqrt{N}}, \cdots, \frac{|\mathbf{x}|}{\sqrt{N}})^T$. Then, there exists a Rotation $\mathcal{R}$ (relative to $\mathbf{R}$) on $\mathfrak{B}(\mathbb{R}^N)$ in which the orthogonal matrix $\mathbf{R}$ satisfied the condition $\mathbf{Ry} = \mathbf{x}$. Thus, by (2.5), we conclude from (3.11) that

$$\begin{aligned}
\mathcal{W}_{\mathbf{x}}^{(N)}\left(\tau_{\geq\epsilon}^{(N)} > T\right) &= \mathcal{R} * \mathcal{W}_{\mathbf{y}}^{(N)}\left(\tau_{\geq\epsilon}^{(N)} > T\right) \\
&= \mathcal{W}_{\mathbf{y}}^{(N)}\left(\left\{\Psi : \tau_{\geq\epsilon}^{(N)}(\mathcal{R}\Psi) > T\right\}\right) \\
&= \mathcal{W}_{\mathbf{y}}^{(N)}\left(\left\{\Psi : \tau_{\geq\epsilon}^{(N)}(\Psi) > T\right\}\right) \\
&\geq e^{-\frac{\pi^2}{2}\cdot\frac{N^2 T}{\epsilon^2}} \cos^N\left(\pi\left(\frac{|\mathbf{x}|}{\epsilon} - \frac{1}{2}\right)\right).
\end{aligned}$$

Finally, by the same argument as above, we also obtain (3.6) and this example is completely established.

Remark. We use $E^P[X, A]$ to denote the expected value under P of X over the set A. Taking $x = \frac{r}{2}$ in (3.3) and thereby obtain

$$\int_{\left(\tau_{\geq r}^{(1)} > T\right)} \cos\left(\pi\left(\frac{\Psi(T)}{r} - \frac{1}{2}\right) + k\pi\right)\mathcal{W}_{\frac{r}{2}}(d\Psi) = (-1)^k e^{-\frac{\pi^2}{2}\frac{T}{r^2}}.$$

Thus,

$$\begin{aligned}
&E^{\mathcal{W}_x}\left[\cos\left(\pi\left(\frac{\pi_T}{r} - \frac{1}{2}\right) + k\pi\right),\ \tau_{\geq r}^{(1)} > T\right] \\
&\quad = (-1)^k \cos\left(\pi\left(\frac{x}{r} - \frac{1}{2}\right) + k\pi\right) E^{\mathcal{W}_{\frac{r}{2}}}\left[\cos\left(\pi\left(\frac{\pi_T}{r} - \frac{1}{2}\right) + k\pi\right),\ \tau_{\geq r}^{(1)} > T\right],
\end{aligned}$$

for any $r > 0,\ T > 0,\ k \in Z$ and $x \in B_1(0; r)$.

Similarly, by (3.4) (with $x = 0$), we have

$$\begin{aligned}
&E^{\mathcal{W}_x}\left[\cos\left(\frac{\pi}{2}\cdot\frac{\pi_T}{r} + k\pi\right),\ \tau_{\geq r}^{(1)} > T\right] \\
&\quad = (-1)^k \cos\left(\frac{\pi}{2}\cdot\frac{x}{r} + k\pi\right) E^{\mathcal{W}}\left[\cos\left(\frac{\pi}{2}\cdot\frac{\pi_T}{r} + k\pi\right),\ \tau_{\geq r}^{(1)} > T\right],
\end{aligned}$$

for any $r > 0,\ T > 0,\ k \in Z$ and $x \in B_1(0; r)$.

Remark. One could define the sub sets of $B_N(0; \frac{\epsilon}{\sqrt{N}})$ under which we will obtain the better estimates than the inequalities of (3.5) and (3.6). Namely, letting $\mathbf{x} \in B_N(0; \frac{\epsilon}{\sqrt{N}})$, for each $\epsilon > 0$, then

$$0 \leq \frac{\pi}{2}\cdot\frac{|\mathbf{x}|}{\epsilon} < \frac{\pi}{2} \quad and \quad -\frac{\pi}{2} \leq \pi\frac{|\mathbf{x}|}{\epsilon} - \frac{\pi}{2} < \frac{\pi}{2}.$$

This implies that,

$$\begin{cases} \cos\left(\pi\left(\frac{|\mathbf{x}|}{\epsilon}-\frac{1}{2}\right)\right)=1, \\ \mathbf{x}\in B_N(0;\ \frac{\epsilon}{\sqrt{N}}) \end{cases} \iff \begin{cases} \mathbf{x}\in \delta B_N(0;\ \frac{\epsilon}{2}), \\ N=1,2,3 \end{cases} \tag{3.12}$$

where $\delta B_N(0;\ \frac{\epsilon}{2})$ is the boundary of $B_N(0;\ \frac{\epsilon}{2})$

and

$$\begin{cases} \cos(\frac{\pi}{2}\cdot\frac{|\mathbf{x}|}{\epsilon})=1, \\ \mathbf{x}\in B_N(0;\ \frac{\epsilon}{\sqrt{N}}) \end{cases} \iff \mathbf{x}\equiv 0. \tag{3.13}$$

Then, it shows from (3.5)(3.12) that

$$\mathcal{W}_{\mathbf{x}}^{(N)}\left(\tau_{\geq\epsilon}^{(N)}>T\right)\geq e^{-\frac{\pi^2}{2}\cdot\frac{N^2T}{\epsilon^2}},$$

for $N=1,2,3;\ \epsilon>0,\ T>0$ and $\mathbf{x}\in\delta B_N(0;\ \frac{\epsilon}{2})$.

Similarly, by (3.6)(3.13), we obtain the certain result of Theorem 7.2.4 in [2],

$$\mathcal{W}^{(N)}\left(\tau_{\geq\epsilon}^{(N)}>T\right)\geq e^{-\frac{\pi^2}{8}\cdot\frac{N^2T}{\epsilon^2}},\ \epsilon>0,\ T>0.$$

Example 3.3. Let us consider two $\{\mathfrak{B}_t^N:\ t\in[0,\infty)\}$-stopping times as follows

$$\sigma(\Psi)=\inf\{s\geq 0: |\ \Psi(s)\ |\leq\frac{r}{2}\} \tag{3.14}$$

and

$$\tau(\Psi)=\inf\{s\geq\sigma(\Psi): |\ \Psi(s)\ |\geq r\}, \tag{3.15}$$

for each $r>0$ and every $\Psi\in\mathfrak{B}(\mathbb{R}^N)$. Let $t\in(0,\infty)$ be fixed and $\mathbf{x}\in B_N(0;r)$. Define $\eta:\ (\tau<+\infty)\longrightarrow[0,\ +\infty]$ by

$$\eta(\Psi)=(t-\tau(\Psi))\vee 0. \tag{3.16}$$

Then η is a $\mathfrak{B}_\tau^N$-measure function. Furthermore, taking $A=(\tau\leq t)$, it is obvious that $A\subset(\eta<\infty)$. The following notations will be used:

$$\mathbf{R}B=\{\mathbf{R}\mathbf{y}:\ \mathbf{y}\in B\}\ \textit{and}\ B+\mathbf{z}=\{\mathbf{y}+\mathbf{z}:\ \mathbf{y}\in B\},\mathbf{z}\in\mathbb{R}^N,\ B\in\mathfrak{B}_{\mathbb{R}^N}.$$

In this example, we shall prove that the Wiener paths satisfy the following properties.

(a) For any orthogonal matrix $\mathbf{R}$ *be given, and every* $B\in\mathfrak{B}_{\mathbb{R}^N}$.

$$\begin{aligned}\mathcal{W}_{\mathbf{x}}^{(N)}&\left(\left\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\leq t,\ \Psi(t)\in B\right\}\right)\\ &=\mathcal{W}_{\mathbf{x}}^{(N)}\left(\left\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\leq t,\ \Psi(t)\in\mathbf{R}^T(B-\Psi(\tau))+\Psi(\tau)\right\}\right),\end{aligned} \tag{3.17}$$

(b)

$$\mathcal{W}_{\mathbf{x}}^{(N)}\left(\left\{\Psi \in \mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi) \leq t,\ \mid \Psi(t) \mid < r\right\}\right) \\ \leq \frac{\mathcal{W}_{\mathbf{x}}^{(N)}(\tau \leq t)}{2} \leq \mathcal{W}_{\mathbf{x}}^{(N)}\left(\left\{\Psi \in \mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi) \leq t,\ \mid \Psi(t) \mid \geq r\right\}\right) \tag{3.18}$$

and

$$\mathcal{W}_{\mathbf{x}}^{(N)}(\tau \leq t) = \mathcal{W}_{\mathbf{x}}^{(N)}\left(\left\{\Psi \in \mathfrak{B}(\mathbb{R}^N):\ \sigma(\Psi) \leq t, \mid 2\Psi(\tau) - \Psi(t) \mid < r\right\}\right) \\ + \mathcal{W}_{\mathbf{x}}^{(N)}\left(\left\{\Psi \in \mathfrak{B}(\mathbb{R}^N):\ \sigma(\Psi) \leq t, \mid \Psi(t) \mid \geq r\right\}\right). \tag{3.19}$$

Indeed, first to prove the assertion (a), from (3.14)-(3.16) applying Corollary 2.4, we have

$$\mathcal{W}_{\mathbf{x}}^{(N)}\Big(A \cap \{\Psi : \Psi(t) \in B\}\Big) = \mathcal{W}_{\mathbf{x}}^{(N)}\left(\left\{\Psi :\ \tau(\Psi) \leq t,\ \Psi(t) \in B\right\}\right) \\ = \int_{(\tau \leq t)} \mathcal{W}_{\Psi(\tau)}^{(N)}\left(\left\{\Phi : \Phi\big(t - \tau(\Psi)\big) \in B\right\}\right)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi). \tag{3.20}$$

Moreover, for each $\Psi \in (\tau \leq t)$, again by (2.5) we see that

$$\mathcal{W}_{\Psi(\tau)}^{(N)}\left(\left\{\Phi : \Phi\big(t - \tau(\Psi)\big) \in B\right\}\right) = \mathcal{R} * \mathcal{W}_{\mathbf{R}^T\Psi(\tau)}^{(N)}\left(\left\{\Phi : \Phi\big(t - \tau(\Psi)\big) \in B\right\}\right) \\ = \mathcal{W}_{\mathbf{R}^T\Psi(\tau)}^{(N)}\left(\left\{\Phi : \mathbf{R}\Phi\big(t - \tau(\Psi)\big) \in B\right\}\right) \\ = \mathcal{W}_{\mathbf{R}^T\Psi(\tau)}^{(N)}\left(\left\{\Phi : \Phi\big(t - \tau(\Psi)\big) \in \mathbf{R}^T B\right\}\right) \tag{3.21}$$

for any Rotation $\mathcal{R}$ (relative to $\mathbf{R}$) be given. Next, apply Corollary 2.5 with respect to $F(\Psi) = \mathcal{I}_A(\Psi)$ and $f(\mathbf{y}) = \mathcal{I}_{\mathbf{R}^T B}(\mathbf{y})$ to see that

$$\int_{(\eta<\infty)} \mathcal{I}_A(\Psi)\mathcal{I}_{\mathbf{R}^T B}\Big(\Psi(\tau+\eta) + \mathbf{R}^T\Psi(\tau) - \Psi(\tau)\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi) \\ = \int_{(\eta<\infty)} \mathcal{I}_A(\Psi)\left(\int_{\mathbb{R}^N} \mathcal{I}_{\mathbf{R}^T B}\big(\mathbf{y} + \mathbf{R}^T\Psi(\tau)\big)\gamma_{\eta(\Psi)}^N(d\mathbf{y})\right)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi).$$

Using Lemma 2.2, the above relation becomes

$$\mathcal{W}_{\mathbf{x}}^{(N)}\Big(A\cap\Big\{\Psi:\Psi(t)+\mathbf{R}^T\Psi(\tau)-\Psi(\tau)\in\mathbf{R}^TB\Big\}\Big)$$
$$=\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ \Psi(t)\in\mathbf{R}^T(B-\Psi(\tau))+\Psi(\tau)\Big\}\Big)$$
$$=\int_{(\tau\le t)}\Big(\int_{\mathbb{R}^N}\mathcal{I}_{\mathbf{R}^TB}(\mathbf{y})\gamma^N_{\eta(\Psi)}(\mathbf{y}-\mathbf{R}^T\Psi(\tau))d\mathbf{y}\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi)$$
$$=\int_{(\tau\le t)}\Big(\int_{\mathfrak{B}(\mathbb{R}^N)}\mathcal{I}_{\mathbf{R}^TB}\Big(\Phi(\eta(\Psi))\Big)\mathcal{W}^{(N)}_{\mathbf{R}^T\Psi(\tau)}(d\Phi)\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi)$$
$$=\int_{(\tau\le t)}\mathcal{W}^{(N)}_{\mathbf{R}^T\Psi(\tau)}\Big(\Big\{\Phi:\Phi(t-\tau(\Psi))\in\mathbf{R}^TB\Big\}\Big)\mathcal{W}_{\mathbf{x}}^{(N)}(d\Psi).\qquad(3.22)$$

Hence, by combinate (3.20),(3.21) and (3.22) we obtain (3.17).

Now, in particular, taking $B=B_N(0;r)$ and choosing $\mathbf{R}=-I_N$ (with I_N is the unit matrix of order to N), from (3.17) it implies that

$$\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ \Psi(t)\ |<r\Big\}\Big)$$
$$=\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ \Psi(t)\in 2\Psi(\tau)-B\Big\}\Big)$$
$$=\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ 2\Psi(\tau)-\Psi(t)\ |<r\Big\}\Big).\qquad(3.23)$$

Furthermore,

$$\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ 2\Psi(\tau)-\Psi(t)\ |<r\Big\}\subset\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ \Psi(t)\ |\ge r\Big\}\qquad(3.24)$$

and by the definitions of τ and σ, we have

$$\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ \Psi(t)\ |\ge r\Big\}=\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \sigma(\Psi)\le t,\ |\ \Psi(t)\ |\ge r\Big\},\qquad(3.25)$$

and

$$\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ 2\Psi(\tau)-\Psi(t)\ |<r\Big\}$$
$$=\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \sigma(\Psi)\le t,\ |\ 2\Psi(\tau)-\Psi(t)\ |<r\Big\}.\qquad(3.26)$$

Besides that,

$$\mathcal{W}_{\mathbf{x}}^{(N)}(\tau\le t)=\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ \Psi(t)\ |<r\Big\}\Big)$$
$$+\mathcal{W}_{\mathbf{x}}^{(N)}\Big(\Big\{\Psi\in\mathfrak{B}(\mathbb{R}^N):\ \tau(\Psi)\le t,\ |\ \Psi(t)\ |\ge r\Big\}\Big),\qquad(3.27)$$

this together with (3.23) and (3.24) it implies (3.18). Finally, by (3.23), (3.25), (3.26), (3.27) we also get (3.19). The example is complete.

References

[1] D. W. Stroock, *Lectures on Stochstic Analysis: Diffusion Theory*,Cambridge University Press, 1987.

[2] D. W. Stroock, *Probability Theory, an Analytic View*, Cambridge University Press, 1993.

[3] N. V. Krylov, *Introduction to the Theory of Diffusion Processes*, American Mathematical Society, Providence, Rhode Island, 1995.