HILBERT-KUNZ MULTIPLICITY, PHANTOM PROJECTIVE DIMENSION BY SPECIALIZATION

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Abstract. The ground field k is assumed to be infinite. We denote by K a field extension of k. Let $x = (x_1, ..., x_n)$ be indeterminates. Let $u = (u_1, ..., u_m)$ be a family of indeterminates, which are considered as parameters. The specialization of an ideal I of $R = k(u)[x]$ with respect to the substitution $u \to \alpha = (\alpha_1, ...\alpha_m) \in K^m$ was defined as the ideal I_α, which is generated by the set $\{f(\alpha, x)|f(u, x) \in I \cup k[u, x]\}$. The theory of specialization of ideals was introduced by W. Krull. Krull has showed that the ideal I_α inherits most of basic properties if I and it was used to prove many important results in algebra and in algebraic geometry. In a paper, we introducedand studied specializations of finitely generated modules over a local ring R_P, where P is a prime ideal of R, and showed that the multiplicity of a module is preserved through a specialization. Now, the problem of concern is the preservation of Hilbert-Kunz multiplicity and of finite phantom projective dimension of modules through specialization when k has positive prime characteristic p. The purpose of this paper is to prove the preservation of Hilbert-Kunz multiplicity, mixed multiplicity and of finite phantom projective dimension of a module through specializations.

0. Introduction

The purpose of this paper is to prove the preservation of Hilbert-Kunz multiplicity, mixed multiplicity and of finite phantom projective dimension of a module through specializations. First of all we fix some notations that will be used throughout this paper. The groundfield k is always assumed to be infinite and has positive prime characteristic p. We denote by K a field extension of k and $K \supseteq k^p$. Aggregates such as $x_1, \ldots, x_n$ or $\alpha_1, \ldots, \alpha_m$, where $\forall \alpha_i \in K$, will often be written x or α. Accordingly, the ring or field extensions $k[x_1, \ldots, x_n]$ or $k(\alpha_1, \ldots, \alpha_m)$ will be written $k[x]$ or $k(\alpha)$, with evident variants of these designations. For short we set $R = k(u)[x]$ and $R_\alpha = k(\alpha)[x]$. In this paper, we shall say that a property holds for almost all α if it holds for all α except perhaps those lying on a proper algebraic subvariety of K^m. For convenience we often obmit the phrase "for almost all α" when we are working with specializations.

The theory of specialization of ideals was introduced by W. Krull [3]. Krull defined the specialization of an ideal I of $R = k(u)[x]$ with respect to the substitution $u \to \alpha$ as the ideal I_α, which is generated by the set $\{f(\alpha, x) \mid f(u, x) \in I \cap k[u, x]\}$. The ideal I_α inherits most of basic properties of I and it was used to prove many important results in algebra and in algebraic geometry.

Let P be a prime ideal of R. In [8], we introduced and studied specializations of finitely generated modules over a local ring R_P and the multiplicity of a module is preserved

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX through a specialization was also be showed in [9]. Now, the problem of concern is the preservation of Hilbert-Kunz multiplicity and of finite phantom projective dimension of modules through specializations when k has posive prime characteristic p.

The paper is divided in three sections. In section 1 we will discuss the notion of Hilbert-Kunz multiplicity, which has been introduced by Kunz [5] in 1969, and has been studied in detail by Monsky [7], Watanabe and Yoshida [12]. We will also discuss the preservation of the notion of mixed multiplicity, which was introduced by Bernard Teissier, (see, for example, [10]). Section 2 will present our discuss tight closure of ideals and of modules by specializations. While preservation of finite phantom projective dimension of a module [1], [2], by specializations will be studied in section 3.

1. Hilbert-Kunz multiplicity by specialization

We first recall some basic facts from [8] about the specializations of ideals and modules. Let P be a prime ideal of R and let S denotes the local ring R_P. It is supposed that $\wp$ is an arbitrary associated prime ideal of P_α, and we set $S_\alpha = (R_\alpha)_\wp$. The notion S_α is not unique. However, all local rings S_α have the same dimension as S. Here the maximal ideals PS of S and $\wp S_\alpha$ of S_α will be denoted by $\mathfrak{m}$ and $\mathfrak{n}$, respectively. We commence with the following results, which are useful late.

Lemma 1.1. [11, Lemma 1.5] *Let $I \subseteq P$ be an ideal of R. Then, for almost all α,*
(i) *$I_\alpha S_\alpha$ is unmixed if IS is unmixed,*
(ii) *$ht\ I_\alpha S_\alpha = ht\ IS$.*

We start by recalling the definition of a specialization of a finitely generated S-module. Let M be a finitely generated S-module. Suppose that $S^h \xrightarrow{\phi} S^t \longrightarrow M \longrightarrow 0$ is a finite free presentation of M, where the matrix of ϕ is (a_{ij}) with all $a_{ij} \in S$. We know that an arbitrary element $f \in R$ may be written in the form

$$f = \frac{p(u,x)}{q(u)},\ p(u,x) \in k[u,x],\ q(u) \in k[u] \setminus \{0\}.$$

For any α such that $q(\alpha) \neq 0$ we define $f_\alpha := p(\alpha,x)/q(\alpha)$. For every element

$$a = \frac{f}{g} \in S,\ f,g \in R,\ g \neq 0,$$

we define $a_\alpha := f_\alpha/g_\alpha$ if $g_\alpha \neq 0$. Then a_α is uniquely determined and belongs to S_α for almost all α. For almost all α, there is a homomorphism $\phi_\alpha : S_\alpha^q \longrightarrow S_\alpha^p$ given by the matrix $((a_{ij})_\alpha)$. As the definition of a specialization of module, we obtain a finite free presentation

$$S_\alpha^q \xrightarrow{\phi_\alpha} S_\alpha^p \longrightarrow M_\alpha \longrightarrow 0,$$

where $M_\alpha = \mathrm{Coker}\phi_\alpha$, see [8]. The S_α-module M_α is called a specialization of M.

Let $\mathbf{G}_\bullet :\ 0 \longrightarrow S^{d_\ell} \xrightarrow{\phi_\ell} S^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S^{d_1} \xrightarrow{\phi_1} S^{do} \longrightarrow M \longrightarrow 0$ be a free resolution of M. Then, the below complex will be obtained:

$$(\mathbf{G}_\bullet)_\alpha :\ 0 \longrightarrow S_\alpha^{d_\ell} \xrightarrow{(\phi_\ell)_\alpha} S_\alpha^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S_\alpha^{d_1} \xrightarrow{(\phi_1)_\alpha} S_\alpha^{do} \longrightarrow M_\alpha \longrightarrow 0.$$

By the following lemma we shall see that the specialization of a minimal free resolution of modules is again minimal.

Lemma 1.2. *If $\mathbf{G}_\bullet$ is a minimal free resolution of M, then $(\mathbf{G}_\bullet)_\alpha$ is also a minimal resolution of M_α for almost all α.*

Proof Let $\mathbf{G}_\bullet : 0 \longrightarrow S^{d_\ell} \xrightarrow{\phi_\ell} S^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S^{d_1} \xrightarrow{\phi_1} S^{d_0} \longrightarrow M \longrightarrow 0$ be a minimal free resolution of M, where ϕ_t is given by the matrix $A_t = (a_{tij})$ with all enties $a_{tij} \in \mathfrak{m}$ for all $t = 1, \ldots, \ell$. By [8, Lemma 2.1], the complex

$$(\mathbf{F}_\bullet)_\alpha : 0 \longrightarrow S_\alpha^{d_\ell} \xrightarrow{(\phi_\ell)_\alpha} S_\alpha^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S_\alpha^{d_1} \xrightarrow{(\phi_1)_\alpha} S_\alpha^{d_0} \longrightarrow M_\alpha \longrightarrow 0$$

is a free resolution of M_α. Since $(A_t)_\alpha = ((a_{tij})_\alpha)$ is the representing matrix of $(\phi_t)_\alpha$ and since $(a_{tij})_\alpha \in \mathfrak{m}_\alpha$, the free resolution $(\mathbf{G}_\bullet)_\alpha$ of M_α is minimal for almost all α.

The dimension of S is denoted by d. Set $q = p^e$ with $e \in \mathbb{N}$. We write $S^{1/q}$ for the ring obtained by adjoining all qth roots of elements of S. We write $S^\infty = \bigcup_q S^{1/q}$. Denote by F^e the Frobenius functor. The inclusion map $S \subseteq S^{1/q}$ is isomorphic with the map $F^e : S \longrightarrow S$, where $F^e(a) = a^q$. Let $\mathfrak{a}$ be an ideal of S. Let $\mathfrak{a}^{[q]}$ denote the ideal generated by the qth powers of all elements of $\mathfrak{a}$. Note that if T denotes a set of generators for $\mathfrak{a}$, the $\{t^q \mid t \in T\}$ generates $\mathfrak{a}^{[q]}$.

Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal of S and M a finitely generated S-module. We know that the usual multiplicity of M with respect to $\mathfrak{q}$ is defined as the number

$$e(\mathfrak{q}, M) = \lim_{h \to \infty} \frac{\ell(M/\mathfrak{q}^h M).d!}{h^d}.$$

In [7],[12], the *Hilbert-Kunz multiplicity* of M with respect to $\mathfrak{q}$ is defined as follows

$$e_{HK}(\mathfrak{q}, M) = \lim_{e \to \infty} \frac{\ell(M/\mathfrak{q}^{[p^e]} M)}{p^{de}}.$$

Note that the limit of right-hand side always exists and $e_{HK}(\mathfrak{q}, M) \geq 0$ and equality holds if and only if $\dim M < \dim S$. If $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ is a short exact sequence of finitely generated S-modules, then $e_{HK}(\mathfrak{q}, M) = e_{HK}(\mathfrak{q}, L) + e_{HK}(\mathfrak{q}, N)$, see [7], [12]. In particular, we shall write as $e(\mathfrak{q})$ instead of $e(\mathfrak{q}, S)$, and $e_{HK}(\mathfrak{q})$ instead of $e_{HK}(\mathfrak{q}, S)$. From Lech's lemma follows $e_{HK}(\mathfrak{q}) = e(\mathfrak{q})$ for every parameter ideal $\mathfrak{q}$. In [9], we proved the preservation of the multiplicity of a module through a specialization. Now we will prove that the Hilbert-Kunz multiplicity of M with respect to $\mathfrak{q}$ is preserved through a specialization, too.

Theorem 1.3. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal and M a finitely generated S-module of dimension d. Then, for almost all α, we have $e_{HK}(\mathfrak{q}_\alpha, M_\alpha) = e_{HK}(\mathfrak{q}, M)$.*

Proof. Let $\mathbf{G}_\bullet : 0 \longrightarrow S^{d_\ell} \xrightarrow{\phi_\ell} S^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S^{d_1} \xrightarrow{\phi_1} S^{d_0} \longrightarrow M \longrightarrow 0$ be a minimal free resolution of M. By Lemma 1.2, the complex

$$(\mathbf{G}_\bullet)_\alpha : 0 \longrightarrow S_\alpha^{d_\ell} \xrightarrow{(\phi_\ell)_\alpha} S_\alpha^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S_\alpha^{d_1} \xrightarrow{(\phi_1)_\alpha} S_\alpha^{d_0} \longrightarrow M_\alpha \longrightarrow 0$$

is a minimal free resolution of M_α. We first observe that the following equalities

$$e_{HK}(\mathfrak{q}, M) = \sum_{i=0}^{\ell} (-1)^{i+1} e_{HK}(\mathfrak{q}, S^{d_i}), \; e_{HK}(\mathfrak{q}_\alpha, M_\alpha) = \sum_{i=0}^{\ell} (-1)^{i+1} e_{HK}(\mathfrak{q}_\alpha, S_\alpha^{d_i})$$

follow from the additivity property of the Hilbert-Kunz multiplicity [7]. Since $\dim M_\alpha = \dim M$ by [8, Theorem 2.6], to obtain $e_{HK}(\mathfrak{q}_\alpha, M_\alpha) = e_{HK}(\mathfrak{q}, M)$, it is therefore sufficient to verify that $e_{HK}(\mathfrak{q}, S^{d_i}) = e_{HK}(\mathfrak{q}_\alpha, S_\alpha^{d_i})$. By definition we get

$$e_{HK}(\mathfrak{q}, S^{d_i}) = d_i.e_{HK}(\mathfrak{q}) = d_i.e(\mathfrak{q}).$$

Since $e(\mathfrak{q}_\alpha) = e(\mathfrak{q})$ by [9, Theorem 1.6], we have $e_{HK}(\mathfrak{q}, S^{d_i}) = d_i.e(\mathfrak{q}) = d_i.e(\mathfrak{q}_\alpha) = e_{HK}(\mathfrak{q}_\alpha, S_\alpha^{d_i})$. This completes the proof.

Because $e_{HK}(\mathfrak{q}^{[q]}, M) = q^d.e_{HK}(\mathfrak{q}, M)$ and $e_{HK}(\mathfrak{q}_\alpha^{[q]}, M_\alpha) = q^d.e_{HK}(\mathfrak{q}_\alpha, M_\alpha)$ by [7], as an immediate consequence of Theorem 1.3 we have the following corollary.

Corollary 1.4. *Let $\mathfrak{q}$ be an $\mathfrak{m}$-primary ideal and M a finitely generated S-module. Then, for almost all α, we have $e_{HK}(\mathfrak{q}_\alpha^{[q]}, M_\alpha) = e_{HK}(\mathfrak{q}^{[q]}, M)$ for all $q = p^e, e \geq 1$.*

We now recall the notion of *mixed multiplicity* [10]. Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathfrak{m}$-primary ideals of S. In the case $d = 2, M = S$, the number

$$e(\mathfrak{a}|\mathfrak{b}) = \frac{1}{2}\{e(\mathfrak{a}\mathfrak{b}) - e(\mathfrak{a}) - e(\mathfrak{b})\}$$

is called the *mixed multiplicity* of $\mathfrak{a}$ and $\mathfrak{b}$. We shall see that the mixed multiplicity is unchanged through a specialization.

Proposition 1.5. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathfrak{m}$-primary ideals of S. Then, for almost all α, we have $e(\mathfrak{a}_\alpha^{[q]}|\mathfrak{b}_\alpha^{[q]}) = e(\mathfrak{a}^{[q]}|\mathfrak{b}^{[q]})$ for all $q = p^e, e \geq 1$.*

Proof. Upon simple computation, from the above definition we get

$$\begin{aligned} e(\mathfrak{a}^{[q]}|\mathfrak{b}^{[q]}) &= \frac{1}{2}\{e((\mathfrak{a}\mathfrak{b})^{[q]}) - e(\mathfrak{a}^{[q]}) - e(\mathfrak{b}^{[q]})\} \\ &= \frac{q^2}{2}\{e(\mathfrak{a}\mathfrak{b}) - e(\mathfrak{a}) - e(\mathfrak{b})\} = q^2 e(\mathfrak{a}|\mathfrak{b}). \end{aligned}$$

By definition, $\mathfrak{a}_\alpha$ and $\mathfrak{b}_\alpha$ are $\mathfrak{n}$-primary ideals. In the same way above, we get

$$e(\mathfrak{a}_\alpha^{[q]}|\mathfrak{b}_\alpha^{[q]}) = q^2 e(\mathfrak{a}_\alpha|\mathfrak{b}_\alpha).$$

Since $e(\mathfrak{q}_\alpha) = e(\mathfrak{q})$ for every parameter ideal $\mathfrak{q}$ by [9, Theorem 1.6], $e(\mathfrak{a}_\alpha|\mathfrak{b}_\alpha) = e(\mathfrak{a}|\mathfrak{b})$. Then the proof is completed.

2. Tight closure for ideals and for modules by specialization

In this section we shall study the specializations of tight closure for ideals and for finitely generated S-modules and shows that many properties of modules will be preserved by a substitution $u \to \alpha$. We first will recall some definitions of the tight closure theory for ideals and for modules.

We set $S^0 = S \setminus \{0\}$, and $S_\alpha^0 = S_\alpha \setminus \{0\}$. Let $\mathfrak{a}$ be an ideal of S. An element $a \in S$ is said to be in the *tight closure* $\mathfrak{a}^*$ of $\mathfrak{a}$ if there exists an element $c \in S^0$ such that for all large q we have $ca^q \in \mathfrak{a}^{[q]}$. If $\mathfrak{a}^* = \mathfrak{a}$, we say that $\mathfrak{a}$ is *tightly closed*. Since S is a local regular ring, as an immediate consequence of [2, (4.4) Theorem], we have the following lemma.

Lemma2.1. *Every ideal of S is tightly closed.*

Proposition 2.2. *If $\mathfrak{a}$ is an ideal of S, then $(\mathfrak{a}^*)_\alpha = (\mathfrak{a}_\alpha)^*$ for almost all α.*

Proof. Since S and S_α are local regular rings, we have $\mathfrak{a}^* = \mathfrak{a}$ and $(\mathfrak{a}_\alpha)^* = \mathfrak{a}_\alpha$ by Lemma 2.1. Then, we obtain $(\mathfrak{a}^*)_\alpha = \mathfrak{a}_\alpha = (\mathfrak{a}_\alpha)^*$ for almost all α.

For an ideal $\mathfrak{a}$ of S we denote it's integral closure by $\overline{\mathfrak{a}}$. Then we have:

Proposition 2.3. *If $\mathfrak{a}$ is an $\mathfrak{m}$-primary ideal, then $\overline{(\overline{\mathfrak{a}})_\alpha} = \overline{\mathfrak{a}_\alpha}$ for almost all α.*

Proof. Because $\mathfrak{a} \subseteq \overline{\mathfrak{a}}$, there is $\mathfrak{a}_\alpha \subseteq (\overline{\mathfrak{a}})_\alpha$. By [9, Theorem 1.6], $e(\mathfrak{a}_\alpha) = e(\mathfrak{a}), e((\overline{\mathfrak{a}})_\alpha) = e(\overline{\mathfrak{a}})$. From $e(\mathfrak{a}) = e(\overline{\mathfrak{a}})$ follows $e(\mathfrak{a}_\alpha) = e((\overline{\mathfrak{a}})_\alpha)$. Since S, S_α are localizations of polynomial rings, they are quasi-unmixed rings by [6, Theorem 31.6]. By [3, Theorem 4.3], we get $\overline{(\overline{\mathfrak{a}})_\alpha} = \overline{\mathfrak{a}_\alpha}$.

Note also that we can identify $F^e(S^s)$ with S^s in such a way that if $z = (a_1, \dots, a_s) \in S^s$ then $z^q = (a_1^q, \dots, a_s^q)$. Suppose that the homomorphism $\phi : S^s \longrightarrow S^h$ between finitely generated free S-modules is described by an $h \times s$-matrix (a_{ij}), then the map $F^e(\phi) : S^s \longrightarrow S^h$ is described by the matrix (a_{ij}^q). The S-module M is always assumed to be finitely generated. Let $S^s \xrightarrow{\phi} S^h \longrightarrow M \longrightarrow 0$ be a finite free presentation of M. Since F^e is exact, the sequence $S^s \xrightarrow{F^e(\phi)} S^h \longrightarrow M^{[q]} \longrightarrow 0$ is a finite free presentation of $F^e(M) = M^{[q]} = \text{Coker}F^e(\phi)$.

We observe that there is a canonical map $M \to M^{[q]}$ that sends z to $1 \otimes z$. If $z \in M$, we shall write z^q for the image of z in $M^{[q]}$. Let $N \subseteq M$ be a submodule of M. We say that $z \in M$ is in the *tight closure* N^* of N if there exist $c \in S^0$ and and integer q' such that $cz^q \in N^{[q]}$ for all $q \geq q'$. If $N = N^*$, we say that N is *tightly closed* (in M). The following proposition is the generalization of Proposition 2.2.

Proposition 2.4. *For every submodule N of the finitely generated S-module M, there is $(N^*)_\alpha = (N_\alpha)^*$ for almost all α.*

Proof. Since S is regular ring, from [2, (8.7) Proposition] we know that every submodule N of M is tightly closed. Since $N^* = N$ and $(N_\alpha)^* = N_\alpha$, we get $(N_\alpha)^* = N_\alpha = (N^*)_\alpha$ for almost all α.

The following proposition shows that the operation $^{[q]}$ commutes with specialization.

Proposition 2.5. *Let M be a finitely generated S-module. Then $(M^{[q]})_\alpha \cong (M_\alpha)^{[q]}$ for almost all α.*

Proof. Assume that $S^s \xrightarrow{\phi} S^h \longrightarrow M \longrightarrow 0$ is a finite free presentation of M. Then $S^s \xrightarrow{F^e(\phi)} S^h \longrightarrow M^{[q]} \longrightarrow 0$ and $S^s_\alpha \xrightarrow{\phi_\alpha} S^h_\alpha \longrightarrow M_\alpha \longrightarrow 0$ are finite free presentations of $M^{[q]}$ and M_α, respectively. From these exact sequences, we obtain two exact sequences $S^s_\alpha \xrightarrow{F^e(\phi)_\alpha} S^h_\alpha \longrightarrow (M^{[q]})_\alpha \longrightarrow 0$ and $S^s_\alpha \xrightarrow{F^e(\phi_\alpha)} S^h_\alpha \longrightarrow (M_\alpha)^{[q]} \longrightarrow 0$ as finite free presentations of $(M^{[q]})_\alpha$ and $(M_\alpha)^{[q]}$, respectively. Upon simple computation, we see that $F^e(\phi)_\alpha$ and $F^e(\phi_\alpha)$ have the same matrix $((a_{ij}^q)_\alpha)$. Hence $(M^{[q]})_\alpha \cong (M_\alpha)^{[q]}$ for almost all α.

This isomorphism enables us to identify the module $(M^{[q]})_\alpha$ with the module $(M_\alpha)^{[q]}$.

3. Finite phantom projective dimension by specialization

Now we want to discuss the concept of finite phantom projective dimension in [1]. First we will need several definitions. Let

$$\mathbf{G}_\bullet : 0 \longrightarrow G_\ell \xrightarrow{\phi_\ell} G_{\ell-1} \longrightarrow \cdots \longrightarrow G_1 \xrightarrow{\phi_1} G_0 \longrightarrow 0$$

be a finite complex of finitely generated free S-modules. The complex $\mathbf{G}_\bullet$ is called to have *phantom homology* if $\mathrm{Ker}\phi_i \subseteq (\Im\phi_{i+1})^*$ for all $i \geq 1$. The complex $\mathbf{G}_\bullet$ is said to be *stably phantom acyclic* if the complex $F^e(\mathbf{G}_\bullet)$ has phantom homology for $i \geq 1$ and for all integers $e \geq 0$. A finite stably phantom acyclic complex $\mathbf{G}_\bullet$ of projective modules of constant rank is said to be a *finite phantom projective resolution* of its augmentation $M = H_0(\mathbf{G}_\bullet)$. Then the module M is said to have *finite phantom projective dimension* ℓ, which is denoted by $\mathrm{ppd}M$ and is defined to be the length of the shortest complex $\mathbf{G}_\bullet$. M is *phantom perfect* if $\mathrm{ppd}M = \min\{\ \mathrm{ht}(\mathrm{Ann}M)(S/\mathfrak{q}) \mid \mathfrak{q}$ is a minimal prime of $S\}$, see [1]. The following proposition shows that the finite phantom projective dimension of a module is unchanged through a specialization.

Theorem 3.1. *Let M be a finitely generated S-module. Then $\mathrm{ppd}M_\alpha = \mathrm{ppd}M$ for almost all α.*

Proof. We denote $\mathrm{ppd}M$ by ℓ. Without loss of generality we may assume that

$$\mathbf{G}_\bullet : 0 \longrightarrow S^{d_\ell} \xrightarrow{\phi_\ell} S^{d_{\ell-1}} \longrightarrow \cdots \longrightarrow S^{d_1} \xrightarrow{\phi_1} S^{d_0} \longrightarrow M \longrightarrow 0$$

is a minimal finite phantom free resolution of M with minimal length. By Lemma 1.2, the complex $(\mathbf{G}_\bullet)_\alpha$ is a minimal free resolution of M_α. By Phantom acyclicity criterion, e.g. [1, (5.3) Theorem], $(\mathbf{G}_\bullet)_\alpha$ is a minimal finite phantom free resolution of M_α. Hence, there is $\mathrm{ppd}M_\alpha = \ell = \mathrm{ppd}M$ for almost all α.

Corollary 3.2. *Let M be a finitely generated S-module. Then $\mathrm{ppd}(M_\alpha)^{[q]} = \mathrm{ppd}M^{[q]}$ for almost all α.*

Proof. Because S, S_α are regular rings, the functor F^e is exact. Then, $\mathrm{ppd}M^{[q]} = \mathrm{ppd}M$ and $\mathrm{ppd}(M_\alpha)^{[q]} = \mathrm{ppd}M_\alpha$. Since $(M^{[q]})_\alpha = (M_\alpha)^{[q]}$ by Proposition 2.5 and $\mathrm{ppd}(M^{[q]})_\alpha = \mathrm{ppd}M^{[q]}$ by Theorem 3.1, therefore $\mathrm{ppd}(M_\alpha)^{[q]} = \mathrm{ppd}M^{[q]}$ for almost all α.

Corollary 3.3. *Let M be a finitely generated S-module. If M is phantom perfect, so is M_α for almost all α.*

Proof. Assume that M is phantom perfect. Then

$$\mathrm{ppd}M = \min\{\ \mathrm{ht}(\mathrm{Ann}M)(S/\mathfrak{q}) \mid \mathfrak{q} \text{ is a minimal prime of } S\}.$$

Since S is locally equidimensional, there is

$$\min\{\ \mathrm{ht}(\mathrm{Ann}M)(S/\mathfrak{q}) \mid \mathfrak{q} \text{ is a minimal prime of } S\} = \mathrm{ht}\mathrm{Ann}M.$$

Since $\mathrm{Ann}M_\alpha = (\mathrm{Ann}M)_\alpha$ by [8, Theorem 2.6], $\mathrm{ht}\mathrm{Ann}M_\alpha = \mathrm{ht}\mathrm{Ann}M$ by lemma 1.1. Therefore $\mathrm{ppd}M_\alpha = \mathrm{ht}\mathrm{Ann}M_\alpha$ follows from Theorem 3.1. Hence, M_α is phantom perfect.

An element $z \in S$ is said to be a *phantom nonzerodivisor* on M if for all $t \geq 1$ and all $e \in \mathbb{N}$ one has that the annihilator of z^t in $F^e(M)$ is contained in the tight closure of 0 in $F^e(M)$. M is said to have *phantom depth* 0 if there is no element of $\mathfrak{m}$ that is a phantom nonzerodivisor on M.

Corollary 3.4. *Let M be a finitely generated S-module. If M has phantom depth 0, then M_α has again phantom depth 0 for almost all α.*

Proof. By [1, (5.9) Proposition] we know that M has phantom depth 0 if and only if $\mathrm{ppd}M = \mathrm{htm}$. By Theorem 3.1, $\mathrm{ppd}M_\alpha = \mathrm{ppd}M$. By Lemma 1.1, $\mathrm{htm}_\alpha = \mathrm{htm}$. Then $\mathrm{ppd}M_\alpha = \mathrm{htm}_\alpha$. Using [1, (5.9) Proposition] again, M_α has phantom depth 0.

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