THE CONTROLLABILITY OF DEGENERATE SYSTEM DESCRIBED BY RIGHT INVERTIBLE OPERATORS

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Abstract. *The controllability of a linear system described by right invertible operators was studied by many authors. However, for the degenerate system, the problem has not been so far considered. In this paper, the controllability of these systems is studied.*

0. Introduction

The theory of right invertible operators was started in 1972 with works of D. Przeworska- Rolewicz and then has been developed by M. Tasche, H. von Trotha, Z. Binderman and many other Mathematicians (see [7]). With the appearing of this theory, the initial, boundary and mixed boundary value problems have been considered. Since 1977- 1978, Nguyen Dinh Quyet, in series of articles, has introduced the controllability of linear systems described by right invertible operators in the case of a resolving operator being invertible (see[2, 3]). The results related to the controllability of linear systems were generalized by Pogorzelec for the case of one-sized invertible resolving operarors. In 1992, Nguyen Van Mau, in his study, introduced the controllability of general system and studied the controllability of linear systems described by generalized invertible operators (see [5]). However, for the degenerate systems, the problem has not been so far investigated. In this paper, the controllability of the degenerate system described by right invertible operators is studied.

1. Some fundamental notions

Let X be a linear space over a field $\mathcal{F}$ of scalars ($\mathcal{F} = \mathbb{R}$ or $\mathbb{C}$).

Denote by $L(X)$ the set of all linear operators with domains and ranges contained in X and $L_0(X) = \{A \in L(X) : \text{dom}A = X\}$.

Definition 1.1. [7] *An operator $D \in L(X)$ is said to be a right invertible operator if there is an operator $R \in L_0(X)$ such that $\text{Im}R \subset \text{dom}D$ and*

$$DR = I, \tag{1.1}$$

where I is identity operator. In this case, R is called a right inverse operator of D. The set of all right invertible operators belonging to $L(X)$ will be denoted by $R(X)$. If $D \in R(X)$, we denote $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$.

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Proposition 1.2. [7] *If $D \in R(X)$, then for every $R \in \mathcal{R}_D$ we have*

$$\mathrm{dom}D = RX \oplus \ker D. \tag{1.2}$$

Definition 1.2. [7] *An operator $F \in L_0(X)$ is said to be an initial operator for D corresponding to $R \in \mathcal{R}_D$ if $F^2 = F, FX = \ker D$ and $FR = 0$ on $\mathrm{dom}R$. The set of all initial operators for D will be denoted by $\mathcal{F}_D$.*

Definition 1.3. [7] *Suppose that $D \in R(X)$ and $R \in \mathcal{R}_D$. An operator $A \in L_0(X)$ is said to be stationary if $DA - AD = 0$ on $\mathrm{dom}D$ and $RA - AR = 0$.*

Theorem 1.1. [7] *Suppose that $D \in R(X)$. A necessary and sufficient condition for an operator $F \in L_0(X)$ to be an initial operator for D corresponding to $R \in \mathcal{R}_D$ is that*

$$F = I - RD. \tag{1.3}$$

Definition 1.4. [7] *An operator $V \in L_0(X)$ is said to be a left invertible operator if there is an operator $L \in L(X)$ such that $\mathrm{Im}V \subset \mathrm{dom}L, LV = I$. We denote $\Lambda(X)$ the set of all left invertible operators belonging to $L(X)$ and by $\mathcal{L}_V$ the set of all left inverses of $V \in \Lambda(X)$.*

Definition 1.5. [5] *An operator $V \in L(X)$ is said to be generalized invertible if there is an operator $W \in L(X)$ (called a generalized inverse of V) such that:*

$$\mathrm{Im}V \subset \mathrm{dom}W\ ,\ \mathrm{Im}W \subset \mathrm{dom}V \text{ and } VWV = V \text{ on } \mathrm{dom}V.$$

The set of all generalized invertible operators in $L(X)$ will be denoted by $W(X)$. If $V \in W(X)$, we denote by $\mathcal{W}_V$ the set of all generalized inverses of V.

Proposition 1.2. [5] *Suppose that $V \in W(X)$ and $W \in \mathcal{W}_V$, then*

$$\mathrm{dom}V = WV(\mathrm{dom}V) \oplus \ker V \tag{1.4}$$

Theorem 1.2. [5] *Suppose that $A, B \in L(X), \mathrm{Im}A \subset \mathrm{dom}B$ and $\mathrm{Im}B \subset \mathrm{dom}A$, then $I - AB$ is right invertible (left invertible, invertible, generalized invertible) if and only if so is $I - BA$. Moreover, if we denote by $R_{AB}(L_{AB}, W_{AB})$ a right inverse (left inverse, generalized inverse) of $I - AB$, then there exists $R_{BA} \in \mathcal{R}_{I-BA}(L_{BA} \in \mathcal{L}_{I-BA}, W_{BA} \in \mathcal{W}_{I-BA}$, respectively such that:*

(i) $R_{AB} = I + AR_{BA}B, \quad R_{BA} = I + BR_{AB}A$,
(ii) $L_{AB} = I + AL_{BA}B, \quad L_{BA} = I + BL_{AB}A$,
(iii) $(I - AB)^{-1} = I + A(I - BA)^{-1}B, \quad (I - BA)^{-1} = I + B(I - AB)^{-1}A$,
(iv) $W_{AB} = I + AW_{BA}B, \quad W_{BA} = I + BW_{AB}A$.

The theory of right invertible operators and thier applications can be seen in [5, 7].

2. Degenerate systems

Definition 2.1. *Suppose that $D \in R(X)$, $\dim\ker D \neq 0$ and $A, B \in L_0(X)$, with $A \neq 0$ non-invertible. Then a linear system of the form*

$$ADx = Bx + y \quad , \quad y \in X \tag{2.1}$$

is said to be a degenerate system.

Proposition 2.1. *Suppose that $D \in R(X)$, $\dim\ker D \neq 0$; F is an initial operator for D corresponding to $R \in \mathcal{R}_D$ and $A, B \in L_0(X)$, with $A \neq 0$ non-invertible. Then the following identities hold on $\mathrm{dom}D$:*

$$AD - B = D\{I - R[(I - A)D + B]\}, \tag{2.2}$$
$$(AD - B)R = A - BR, \tag{2.3}$$
$$AD - B = (A - BR)D - BF. \tag{2.4}$$

Proof. (i) On $\mathrm{dom}D$ we have

$$D\{I - R[(I - A)D + B]\} = D - DR[(I - A)D + B] = D - (I - A)D - B$$
$$= D - D + AD - B = AD - B.$$

The proofs of (2.3) and (2.4) are completely similar.

Proposition 2.2. *Suppose that all the assumptions of Proposition 2.1 are satisfied. If $A - BR$ is right invertible (left invertible, invertible, generalized invertible), then so is $I - R[(I - A)D + B]$. Moreover, if $R_{AB}(L_{AB}, (A - BR)^{-1}, W_{A,B})$ is right inverse (left inverse, inverse, generalized inverse) of $A - BR$, then there exists $R_0 \in \mathcal{R}_{I-R[(I-A)D+B]}(L_0 \in \mathcal{L}_{I-R[(I-A)D+B]}, W_0 \in \mathcal{W}_{I-R[(I-A)D+B]}$ respectively, such that:*

(i) $R_0 = I + RR_{AB}[(I - A)D + B]$,
(ii) $L_0 = I + RL_{AB}[(I - A)D + B]$,
(iii) $\{I - R[(I - A)D + B]\}^{-1} = I + R(A - BR)^{-1}[(I - A)D + B]$,
(iv) $W_0 = I + RW_{A,B}[(I - A)D + B]$.

Proof. We will prove the case (iv). We have $I - [(I - A)D + B]R = A - BR$. Suppose that $A - BR \in W(X)$ and $W_{A,B} \in \mathcal{W}_{A-BR}$. Then $I - R[(I - A)D + B] \in W(X)$ (by Theorem 1.2). Moreover, there exists $W_0 = I + RW_{A,B}[(I - A)D + B]$ is a generalized inverse of $I - R[(I - A)D + B]$.

An operator $A - BR$ is said to be *a resolving operator* for the system (2.1), if $A - BR$ is invertible, then the system (2.1) is said to be well-determined. Otherwise, it is ill-determined.

Theorem 2.1. *Suppose that all assumptions of Proposition 2.1 are satisfied. Then, we have:*

(i) If $A - BR \in R(X)$ and $R_{AB} \in \mathcal{R}_{A-BR}$, then all solutions of the system (2.1) are given by

$$x = \{I + RR_{AB}[(I - A)D + B]\}(Ry + z) + \tilde{z}, \tag{2.5}$$

where $z \in \ker D, \tilde{z} \in \ker\{I - R[(I-A)D + B]\}$,

(ii) *If $A - BR \in \Lambda(X)$ and $L_{AB} \in \mathcal{L}_{A-BR}$, then all solutions of the system (2.1) are given by*

$$x = \{I + RL_{AB}[(I-A)D + B]\}(Ry + z)\,,\ z \in \ker D, \tag{2.6}$$

(iii) *If $A - BR$ is invertible, then all solutions of the system (2.1) are given by*

$$x = \{I + R(A - BR)^{-1}[(I-A)D + B]\}(Ry + z)\,,\ z \in \ker D, \tag{2.7}$$

(iv) *If $A - BR \in W(X)$ and $W_{A,B} \in \mathcal{W}_{A-BR}$, then all solutions of the system (2.1) are given by*

$$x = \{I + RW_{A,B}[(I-A)D + B]\}(Ry + z) + \tilde{z}\,, \tag{2.8}$$

where $z \in \ker D\,,\ \tilde{z} \in \ker\{I - R[(I-A)D + B]\}$.

Proof. Since both one-sided invertible and invertible operators are generalized invertible, it is sufficient to consider the case (iv). According to equality the (2.2) in Proposition 2.1, we see that (2.1) is equivalent to $D\{I - R[(I-A)D + B]\}x = y$. Hence,

$$\{I - R[(I-A)D + B]\}x = Ry + z\,,\ z \in \ker D, \tag{2.9}$$

By the assumption, $A - BR \in W(X)$ and $W_{A,B} \in \mathcal{W}_{A-BR}$. Thus, Proposition 2.2 implies that $I - R[(I-A)D + B] \in W(X)$ and there exists a generalized invertible operator $W_0 = I + RW_{A,B}[(I-A)D + B]$. Therefore, (2.9) obtaines that all solutions of (2.1) are given by $x = \{I + RW_{A,B}[(I-A)D + B]\}(Ry + z) + \tilde{z}$.

3. The initial value problem

Suppose that $D \in R(X)$, $\dim\ker D \neq 0$; F is an initial operator for D corresponding to $R \in \mathcal{R}_D$; and $A, B \in L_0(X)$, with $A \neq 0$ non-invertible. In this section, we consider the initial value problem for degenerate system $(DS)_0$ of the form:

$$ADx = Bx + y\ ,\quad y \in X \tag{3.1}$$

$$Fx = x_0\ ,\quad x_0 \in \ker D. \tag{3.2}$$

Theorem 3.1. *Suppose that all the assumptions of Proposition 2.1 are satisfied and $Ry + x_0 \in \{I - R[(I-A)D + B]\}domD$. Then, we have:*

(i) *If $A - BR \in R(X)$ and $R_{AB} \in \mathcal{R}_{A-BR}$, then all solutions of the problem (3.1)-(3.2) are given by*

$$x = \{I + RR_{AB}[(I-A)D + B]\}(Ry + x_0) + \tilde{z}\,, \tag{3.3}$$

where $\tilde{z} \in \ker\{I - R[(I-A)D + B]\}$.

(ii) *If $A - BR \in \Lambda(X)$ and $L_{AB} \in \mathcal{L}_{A-BR}$, then all solutions of the problem (3.1)-(3.2) are given by*

$$x = \{I + RL_{AB}[(I-A)D + B]\}(Ry + x_0). \tag{3.4}$$

(iii) *If $A - BR$ is invertible, then solution of the problem (3.1)-(3.2) are given by*

$$x = \{I + R(A - BR)^{-1}[(I-A)D + B]\}(Ry + x_0). \tag{3.5}$$

(iv) *If $A - BR \in W(X)$ and $W_{A,B} \in \mathcal{W}_{A-BR}$, then all solutions of the problem (3.1)-(3.2) are given by*

$$x = \{I + RW_{A,B}[(I - A)D + B]\}(Ry + x_0) + \tilde{z} , \tag{3.6}$$

l where $\tilde{z} \in \ker\{I - R[(I - A)D + B]\}$.

Proof. According to the proof of Theorem 2.1, from (3.1) we have

$$\{I - R[(I - A)D + B]\}x = Ry + z , \; z \in \ker D. \tag{3.7}$$

Thus, acting on both sides of this equality by operator F , we find that $Fx - FR[(I - A)D + B]x = FRy + Fz$. Hence $x_0 = Fx = Fz = z$. Therefore,

$$\{I - R[(I - A)D + B]\}x = Ry + x_0 . \tag{3.8}$$

By our assumption, $A - BR \in W(X)$ implies that $I - R[(I - A)D + B] \in W(X)$ and there exists it's generalized inverse $W_0 = I + RW_{A,B}[(I - A)D + B]$, with condition $Ry + x_0 \in \{I - R[(I - A)D + B]\}\text{dom}D$, we have all solutions of the problem (3.1)-(3.2) is given by

$$x = \{I + RW_{A,B}[(I - A)D + B]\}(Ry + x_0) + \tilde{z} , \; \tilde{z} \in \ker\{I - R[(I - A)D + B]\}.$$

Theorem 3.2. *Suppose that A, B are stationary operators and $A - BR$ is invertible. Then, the initial value problem (3.1)-(3.2) has a unique solution*

$$x = (A - BR)^{-1}(Ry + x_0) . \tag{3.9}$$

Proof. By the assumption A, B are stationary operators, $AD - B = D(A - BR)$ and (3.1) becomes $D(A - BR)x = y$, this implies that $(A - BR)x = Ry + z$, $z \in \ker D$. The condition (3.2) finds that $z = x_0$. Moreover, $A - BR$ is invertible. Thus, the solution of the problem (3.1)-(3.2) is unique and given by

$$x = (A - BR)^{-1}(Ry + x_0) .$$

Example 3.1. *Suppose that X is the space (s) of all real sequences $\{x_n\}, n = 0, 1, 2, \cdots$ with addition and multiplication by scalars defined as follows: If $x = \{x_n\}$, $y = \{y_n\}, \lambda \in \mathbb{R}$ then $x + y = \{x_n + y_n\}, \lambda x = \{\lambda x_n\}, n = 0, 1, 2, \cdots$ Write $D\{x_n\} = \{x_{n+1} - x_n\}$, $R\{x_n\} = \{y_n\}$, $y_0 = 0$, $y_n = \sum_{k=0}^{n-1} x_k, n \geqslant 1$ and $F\{x_n\} = \{z_n\}, z_n = x_0, n = 0, 1, 2, \cdots$ It is easy to verify that $D \in R(X), R \in \mathcal{R}_D$, F is an initial operator for D corresponding to R and $\ker D = \{z = \{z_n\}, z_n = c, n \in \mathbb{N}, c \in \mathbb{R}\}$. Consider the degenerate system $(DS)_0$ of the form:*

$$ADx = Bx + y \quad , \quad y \in X \tag{3.10}$$

$$Fx = \overline{x}_0 \quad , \quad \overline{x}_0 \in \ker D, \tag{3.11}$$

where A, B are defined by $A\{x_n\} = \{x_{n+1}\}$, $B\{x_n\} = \{x_{n+2} - x_n\}$ and $y = \{y_n\} \in X$, $\overline{x}_0 = \{x_0\} \in \ker D$ are given. We conclude that $A \neq 0$ is non-invertible, the resolving operator $A - BR$ is invertible, $A - BR = (A - BR)^{-1} = -I$. By Theorem 3.1, the solution of the system $(DS)_0$ is of the form:

$$\begin{aligned} x &= \{I + R(A - BR)^{-1}[(I - A)D + B]\}(Ry + \overline{x}_0) \\ &= \{I - R[(I - A)D + B]\}(Ry + \overline{x}_0) \\ &= \{x_0, x_0 - y_0, x_0 - y_0 - y_1, x_0 - y_0 - y_1 - y_2, \cdots\}. \end{aligned}$$

Example 3.2. *Suppose that X, D, R and F are defined as in Example 3.1; Write* $A\{x_n\} = \{2x_0+x_1, 0, x_2+x_3, x_3+x_4, \cdots\}$, $B\{x_n\} = \{x_2-x_0, 0, x_4-x_2, x_5-x_3, \cdots\}$. *Clear,* $A \neq 0$ *and is non-invertible, since* $\ker A = \{x_0, -2x_0, x_2, -x_2, x_2, -x_2, \cdots\} \neq \{0\}$ *and* $AX \neq X$
Let $y = \{y_n\} \in X$ *and* $\overline{x}_0 = \{x_0\} \in \ker D$.

Now we consider the degenerate system $(DS)_0$ *of the form:*

$$ADx = Bx + y \tag{3.12}$$

$$Fx = \overline{x}_0 \quad , \qquad \overline{x}_0 \in \ker D \tag{3.13}$$

It is easy to verify that the resolving operator $A - BR$ *is generalized invertible. Indeed,* $(A - BR)I(A - BR)\{x_n\} = (A - BR)\{x_n\}$, *i.e.* $A - BR \in W(X)$ *and* $I \in \mathcal{W}_{A-BR}$. *Moreover,* $\ker\{I - R[(I - A)D + B]\} = \{\{0, 0, x_2, x_3, x_4, \cdots\}, x_n \in \mathbb{R}, n = 2, 3, 4, \cdots\}$. *By (3.6) , the solution of the problem* $(DS)_0$ *is given by*

$$\begin{aligned} x &= \{I + R[(I - A)D + B]\}(Ry + \overline{x}_0) + \tilde{z} \ , \ \tilde{z} \in \ker\{I - R[(I - A)D + B]\} \\ &= \{x_0, x_0 + y_0, x_0 + y_0 + 2y_1 + \tilde{x}_2, x_0 + y_0 + 2y_1 + 2y_2 + \tilde{x}_3, \cdots\}. \end{aligned}$$

4. Controllability of the degenerate system

Let X and U be linear spaces over the same field $\mathcal{F}(\mathcal{F} = \mathbb{R}$ or $\mathbb{C})$. Suppose that $D \in R(X)$, $\dim\ker D \neq 0$; $F \in \mathcal{F}_D$ is an initial operator for D corresponding to $R \in \mathcal{R}_D$; and $A, B \in L_0(X)$, with $A \neq 0$ non-invertible and $C \in L_0(U, X)$. We consider the problem $(DS)_0$:

$$\begin{cases} ADx &= Bx + Cu \text{, with condition } RCU \oplus \{x_0\} \subset \{I - R[(I - A)D + B]\} \operatorname{dom} D \quad (4.1) \\ Fx &= x_0 \quad , \quad x_0 \in \ker D. \quad (4.2) \end{cases}$$

The spaces X and U are called the space of states and the space of controls, respectively. Elements $x \in X$ and $u \in U$ are called states and controls, respectively. The element $x_0 \in \ker D$ is called an initial state. A pair $(x_0, u) \in (\ker D) \times U$ is called an input.

In section 3, we have proved that the problem (4.1)-(4.2) is equivalent to the equation

$$\{I - R[(I - A)D + B]\}x = RCu + x_0 \, . \tag{4.3}$$

Hence, the inclution $RCU \oplus \{x_0\} \subset \{I - R[(I - A)D + B]\} \operatorname{dom} D$ is a necessary and sufficient condition for the problem (4.1)-(4.2) have solution for every $u \in U$. Denote by $\Phi_i (i = 1, 2, 3, 4)$ the following sets, defined for every $x_0 \in \ker D$, $u \in U$:

(i) If $A - BR \in R(X)$ and $R_{AB} \in \mathcal{R}_{A-BR}$, then:

$$\begin{aligned} \Phi_1(x_0, u) = \{x = T_1(RCu + x_0) + \tilde{z} \ , \ \tilde{z} \in \ker\{I - R[(I - A)D + B]\} \, , \\ T_1 = I + RR_{AB}[(I - A)D + B] \, \}. \end{aligned} \tag{4.4}$$

(ii) If $A - BR \in \Lambda(X)$ and $L_{AB} \in \mathcal{L}_{A-BR}$, then:

$$\Phi_2(x_0, u) = \{x = T_2(RCu + x_0) \, , T_2 = I + RL_{AB}[(I - A)D + B] \, \}. \tag{4.5}$$

(iii) If $A - BR$ is invertible, then:

$$\Phi_3(x_0, u) = \{x = T_3(RCu + x_0)\,,\ T_3 = I + R(A - BR)^{-1}[(I - A)D + B]\,\}. \quad (4.6)$$

(iv) If $A - BR \in W(X)$ and $W_{A,B} \in \mathcal{W}_{A-BR}$, then:

$$\Phi_4(x_0, u) = \{x = T_4(RCu + x_0) + \tilde{z}\,,\ \tilde{z} \in \ker\{I - R[(I - A)D + B]\}\,,$$
$$T_4 = I + RW_{A,B}[(I - A)D + B]\,\}. \quad (4.7)$$

Note that $\Phi_i (i = 1, 2, 3, 4)$ are sets of all solutions of the problem (4.1)-(4.2) in the corresponding case.

Definition 4.1. *Suppose that we are given a system $(DS)_0$ and the sets $\Phi_i(x_0, u)$ of the forms (4.4)-(4.7) . A state $x \in X$ is said to be (i)- reachable $(i = 1, 2, 3, 4)$ from an initial state $x_0 \in \ker D$ if for every $T_i(T_1 = I+RR_{AB}[(I-A)D+B]\,,\ T_2 = I+RL_{AB}[(I-A)D+B]$, $T_3 = I + R(A - BR)^{-1}[(I - A)D + B]$, $T_4 = I + RW_{A,B}[(I - A)D + B])$ there exists a control $u \in U$ such that $x \in \Phi_i(x_0, u)$.*

$$\text{Write}\quad \mathrm{Rang}_{U,x_0}\Phi_i = \bigcup_{u\in U} \Phi_i(x_0, u)\,,\ x_0 \in \ker D\,,\ (i = 1, 2, 3, 4).$$

It is easy to see that $\mathrm{Rang}_{U,x_0}\Phi_i$ is (i)- reachable from $x_0 \in \ker D$ by means of controls $u \in U$, and these sets are contained in dom$D$.

Lemma 4.1. *Suppose that $T_i(i = 1, 2, 3, 4)$ are defined as in (4.4)-(4.7) , then:*

$$T_i(RCU \oplus \{x_0\}) + \ker\{I - R[(I - A)D + B]\}$$
$$= T_iRCU \oplus \{T_ix_0\} \oplus \ker\{I - R[(I - A)D + B]\}. \quad (4.8)$$

Proof. It is sufficient to prove the case $i = 4$.

By our assumption, $I - R[(I - A)D + B] \in W(X)$ and $T_4 \in \mathcal{W}_{I-R[(I-A)D+B]}$. Therefore, Proposition 1.2 implies that $X = T_4\{I - R[(I - A)D + B]\}X \oplus \ker\{I - R[(I - A)D + B]\}$. On the other hand $RCU \oplus \{x_0\} \subset \{I - R[(I - A)D + B]\}$ dom$D$, there exists $E \subset$ dom$D$ such that $RCU \oplus \{x_0\} = \{I - R[(I - A)D + B]\}E \subset \{I - R[(I - A)D + B]\}$ dom$D$. Hence,

$$T_4(RCU \oplus \{x_0\}) + \ker\{I - R[(I - A)D + B]\}$$
$$= T_4\{I - R[(I - A)D + B]\}E + \ker\{I - R[(I - A)D + B]\}$$
$$= T_4\{I - R[(I - A)D + B]\}E \oplus \ker\{I - R[(I - A)D + B]\}$$
$$= T_4(RCU \oplus \{x_0\}) \oplus \ker\{I - R[(I - A)D + B]\}\,.$$

We will prove the equality $T_4(RCU \oplus \{x_0\}) = T_4RCU \oplus \{T_4x_0\}$. Indeed, let $x \in (T_4RCU) \cap \{T_4x_0\}$, i.e. there exists $u \in U$, $t \in \mathcal{F}$ such that $x = T_4RCu = T_4tx_0$, or $T_4(RCu - tx_0) = 0$. By our assumption $RCU \oplus \{x_0\} \subset \{I - R[(I - A)D + B]\}domD$, there exists $v \in$ dom$D$ such that $RCu - tx_0 = \{I - R[(I-A)D + B]\}v$, this implies that $0 = T_4(RCu - tx_0) = T_4\{I - R[(I-A)D + B]\}v$, or

$$\begin{aligned} 0 &= \{I - R[(I-A)D + B]\}T_4\{I - R[(I-A)D + B]\}v \\ &= \{I - R[(I-A)D + B]\}v \quad = RCu - tx_0\,. \end{aligned}$$

Hence, $0 = Dtx_0 = DRCu = Cu$, $tx_0 = 0$ and $x = T_4RCu = T_4tx_0 = 0$.

Remark If either $A - BR \in \Lambda(X)$ or $A - BR$ is an invertible operator, then $\ker\{I - R[(I-A)D + B]\} = \{0\}$. Thus the formula (4.8) takes the form:

$$T_i(RCU \oplus \{x_0\}) = T_iRCU \oplus \{T_ix_0\}\,,\ (i = 1, 2, 3, 4). \tag{4.9}$$

Corollary 4.1.

$$\text{Rang}_{U,x_0}\Phi_i = T_iRCU \oplus \{T_ix_0\} \oplus \ker\{I - R[(I-A)D + B]\}. \tag{4.10}$$

Corollary 4.2. *A state x is (i)- reachable from a given initial state $x_0 \in \ker D$ if and only if $x \in T_iRCU \oplus \{T_ix_0\} \oplus \ker\{I - R[(I-A)D + B]\}$, $(i = 1, 2, 3, 4)$.*

Definition 4.2. *Let be given a degenerate system $(DS)_0$ of the form (4.1)-(4.2) and let $F_i \in \mathcal{F}_D (i = 1, 2, 3, 4)$ be arbitrary initial operators (not necessarily different).*

(i) A state $x_1 \in \ker D$ is said to be F_i- reachable from an initial state $x_0 \in \ker D$ if there exists a control $u \in U$ such that $x_1 \in F_i\Phi_i(x_0, u)$. The state x_1 is called a final state.

(ii) The system $(DS)_0$ is said to be F_i- controllable if for every initial state $x_0 \in \ker D$,

$$F_i(\ \text{Rang}_{U,x_0}\Phi_i) = \ker D. \tag{4.11}$$

(iii) The system $(DS)_0$ is said to be F_i- controllable to $x_1 \in \ker D$ if

$$x_i \in F_i(\text{Rang}_{U,x_0}\Phi_i) \tag{4.12}$$

for every initial state $x_0 \in \ker D$.

Lemma 4.2. *Let be given a degenerate system $(DS)_0$ and an initial operator $F_i \in \mathcal{F}_D$. Suppose that the system $(DS)_0$ is F_i- controllable to zero and that*

$$F_i(T_i \ker D + \ker\{I - R[(I-A)D + B]\}) = \ker D. \tag{4.13}$$

Then every state $x_1 \in \ker D$ is F_i- controllable to zero.

Proof. It is sufficient to prove the case $i = 4$. Let $x_1 \in \ker D$, since the assumption $(DS)_0$ is F_4- controllable to zero. Thus, $0 \in F_4(\ \text{Rang}_{U,x_0}\Phi_4)$ for every $x_0 \in \ker D$, i.e. there exists a control $u_0 \in U$ and $z_0 \in \ker\{I - R[(I-A)D + B]\}$ such that $F_4[T_4(RCu_0 + x_0) + z_0] = 0$, or

$$F_4(T_4RCu_0 + z_0) = -F_4T_4x_0. \tag{4.14}$$

The condition (4.13) implies that for every $x_1 \in \ker D$, there exists $x_2 \in \ker D$ and $z_1 \in \ker\{I - R[(I-A)D + B]\}$ such that

$$F_4(T_4 x_2 + z_1) = x_1. \tag{4.15}$$

On the other hand, by formula (4.14), for every $x_2 \in \ker D$, there exists $u_0' \in U$ and $z_0' \in \ker\{I - R[(I-A)D + B]\}$ such that

$$F_4(T_4 RCu_0' + z_0' + z_1) = F_4(T_4 x_2 + z_1). \tag{4.16}$$

So that (4.15) and (4.16) imply $F_4(T_4 RCu_0' + z_1') = x_1$, with $z_1' = z_0' + z_1 \in \ker\{I - R[(I-A)D + B]\}$. This proves that every state $x_1 \in \ker D$ is F_4- reachable from zero.

Theorem 4.1. *Suppose that all assumptions of Lemma 4.2 are satisfied. Then the degenerate system $(DS)_0$ is F_i- controllable.*

Proof. Suppose that $A - BR \in W(X)$. By our assumption there exists $u_0 \in U$ and $z_0 \in \ker\{I - R[(I-A)D + B]\}$ such that

$$F_4[T_4(RCu_0 + x_0) + z_0] = 0. \tag{4.17}$$

On the other hand, by Lemma 4.2, for every $x_1 \in \ker D$ there exists $u_0' \in U$ and $z_1 \in \ker\{I - R[(I-A)D + B]\}$ such that

$$F_4(T_4 RCu_0' + z_1) = x_1. \tag{4.18}$$

Therefore, (4.17) and (4.18) imply that $F_4\{T_4[RC(u_0 + u_0') + x_0] + (z_0 + z_1)\} = x_1$, i.e. the state x_1 is F_4- reachable from initial state x_0. The theorem has been proved.

Theorem 4.2. *Let be given a degenerate system $(DS)_0$ of the form (4.1)-(4.2) and an initial operator $F_i \in \mathcal{F}_D (i = 1, 2, 3, 4)$ and let $T_1 = I + RR_{AB}[(I-A)D + B]$ if $A - BR \in R(X)$, $T_2 = I + RL_{AB}[(I-A)D + B]$ if $A - BR \in \Lambda(X)$, $T_3 = I + R(A-BR)^{-1}[(I-A)D + B]$ if $A - BR$ is invertible and $T_4 = I + RW_{A,B}[(I-A)D + B]$ if $A - BR \in W(X)$. Suppose that $C \in L_0(U \to X, X' \to U')$, $D \in L(X, X')$, and $A, B, R \in L_0(X, X')$. Then, the system $(DS)_0$ is F_i- controllable if and only if*

$$\ker C^* R^* T_i^* F_i^* = \{0\}. \tag{4.19}$$

Proof. It is sufficient to consider the case $i = 4$. Note that in all the cases consider, $F_i T_i RC$ maps U into $\ker D$. The condition (4.19) is equivalent to

$$F_i T_i RCU = \ker D. \tag{4.20}$$

The assumption $RCU \oplus \{x_0\} \subset \{I - R[(I-A)D + B]\}\ \mathrm{dom} D$, implies that

$$\begin{aligned} F_4 T_4 RCU &= F_4 T_4 (RCU \oplus \{x_0\}) - \{F_4 T_4 x_0\} \\ &\subset F_4 T_4 \{I - R[(I-A)D + B]\} \mathrm{dom} D - \{F_4 T_4 x_0\} \\ &\subset F_4 \{T_4 \{I - R[(I-A)D + B]\} \mathrm{dom} D \oplus \ker\{I - R[(I-A)D + B]\}\} \\ &\qquad - \{F_4 T_4 x_0\} - F_4 \ker\{I - R[(I-A)D + B]\} \\ &= F_4 \mathrm{dom} D - \{F_4 T_4 x_0\} - F_4 \ker\{I - R[(I-A)D + B]\} \subset \ker D. \end{aligned}$$

By (4.20), we have $F_4T_4RCU = F_4\text{dom}D - \{F_4T_4x_0\} - F_4\ker\{I-R[(I-A)D+B]\} = \ker D$. Thus, $F_4T_4RCU + \{F_4T_4x_0\} + F_4\ker\{I-R[(I-A)D+B]\} = F_4\text{dom}D = \ker D$. This means that for every $x_1 \in \ker D$, there exists $v \in \text{dom}D$, $u \in U$ and $z_0 \in \ker\{I-R[(I-A)D+B]\}$ such that $x_1 = F_4v = F_4T_4RCu + F_4T_4x_0 + F_4z_0 = F_4[T_4(RCu+x_0)+z_0]$, i.e. x_1 is F_4-reachable from x_0. The arbitrariness of $x_0, x_1 \in \ker D$ implies that

$$F_4(\text{ Rang}_{U,x_0}\Phi_4) = \ker D.$$

Conversely, suppose that $F_4(\text{ Rang}_{U,x_0}\Phi_4) = \ker D$. Choosing $x_0 = 0, z_0 = 0$, we get that $F_4T_4RCU = \ker D$. The proof is completed.

Corollary 4.3. *Suppose that A, B are stationary operators. Then the system $(DS)_0$ is F_3- controllable if and only if*

$$\ker C^* R^* (A - BR)^{*^{-1}} F_3^* = \{0\}. \tag{4.21}$$

Theorem 4.3. *Suppose that the system $(DS)_0$ is F_i- controllable. Then, it is F_i'- controllable for every initial operator $F_i' \in \mathcal{F}_D$.*

Proof. Let F_i be an initial operator for D corresponding to $R \in \mathcal{R}_D$, i.e. $F_iR_i = 0$. On the other hand, for every $x_1 \in \ker D$ and $v \in X$, there exists $x_2 \in \ker D$ such that $x_1 = x_2 + F_i'R_iv$. By our assumption the system $(DS)_0$ is F_i- controllable. Thus, for every $x_0, x_2 \in \ker D$, there exists a control $u \in U$ and $z_0 \in \ker\{I - R[(I-A)D+B]\}$ such that $F_i[T_i(RCu+x_0)+z_0] = x_2$, or $F_i[T_i(RCu+x_0)+z_0] = F_i(x_2+R_iv)$, for some $v \in X$. Hence, $F_i'[T_i(RCu+x_0)+z_0] = F_i'(x_2+R_iv) = x_2 + F_i'R_iv = x_1$. The arbitrariness of $x_0, x_1 \in \ker D$, the proof is completed.

Theorem 4.4. *Let be given a degenerate system $(DS)_0$ and an initial operator $F_i \in \mathcal{F}_D$. Then, the system $(DS)_0$ is F_i- controllable if and only if it is F_i- controllable to every element $v' \in F_iT_iRX$.*

Proof. First, we prove the equality:

$$F_4\{T_4(RX \oplus \ker D) + \ker\{I - R[(I-A)D+B]\}\} = \ker D. \tag{4.22}$$

Indeed, since $\{I - R[(I-A)D+B]\}\text{dom}D \subset \text{dom}D = RX \oplus \ker D$, there exists $E \subset X$ and $Z' \subset \ker D$ such that $RE \oplus Z' = \{I - R[(I-A)D+B]\}\text{dom}D$. This implies that

$$\begin{aligned} &T_4(RE \oplus Z') + \ker\{I - R[(I-A)D+B]\} \\ &\quad = T_4\{I - R[(I-A)D+B]\}\text{dom}D \oplus \ker\{I - R[(I-A)D+B]\} = \text{dom}D. \end{aligned}$$

Hence,

$$\begin{aligned} \ker D = F_4\text{dom}D &= F_4\{T_4(RE \oplus Z') + \ker\{I - R[(I-A)D+B]\}\} \\ &\subset F_4\{T_4(RX \oplus \ker D) + \ker\{I - R[(I-A)D+B]\}\} \subset \ker D, \end{aligned}$$

i.e. the formula (4.22) is has been proved.

Suppose that the system $(DS)_0$ is F_4-controllable to every element $v' \in F_4T_4Rv, v \in X$, i.e. there exists a control $u_0 \in U$ and $z_0 \in \ker\{I - R[(I-A)D+B]\}$ such that

$$F_4[T_4(RCu_0 + x_0) + z_0] = F_4T_4Rv.$$

This implies that

$$F_4\{T_4[RCu_0 + x_0 + x_2] + z_0 + z_1\} = F_4\{T_4(Rv + x_2) + z_1\}, \tag{4.23}$$

where $z_1 \in \ker\{I - R[(I-A)D + B]\}$ and $x_2 \in \ker D$ are arbitrary.

By formula (4.22), for every $x_1 \in \ker D$, there exists $z_1' \in \ker\{I - R[(I-A)D+B]\}$, $v' \in X$ and $x_2' \in \ker D$ such that $x_1 = F_4[T_4(Rv' + x_2') + z_1']$. This equality and (4.23) obtain

$$F_4[T_4(RCu_0' + x_0 + x_2') + z_0 + z_1'\} = x_1. \tag{4.24}$$

On the other hand, the condition $0 \in F_4 T_4 RX$ and our assumption imply that $(DS)_0$ is F_4- controllable to zero, i.e. $0 \in F_4(\text{ Rang}_{U,x_0}\Phi_4)$ for every $x_0 \in \ker D$. Therefore, there exists $u_1 \in U$ and $z_2 \in \ker\{I - R[(I-A)D + B]\}$ such that

$$F_4[T_4(RCu_1 - x_2') + z_2\} = 0. \tag{4.25}$$

If we add (4.24) and (4.25), then $F_4[T_4(RCu_3 + x_0) + z_3\} = x_1$, where $z_3 = z_0 + z_1' + z_2$, $u_3 = u_0' + u_1$. The arbitrariness of x_0, x_1 gives $F_4(\text{ Rang}_{U,x_0}\Phi_4) = \ker D$.

Example 4.1. *Suppose that X is the space (s) of all real sequences $\{x_n\}, n = 0, 1, 2, \cdots$ Write $D\{x_n\} = \{x_{n+1} - x_n\}$, $R\{x_n\} = \{y_n\}$, $y_0 = 0$, $y_n = \sum_{k=0}^{n-1} x_k, n \geqslant 1$ and $F\{x_n\} = \{z_n\}, z_n = x_0, n = 0, 1, 2, \cdots$ It is easy to check that $D \in R(X), R \in \mathcal{R}_D$ and F is an initial operator for D corresponding to R. Moreover,*

$$\ker D = \{z = \{z_n\}, z_n = c, n \in \mathbb{N}, c \in \mathbb{R}\} \neq \{0\}.$$

Suppose that $U = \{\{u_n\} : u_n \in \mathbb{R}, u_n = 0, \forall n \geqslant 1\}$, an operator $C \in L_0(U, X)$ is defined by $C = \alpha I$ (I is an identity operator, $\alpha \in \mathbb{R}$) and $\overline{x}_0 = \{x_0\} \in \ker D$. We consider the degenerate system $(DS)_0$ of the form

$$ADx = Bx + Cu \quad , \quad u \in U \tag{4.26}$$

$$Fx = \overline{x}_0 , \tag{4.27}$$

where A, B are defined as follows $A\{x_n\} = \{x_{n+1}\}$, $B\{x_n\} = \{x_{n+2} - x_n\}$. In Example 3.1, we see that $A \neq 0$ non-invertible, the resolving operator $A - BR$ is invertible and $A - BR = (A - BR)^{-1} = -I$. Therefore, by formula (4.6), the solution of $(DS)_0$ is given

$$\Phi_3(\overline{x}_0, u) = T_3(R\alpha I u + \overline{x}_0),\ T_3 = I - R[(I-A)D + B]$$
$$= \{x_0, x_0 - \alpha u_0, x_0 - \alpha u_0, x_0 - \alpha u_0, \cdots\}. \tag{4.28}$$

Let $R_3 \in \mathcal{R}_D$ be defined by $R_3\{x_n\} = \{-x_0, 0, x_1, x_1 + x_2, x_1 + x_2 + x_3, \cdots\}$. Thus, $F_3\{x_n\} = (I - R_3 D)\{x_n\} = \{x_1\}$. Therefore, for every $\overline{x}_0 = \{x_0\} \in \ker D$, there exists $\overline{u}_0 = \{\frac{x_0}{\alpha}, 0, 0, 0, \cdots\} \in U$ such that $F_3\Phi_3(\overline{x}_0, \overline{u}_0) = F_3\{x_0, 0, 0, 0, \cdots\} = \{0\}$, i.e. the system $(DS)_0$ is F_3- controllable to zero. Moreover, $F_3T_3(\ker D) = \ker D$. By Theorem 4.1, the system $(DS)_0$ is F_3- controllable.

Example 4.2. *Suppose that X is the space (s) of all real sequence $\{x_n\}$. Let D, R, F, A, $B \in L_0(X)$ be defined by $D\{x_n\} = \{x_{n+1} - x_n\}$, $R\{x_n\} = \{y_n\}$, $y_0 = 0$, $y_n = \sum_{k=0}^{n-1} x_k, n \geqslant 1$, $F\{x_n\} = \{x_0\}$, $A\{x_n\} = \{2x_0 + x_1, 0, x_2 + x_3, x_3 + x_4, x_4 + x_5, \cdots\}$ and $B\{x_n\} = \{x_2 - x_0, 0, x_4 - x_2, x_5 - x_3, x_6 - x_4, \cdots\}$. It is easy to see that $D \in R(X)$, $R \in \mathcal{R}_D$, $\ker D = \{z = \{z_n\}, z_n = c, n \in \mathbb{N}, c \in \mathbb{R}\} \neq \{0\}$ and F is an initial operator for D corresponding to R. The operator $A \neq 0$ and non-invertible, since $\ker A = \{x_0, -2x_0, x_2, -x_2, x_2, -x_2, \cdots\} \neq \{0\}$ and $AX \neq X$.*

Let $U = \{\{u_n\} : u_n \in \mathbb{R}, u_n = 0, \forall n \geqslant 2\}$ *and* $C = I \in L_0(U, X)$. *Consider the degenerate system* $(DS)_0$ *of the form:*

$$ADx = Bx + Cu \quad , \quad u \in U \tag{4.29}$$

$$Fx = \overline{x}_0 \quad , \quad \overline{x}_0 \in \ker D \tag{4.30}$$

We have $A - BR \in W(X)$, $I \in \mathcal{W}_{A-BR}$, *since* $(A - BR)I(A - BR)\{x_n\} = (A - BR)\{x_n\}$. *Moreover,* $\ker\{I - R[(I - A)D + B]\} = \{\{0, 0, x_2, x_3, x_4, \cdots\}, x_n \in \mathbb{R}, n = 2, 3, 4, \cdots\}$. *Therefore, by the formula (4.7), the solution of the problem* $(DS)_0$ *is given by*

$$\begin{aligned}\Phi_4(\overline{x}_0, u) &= \{I + R[(I - A)D + B]\}(RCu + \overline{x}_0) + \tilde{z}, \tilde{z} \in \ker\{I - R[(I - A)D + B]\} \\ &= \{x_0, x_0 + u_0, x_0 + u_0 + 2u_1 + \tilde{x}_2, x_0 + u_0 + 2u_1 + \tilde{x}_3, \cdots\}.\end{aligned} \tag{4.31}$$

Let $F_4 \in \mathcal{F}_D$ *be defined by* $F_4\{x_n\} = \{x_1\}$. *Then for every* $\overline{x}_0 = \{x_0\} \in \ker D$, *there exists* $\overline{u}_0 = \{-x_0, u_1, 0, 0, 0, \cdots\} \in U$ *such that* $F_4\Phi_4(\overline{x}_0, \overline{u}_0) = \{0, 0, 0, \cdots\}$. *Thus, the system* $(DS)_0$ *is* F_4*- controllable to zero. Moreover,*

$$F_4(T_4 \ker D + \ker\{I - R[(I - A)D + B]\}) = \ker D, \ \text{with}\, T_4 = I + R[(I - A)D + B].$$

By the Theorem 4.1, the system $(DS)_0$ *is* F_4*- controllable.*

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