# ON THE LOCAL DIMENSIONS OF FRACTAL MEASURES

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**Abstract.** *Let $X_0, X_1, \cdots$ be a sequence of independent, identically distributed random variables each taking values $r_0, r_1, \cdots, r_m$ with equal probability $p = \frac{1}{m+1}$. Let $\mu$ be the probability measure induced by $S = \sum_{i=0}^{\infty} \rho^i X_i$. The aim of this paper is to study some properties of support of $\mu$ and the local dimension of $\mu$ at elements $s \in supp\mu$ in the case: $r_0 = 0, r_1 < r_2 < \cdots < r_m$ and $q$ are integers such that $\frac{r_m}{2} < q \leq m+1$, $r_m - q \in D = \{r_0, r_1, \cdots, r_m\}$; $\rho = \frac{1}{q}$.*

## 1. Introduction and notations

By a *probabilistic system* we mean a sequence $X_0, X_1, \cdots$ of independent, identically distributed random variables each taking values $r_0, r_1, \cdots, r_m$ with respective probabilities $p_0, p_1, \cdots, p_m$. We say that the system is uniformly distributed if $p_i = \frac{1}{m+1}$. For $0 < \rho < 1$, put

$$S = \sum_{i=0}^{\infty} \rho^i X_i \qquad S_n = \sum_{i=0}^{n} \rho^i X_i$$

Let $\mu$ and $\mu_n$ denote the probability distributions of $S$ and $S_n$, respectively. Then $\mu$ is called the *fractal measure* associated with the probabililistic system.

Recall that for $s \in \sup\mu$, the *lower local dimension* $\alpha_*(s)$ of $\mu$ at $s$ is defined by

$$\alpha_*(s) = \lim_{h \to 0^+} inf \frac{\log \mu(B(s,h))}{\log\ h}, \qquad \text{where } B(s,h) = [s-h, s+h].$$

We similarly difine the *upper local dimension* using the upper limit and denote it by $\alpha^*(s)$. If the two limits are equal, then the common value is called the *local dimension* of $\mu$ at $s$ and is denoted by $\alpha(s)$. Roughly speaking, if $\alpha(s)$ exits, then $\mu(B(s,h))$ is approximately proportional to $h^{\alpha(s)}$ for small $h$. Thus $\mu$ can be viewed as a probability measure of degree of singularity $\alpha(s)$. In this sense, the local dimension measures the degree of singularities of $\mu$ locally.

In [4], T. Hu considered the local dimensions of fractal measure $\mu$ in the case $m = 1$ $r_0 = 0$, $r_1 = 1$ and $\rho^{-1} = \frac{1+\sqrt{5}}{2}$ (this number is said to be the golden number). In [5], T. Hu and N. Nguyen studied the problem in the case $r_0 = 0, r_1 = 1, \cdots, r_m = m$, $p_0 = p_1 = \cdots = p_m = \frac{1}{m+1}$, $\rho = \frac{1}{q}$, $2 \leq q \leq m$, $q$ is an integer. It is very difficult to study the above problem in the case that the distances between $r_1, r_2, \cdots, r_m$ are not equal. In [6] only considered the problem in special case: $m = 2$, $r_0 \doteq 0, r_1 = 1, r_2 = 3$; $p_0 = p_1 = p_2 = \frac{1}{3}$ and $\rho = \frac{1}{q} = \frac{1}{3}$.

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The aim of this paper is to study some properties of supp$\mu$ and the local dimension of $\mu$ at elements $s \in$ supp$\mu$ in more general case. The main results of this paper are the theorems 2.7 and 3.1. Our results here extent some results in [5] and [6] (see *proposition 2.4* in [5], *proposition 2.1* and *Main Theorem* in [6]).

All notations and definitions of this paper we refer to [2], [4] and [5]

## 2. Some properties of fractal measure

Throughout of this paper the following assumptions are made: $r_0 = 0, r_1 < r_2 < \cdots < r_m$ are integers such that $\frac{r_m}{2} < q \leq m+1$, $r_m - q \in D = \{r_0, r_1, \cdots, r_m\}$ and $\rho = \frac{1}{q}$. The following proposition was proved in [5].

**Propositions 2.1.** *Let $s_n(0) < s_n(1) < \cdots < s_n(k_n)$ denote the set of all distinct values of supp$\mu_n$. Then we have*

*1. $s_n(0) = 0$ and $s_{n+1}(k_{n+1}) = s_n(k_n) + mq^{-n-1}$ for every $n \in N$*

*2 The distance between two consecutive points in sup $\mu$ is at least $q^{-n}$*

*3. supp$\mu_n \subset$ supp$\mu_{n+1}$ and supp$\mu = \overline{\cup_{n=0}^{\infty} supp\mu_n}$.*

**Proposition 2.2.** *Let*

$$< s_n >= \left\{(x_0, x_1, \cdots, x_n) \in D_m^{n+1},\ \sum_{i=0}^{n} q^{-i}x_i = s_n\right.$$

*Then we have*

$$\mu_n(s_n) = \# < s_n > (m+1)^{-n-1},$$

*where $\# < s_n >$ denotes the cardinality of $s_n$.*

*Proof.* For $x(n) = (x_0, x_1, \cdots, x_n) \in < s_n >$, put

$$A_{x(n)} = \bigcap_{i=1}^{n} \{\omega : \ X_i(\omega) = x_i\ \}.$$

It is easy to see that if $x(n), y(n) \in < s_n >$, $x(n) \neq y(n)$ then $A_{x(n)} \cap A_{y(n)} = \emptyset$ and

$$\mu_n(s_n) = P\{\omega : \ S(\omega) = s_n\ \} = P\{\omega : \ \sum_{i=0}^{n} q^{-i}X_i(\omega) = s_n$$

$$= P\{\omega : \ X_i(\omega) = x_i \ \forall i = \overline{0,n};\ \sum_{i=1}^{n} q^{-i}x_i = s_n\ \}\}$$

$$= P\Big(\bigcup_{x(n)\in<s_n>} A_{x(n)}\Big) = \sum_{x(n)\in<s_n>} P(A_{x(n)}) = \sum_{x(n)\in<s_n>} P\Big(\bigcap_{i=1}^{n}\{\omega : \ X_i(\omega) = x_i\Big)$$

$$= \sum_{x(n)\in<s_n>} \Big(\prod_{i=1}^{n} P\Big(\{\omega : \ X_i(\omega) = x_i\}\Big)\Big)$$

$$= \sum_{x(n)\in<s(n)>} \frac{1}{(m+1)^{n+1}} = \# < s_n > .(m+1)^{-n-1}$$

**Definition 2.3.** Let $s_n \in \text{supp}\mu$, $s_{n+1} \in \text{supp}\mu_{n+1}$, we say that $s_{n+1}$ is *represented through* $s_n$ if there exists $x_{n+1} \in D_m$ such that $s_{n+1} = s_n + q^{-n-1}.x_{n+1}$.

It is easy to see that if $s_{n+1}$ is represented through $s_n$, then

$$\# < s_n > \leq \# < s_{n+1} > .$$

**Lemma 2.4.** *If $s_{n+1} \in \text{supp}\mu_{n+1}$ then there is $s_n \in \mu_n$ such that $s_{n+1}$ is represented through $s_n$ and $0 \leq s_{n+1} - s_n < 2q^{-n}$.*

*Proof.* If $s_{n+1} \in \text{supp}\mu_{n+1}$ then there exists $x(n+1) = (x_0, x_1, \cdots, x_{n+1}) \in D^{n+2}$ such that

$$s_{n+1} = \sum_{i=0}^{n+1} q^{-i}x_i = \sum_{(} i = 0)^n q^{-i}x_i + q^{-n-1}x_{n+1} = s_n + q^{-n-1}$$

and

$$0 \leq s_{n+1} - s_n = q^{-n-1}x_{n+1} \leq q^{-n-1}r_m < q^{-n-1}2q = 2q^{-n}.$$

**Lemma 2.5.** *If $s_{n+1} \in \text{supp}\mu_{n+1}$ then there are at most two points $s_n$ and $s'_n$ in $\text{supp}\mu_n$ such that $s_{n+1}$ is represented through them. In this case $s_n, s'_n$ are two consecutive points in $\text{supp}\mu_n$.*

*Proof.* Suppose that there are three points $t_n < t'_n < t''_n$ in $\text{supp}\mu$ and three elements $x_n$, $x'_n$, $x''_n$ in $D_m$ such that

$$s_{n+1} = t_n + q^{-n-1}x_{n+1}$$
$$s_{n+1} = t'_n + q^{-n-1}x'_{n+1}$$
$$s_{n+1} = t''_n + q^{-n-1}x''_{n+1}.$$

Then

$$s_{n+1} \geq t_n \geq t''_n + 2q^{-n}.$$

Thus

$$s_{n+1} - t''_n \geq 2q^{-n},$$

which is imposible (by Lemma 2.4 ) and the first part of the lemma is proved.

Now, suppose that $s_{n+1}$ is represented through $s_n$ and $s'_n$, $s_n < s'_n$, we have

$$q^{-n} \leq |s_n - s'_n| \leq s_{n+1} - s_n < 2q^{-n}$$

which follows that $|s_n - s'_n| = q^{-n}$ and $s_n$, $s'_n$ are two consecutive points in $\text{supp}\mu_n$.

**Lemma 2.6.** *If $s_{n+1} \in \text{supp}\mu_{n+1}$ is represented through $s_n \in \text{supp}\mu_n$ and $t_n \in \text{supp}\mu_n$ such that $s_n < t_n \leq s_{n+1}$ then $t_n = s_n + q^{-n}$.*

*Proof.* We have

$$2q^{-n} > s_{n+1} - s_n \geq t_n - s_n > 0.$$

This implies

$$t_n - s_n = q^{-n},$$

which completes the proof.

The main result of this section is following theorem.

**Theorem 2.7.** *If $s_n, s'_n$ are two consecutive points in supp$\mu_n$ then*

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} \leq n+1$$

*Proof.* We prove the inequality by induction. Clearly the inequality holds for $n = 0$. Suppose that it is true for $n \leq k$. Let $s_{k+1} > s'_{k+1}$ be two arbitrary consecutative points in supp$\mu_{k+1}$

$$s - k + 1 = s_k + q^{-k-1} x_{k+1}$$
$$s' - k + 1 = s_k + q^{-k-1} x'_{k+1},$$

where $s_k, s'_k \in$ supp$\mu_k$; $x_{k+1}, x'_{k+1} \in D_m$.

Then

$$s'_k \leq s'_{k+1} < s_{k+1}.$$

We consider three case:

a. If $s'_k > s_k$ then $s_{k+1} > s'_k > s_k$. Using lemma 2.6 we get

$$s'_k = s_k + q^{-k}$$

By lemma 2.5, $s_{k+1}$ has at most two representations through $s_k$ and $s'_k$. It follows that

$$\# < s_{k+1} > \leq \# < s_k > + \# < s'_k > .$$

Thus

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} = \frac{\# < s_{k+1} >}{\# s'_{k+1}} \leq \frac{\# < s_k > + \# < s'_k >}{\# < s'_k >}$$
$$= 1 + \frac{\# < s_k >}{\# < s'_k >} = 1 + \frac{\mu_k(s_k)}{\mu_k(s'_k)} \leq 1 + (k+1) = k+2.$$

b. If $s'_k = s_k$ then by lemma 2.5, there exists at most one point $t_k \in$ supp$\mu_k$, $t_k \neq s_k$ such that $s_{k+1}$ is represented through $t_k$ ($s_k$ and $t_k$ are two consecutive points). It follows that

$$\# < s_{k+1} > < \# < s_k > + \# < s'_k > .$$

Thus

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} = \frac{\# < s_{k+1} >}{\mu_{k+1}(s'_{k+1})} \leq \frac{\# < s_k > + \# < s'_k >}{\# < t_k >} =$$
$$= 1 + \frac{\mu_k(t_k)}{\mu_k(s_k)} \leq 1 + (k+1) = k+2.$$

c. If $s'_k < s_k$ then we consider two cases.

$c_1$. If there exists $t'_k \in \text{supp}\mu_k$ such that $s'_k < t'_k < s_k$ then from the inequality

$$s'_{k+1} - s'_k < 2q^{-k} \leq s_k - s'_k,$$

we have

$$s'_{k+1} < s_k \leq s_{k+1}.$$

On the other hand, since $s_{k+1}$ and $s_k$ are two consecutive points, we have $s_{k+1} = s_k$ and

$$s_{k+1} = s_k \geq s'_k + 2q^{-k} > s'_k + q^{-k-1} \geq s'_{k+1}$$
$$s'_{k+1} = s'_k + q^{-k-1} r_m$$
$$s'_{k+1} \geq t'_k.$$

Using lemma 2.6 we get $t'_k = s'_k + q^{-k}$.

This implies

$$s'_{k+1} = s'_k + q^{-k-1} r_m = t'_k - q^{-k} + q^{-k-1} r_m = t'_k + q^{-k-1}(r_m - q).$$

Since $r_m - q \in D_m$, we have $s'_{k+1}$ is represented through $t'_k$. It follows that

$$\# < s'_{k+1} > \geq \# < t'_k > .$$

We now prove that $t'_k$ and $s_k$ are consecutive points in $\text{supp}\mu_k$.

Suppose that there exists $t''_k \in \text{supp}\mu_k$ such that $t'_k < t''_k < s_k$, then $s'_{k+1} \geq t''_k$ (because $s'_{k+1}$ and $s_{k+1}$ are two consecutive points in $\text{supp}\mu_{k+1}$). It follows that

$$s'_{k+1} - s'_k \geq t''_k - s'_k \geq t'_k + q^{-k} - s'_k = 2q^{-k}.$$

It is imposible (by Lemma 2.4). Hence $t'_k$ and $s_k$ are two consecutive points.

By lemma 2.5, $s_{k+1}(= s_k)$ has at most two representations through $s_k$ and $s'_k$ .It follows that

$$\# < s_{k+1} > \leq \# < s_k > + \# < t'_k >$$

and

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} \leq \frac{\# < s_k > + \# < t'_k >}{\# < t'_k >} = 1 + \frac{\# < s_k >}{\# < t'_k >} \leq 1 + k + 1 = k + 2.$$

$c_2$. If does not exists $t'_k \in \text{supp}\mu_k$ such that $s'_k < t'_k < s_k$, then $s'_k$ and $s_k$ are two consecutive points in $\mu_k$. By lemma 2.5, $s_{k+1}$ has at most two representations through $s_k$ and $s'_k$. It follows that

$$\# < s_{k+1} > \leq \# < s_k > \# < s'_k >$$

and

$$\frac{\mu_{k+1}(s_{k+1})}{\mu_{k+1}(s'_{k+1})} \leq k + 2$$

The theorem is proved.

**Corollary 2.8.** *If $s_n, s'_n \in supp\mu_n$ and $|s_n - s'_n| \leq cq^{-n}$ then*

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} \leq (n+1)^c.$$

*Proof.* Let $s'_n = t_0 < t_1 < \cdots < t_k = s_n$ be k+1 consecutive points in supp$\mu_n$. Then by Proposition 2.1 we have $k \leq c$ and

$$\frac{\mu_n(s_n)}{\mu_n(s'_n)} = \frac{\mu_n(t_k)}{\mu_n(t_0)} = \frac{\mu_n(t_k)}{\mu_n(t_{k-1})} \cdot \frac{\mu_n(t_{k-1})}{\mu_n(t_{k-2})} \cdots \frac{\mu_n(t_1)}{\mu_n(t_0)}$$
$$\leq (n+1)(n+1)\cdots(n+1) \leq (n+1)^c.$$

## 3. Local dimensions of fractal

The following theorem is an extension of Proposition 2.1 in [6].

**Theorem 3.1.** *For $s \in supp\mu$, we have*

$$\alpha(s) = \lim_{n\to\infty} \frac{\log \mu_n(s_n)}{n \log q},$$

*provided that the limit exists. Otherwise, by taking the upper and lower limits, respectively, we get the formulas for $\alpha^*(s)$ and $\alpha_*(s)$.*

*Proof.* Suppose that there exists the limit

$$\alpha(s) = \lim_{h\to o^+} \frac{\log \mu(B(s,h))}{\log h}$$

where $B(s,h) = [s-h, s+h]$.

For $h > 0$, take n such that

$$q^{-n-1} < h < q^{-n}.$$

Then

$$\frac{\mu(B(s,q^{-n-1}))}{\log q^{-n}} \leq \frac{\mu(B(s,h))}{\log h} \leq \frac{\mu(B(s,q^{-n}))}{\log q^{-n-1}}.$$

Since

$$|s - s_n| \leq \sum_{i=n+1}^{\infty} q^{-i} r_n = \frac{r_n q^{-n}}{q-1},$$

we have

$$\mu(B(s,q^{-n})) \leq \mu_n(B(s,cq^{-n})) \leq \mu(B(s,2cq^{-n})),$$

where $c = \frac{r_n}{q-1} + 1$ is a constant depending only on m and q. Similarly, we have

$$\mu(B(s,q^{-n})) \geq \mu_n(B(s,c'q^{-n})).$$

Thus

$$\mu_n(B(s, c'q^{-n})) \le \mu(B(s, q^{-n})) \le \mu_n(B(s, cq^{-n})).$$

This implies

$$\frac{\log \mu_n(B(s, cq^{-n}))}{-n \log q} \le \frac{\log \mu(B(s, q^{-n}))}{-n \log q} \le \frac{\log \mu_n(B(s, c'q^{-n}))}{-n \log q}.$$

For $t \in B(s, cq) \cap \mathrm{supp}\mu$, we have

$$|t_n - s_n| \le 2cq^{-n} .$$

By corollary 2.8, we have

$$\mu_n(t_n) \le (n+1)^{2c} \mu_n(s_n).$$

This implies

$$\mu_n(B(s, cq^{-n})) \le (2c+1)(n+1)^{2c} \mu_n(s_n),$$

and

$$\lim_{n\to\infty} \frac{\log(\mu(B(s, q^{-n})))}{-n \log q} \ge \lim_{n\to\infty} \frac{\log \mu_n(s_n)}{-n \log q} = \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{-n \log q}.$$

Similarly, we have

$$\lim_{n\to\infty} \frac{\log(\mu(B(s, q^{-n})))}{-n \log q} \le \lim_{n\to\infty} \frac{|\log \mu_n(s_n)|}{-n \log q}.$$

This completes the proof.

The following corollaries can be proved by the same technique as those in [6] (by using theorema 3.1 and proposition 2.2).

**Corollary 3.2.** *For $s = \sum_{i=0}^{\infty} q^{-i} x_i \in \mathrm{supp}\mu$, we have*

$$\alpha(s) = \frac{\log(m+1)}{\log q} + \lim_{n\to\infty} \frac{\log \# < s_n >}{n \log q},$$

*provided that the limit exists. Otherwise, by taking the upper and lower limits respectively we get the formulas for $\alpha^*(s)$ and $\alpha_*(s)$.*

**Corollary 3.3.**

$$\overline{\alpha} = \alpha^* = \frac{\log(m+1)}{\log q},$$

*where*

$$\overline{\alpha} = \sup\{\alpha(s) : \ s \in \mathrm{supp}\mu\} \ \ \alpha^* = \sup\{\alpha^*(s) : \ s \in \mathrm{supp}\mu\}.$$

**Corollary 3.4.** *(see [6] Main Theorem). For $m = 2$, $r_0 = 0$, $r_1 = 1, r_2 = 3, q = m + 1 = 3$, we have*

$$\overline{\alpha} = \alpha^* = 1 \, .$$

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