VNU. JOURNAL OF SCIENCE, Mathematics - Physics. T.XVIII, No4 - 2002

ON THE ASYMPTOTICAL STABILITY FOR INDEX-k TRACTABLE DAEs

Dao Thi Lien
Teacher's Training College, Thai Nguyen University

Abstract. DAEs arise in various problems in the natural sciences and technology. The stability of DAEs was studied by many authors [3 - 9]. In [9] Tatyana Shtykel proposed a numerical parameter $\chi(A, B)$ characterising the asymptotical stability of the trivial solution of linear system index-1 DAEs

$$AX' + BX = 0,$$

with constant matrix A, B, where A is singular. In this paper we study the same parameter for linear system of index-k DAEs.

1. The index-k tractable DAEs

Consider the differential algebraic equation

$$AX' + BX = 0, \tag{1}$$

where A, B are constant matrices of order m satisfying

$$detA = 0, \quad rank[(cA + B)^{-1}A]^k = r.$$

Definition 1.(see [3]) *The equation (1) is called index-k tractable if the matrix pencil $\{A, B\}$ is regular with index-k.*

Since the matrix pencil is regular index-k and $rank[(cA+B)^{-1}A]^k = r$, there exist invertible matrices W, T such that

$$A = W\begin{pmatrix} I_r & O \\ O & U \end{pmatrix}T^{-1},$$
$$U^k = O,\ U^l \neq O, \quad \text{for all } l < k,$$
$$B = W\begin{pmatrix} -B_1 & O \\ O & I_{m-r} \end{pmatrix}T^{-1},$$

where I_s is the $s \times s$ identity matrix. Let us set

$$Q_0 = T\begin{pmatrix} O & O \\ O & U^{k-1} \end{pmatrix}T^{-1},\ P_0 = I - Q_0 = T\begin{pmatrix} I_r & O \\ O & I_{m-r} - U^{k-1} \end{pmatrix}T^{-1},$$
$$Q_1 = T\begin{pmatrix} O & O \\ O & U^{k-2} \end{pmatrix}T^{-1},\ P_1 = I - Q_1 = T\begin{pmatrix} I_r & O \\ O & I_{m-r} - U^{k-2} \end{pmatrix}T^{-1}, \ldots$$
$$Q_{k-2} = T\begin{pmatrix} O & O \\ O & U \end{pmatrix}T^{-1},\ P_{k-2} = I - Q_{k-2} = T\begin{pmatrix} I_r & O \\ O & I_{m-r} - U \end{pmatrix}T^{-1},$$
$$Q_{k-1} = T\begin{pmatrix} O & O \\ O & I_{m-r} \end{pmatrix}T^{-1},\ P_{k-1} = I - Q_{k-1} = T\begin{pmatrix} I_r & O \\ O & O \end{pmatrix}T^{-1}.$$

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Let

$$\mathcal{A} = A - BQ_{k-2} = W\begin{pmatrix} I_r & O \\ O & O \end{pmatrix} T^{-1},$$
$$N_1 = \ker \mathcal{A},\ S_1 = \{z \in R^m \ :\ BP_{k-2}z \in \operatorname{Im} \mathcal{A}\}.$$

It is clear that Q_{k-1} is canonical projector onto N_1 along S_1 and P_{k-1} is canonical projector onto S_1 along N_1. Denote

$$A_1 = \mathcal{A} + BP_{k-2}Q_{k-1} = W\begin{pmatrix} I_r & O \\ O & I_{m-r} - U \end{pmatrix} T^{-1}.$$

It is easy to see that

$$A_1^{-1} = T\begin{pmatrix} I_r & O \\ O & I_{m-r} + U + ... + U^{k-1} \end{pmatrix} W^{-1}.$$

Multiplying (1) by $P_{k-1}A_1^{-1}$, $Q_0A_1^{-1}$, $Q_1A_1^{-1}, ..., Q_{k-1}A_1^{-1}$, respectively, we obtain:

$$\begin{cases} (P_{k-1}X)' + P_{k-1}A_1^{-1}BP_{k-1}X = 0, \\ Q_0X = 0, \\ (Q_0X)' + Q_1X + Q_0X = 0, \\ \dots \\ (Q_{k-2}X)' + (Q_{k-3}X)' + ... + (Q_0X)' + Q_{k-1}X + Q_{k-2}X + ... + Q_0X = 0. \end{cases} \tag{2}$$

Because of

$$P_{k-1} + Q_0 + ... + Q_{k-1} = T\begin{pmatrix} I_r & O \\ O & I_{m-r} + U + ... + U^{k-1} \end{pmatrix} T^{-1} = K$$

is invertible ,hence the system (2) is equivalent to (1); and from the system(2) we have

$$\begin{cases} (P_{k-1}X)' + P_{k-1}A_1^{-1}BP_{k-1}X = 0, \\ Q_{k-1}X = 0. \end{cases} \tag{3}$$

X is a solution of (1) if and only if $P_{k-1}X$ is the one of (3).

Definition 2.(see [9]) *A matrix valued function $\mathcal{G}(t) = \mathcal{G}(t, A, B) \in C^1$ is called the Green matrix of equation (1) if it satisfies the initial value problem (IVP)*

$$\begin{cases} \frac{d}{dt}\mathcal{G}(t) = M\mathcal{G}(t) \ \ (t > 0), \\ \mathcal{G}(0) = P_{k-1}, \end{cases} \tag{4}$$

where $M = -P_{k-1}A_1^{-1}B$.

It is easy to verify that $M = P_{k-1}M = MP_{k-1}$; and consequently $\mathcal{G}(t) = P_{k-1}e^{tM}$ is the unique solution of the IVP (4).

Therefore the general solution of equation (1) is of the form

$$X(t) = \mathcal{G}(t)X_0 = P_{k-1}e^{tM}X_0,$$

where X_0 is an arbitrary constant vector. Thus, we have proved the following

Theorem 1. *Let $\{A, B\}$ be a regular pencil with index-k, Q_{k-1} the canonical projector onto N_1 along S_1, and $P_{k-1} = I - Q_{k-1}$. Then the initial value problem*

$$\begin{cases} AX' + BX = 0, \\ P_{k-1}(X(0) - X_0) = 0, \end{cases}$$

for all $X_0 \in R^m$ has a unique solution $X(t)$ given by $X(t) = P_{k-1}e^{tM}X_0$ with the matrix $M = -P_{k-1}A_1^{-1}B$.

This theorem seems not new but the method of proof is appropriate for studying the asymptotical stability of index-k tractable DAEs.

2. The criterion of asymptotical stability of the trivial solution of DAEs with index-k

2.1. The asymptotical stability of the trivial solution of DAEs with index-k

Definition 3.(see [3 - 7],[9]) *The trivial solution $X \equiv 0$ of (1) is called stable in the sense of Liapunov if for certain projector Π along the maximal invariant subspace of the matrix pencil $\{A, B\}$ associated with the infinite eigenvalue the IVP*

$$\begin{cases} AX' + BX = 0, \\ \Pi(X(0) - X_0) = 0, \end{cases}$$

for all $X_0 \in R^m$ has a solution $X(t, X_0)$ defined on $[0, +\infty)$. Moreover, for each $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that $||X(t, X_0)|| < \varepsilon$ for all $t \geq 0$ and for all $X_0 \in R^m$ with $||\Pi X_0|| < \delta$. Here we choose $\Pi = P_{k-1}$.

Definition 4.(see [6],[9]) *The trivial solution $X \equiv 0$ of (1) is said to be asymptotically stable in the sense of Liapunov if it is stable and there is a $\delta_0 > 0$ such that for all $X_0 \in R^m$ satisfying the inequality $||\Pi X_0|| < \delta_0$ one gets $X(t, X_0) \to 0$ as $t \to +\infty$.*

Lemma. *If U is a k-nipotent matrix then $det(\lambda U + I_{m-r}) \neq 0$ for all $\lambda \in C$.*

Theorem 2. *The trivial solution $X \equiv 0$ of (1) is asymptotically stable if and only if all finite eigenvalues of the matrix pencil $\{A, B\}$ have negative real parts.*

2.2. The criterion of asymptotical stability

Let all finite eigenvalues of the pencil $\{A, B\}$ with index-k have negative real parts. Assume that the matrices M and P_{k-1} have the structures described above. We consider the Liapunov equation

$$XM + M^*X = -P_{k-1}^*FP_{k-1}, \tag{5}$$

with an unknown matrix X. The matrix F is supposed to be hermitian and positive definite. Since

$$||P_{k-1}e^{tM}|| \leq \gamma(r)(\frac{||B_1||}{\sigma})^{r-1}e^{-t\sigma/2},$$

the following intergral

$$H_k = \int_0^{+\infty} e^{tM^*} P_{k-1}^* F P_{k-1} e^{tM} dt + Q_{k-1}^* F Q_{k-1}$$

converges. On the other hand H_k is hermitian and positive definite and H_k satisfies the equation (5).

Theorem 3. *If the matrix pencil $\{A, B\}$ has index-k and all its finite eigenvalues belong to the negative complex half plane, then $\chi(A, B) = 2\|A - BQ_{k-2}\|\|B\|\|H_k\| < \infty$. Inversly we assume that $\chi(A, B) < \infty$ and H_k is a solution of (5).Then the following inequality is valid*

$$\|X(t)\| \leq \sqrt{2\|A - BQ_{k-2}\|\|B\|\|H_k\|}\, e^{\frac{-t\|A-BQ_{k-2}\|\|B\|}{2\|A-BQ_{k-2}\|\|B\|\|H_k\|\|A_1^{-1}\|^2}} \|P_{k-1}X_0\|$$

$$\leq \sqrt{\chi(A, B)}\, e^{\frac{-t\|A-BQ_{k-2}\|\|B\|}{\chi(A,B)\|A_1^{-1}\|^2}} \|P_{k-1}X_0\|.$$

This means that the trivial solution $X \equiv 0$ is asymptotically stable.

Acknowlegment. We wish to thank Prof. Pham Ky Anh and Dr. Nguyen Huu Du for their valuable suggestions during the preparation of this paper.

References

1. B. P. Demidovish, *Lectures on the Mathematics Theory of Stability*, Nauka, Moscow 1967 (in Russian).
2. S. K. Godunov, *Ordinary differential equations with constant coefficients*, T.V.1 Boundary problems Novosibirk University 1994 (Russian).
3. E. Griepentrog, R. Marz, *Differential Algebraic Equations and their Numerical Treatment*, Teubner - Texte Math. 88, Leipzig 1986.
4. M. Hanke, Rodriguezs and R. Antonio, *Asymptotic properties of regularized differential algebraic equations*, Preprint Nr 95-6, Humboldt Universitat zu Berlin 1995.
5. R. Lamair, R. Marz and M. M. R. Mattheij, *On the stability behaviour of systems obtained by index reduction*, Preprint Nr 92-27, Humboldt Universitat zu Berlin 1992.
6. R. Marz, *Criteria for the trivial of differential algebraic equation with small nonlinear to be asymptotically stable*, Preprint Nr 97-13, Humboldt Universitat zu Berlin 1997.
7. C. Tischendorf, *On stability of solutions of autonomous index-1 tractable and quasilinear index-2 tractable DAEs*, Preprint Nr 91-25, Humboldt Universitat zu Berlin 1991.
8. C. Tischendorf, *Feasibility and stable behaviour of the BDF. Applied to index-2 differential algebraic equations*, Preprint Nr 93-04, Humboldt Universitat zu Berlin 1993.
9. Tatyana Shtykel, *On the criterion of asymptotic stability for index-1 tractable differential algebraic equations*, Preprint Nr 98-6, Humboldt Universitat zu Berlin 1998.