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Local polynomial convexity of union of two graphs with CR isolated singularities

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Abstract. We give sufficient conditions so that the union of two graphs with CR isolated singularities in $\mathbf{C}^2$ is locally polynomially convex at a singularly point. Using this result and some ideas in previous work, we obtain a new result about local approximation continuous function.

1. Introduction

We recall that for a given compact K in $\mathbf{C}^n$, by $\hat{K}$ we denote the polynomial convex hull of K i.e.,

$$\hat{K} = \{z \in \mathbf{C}^n : |p(z)| \leq \|p\|_K \text{ for every polynomial } p \text{ in } \mathbf{C}^n\}.$$

We say that K is polynomially convex if $\hat{K} = K$. A compact K is called locally polynomially convex at $a \in K$ if there exists the closed ball $B(a)$ centered at a such that $B(a) \cap K$ is polynomially convex.

A smooth real manifold $S \subset \mathbf{C}^n$ is said to be *totally real* at $a \in S$ if the tangent plane $T_S(a)$ of S at a contains no complex line. A point $a \in S$ is not totally real that will be called a *CR singularity*. By the result of Wermer, if K is contained in totally real smooth submanifolds of $\mathbf{C}^2$ then K is locally polynomially convex at all point $a \in K$ (see [1], chapter 17). Note that union of two polynomially convex sets which can be not polynomially convex set. Let D be a small closed disk in the complex plane, centered at the origin and

$$M_1 = \{(z, \bar{z}) : z \in D\}; M_2 = \{(z, \bar{z} + \varphi(z)) : z \in D\},$$

where φ is a C^1 function in neighborhood of 0, $\varphi(z) = o(|z|)$. Then M_1, M_2 are totally real(locally contained in a totally real manifold), so that M_1, M_2 are locally polynomially convex at 0. The local polynomially convex hull of $M_1 \cup M_2$ is essentially studied by Nguyen Quang Dieu (see [2,3]).

Let

$$X_1 = \{(z, \bar{z}^n) : z \in D\}, X_2 = \{(z, \bar{z}^n + \varphi(z)) : z \in D\}, \qquad *$$

where $n \geq 1$ is interger and φ is a C^1 function in neighborhood of 0, $\varphi(z) = o(|z|^n)$. If $n > 1$ then X_1 and X_2 is not totally real at 0, so we can not deduce that X_1 and X_2 are locally polynomially at 0 by the Wermer's work. However, using the results about local approximation of De Paepe (see [4]) or the work of Bharali (see [5]), we can conclude that X_1 and X_2 are locally polynomially convex at 0. In this paper, we will investigate the local polynomially hull of $X_1 \cup X_2$ at 0. The ideas of proof takes

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from [2] and [3]. An appropriate tool in this context is Kallin's lemma (see [6,7]): *Suppose X_1 and X_2 are polynomially convex subsets of $\mathbf{C}^n$, suppose there is polynomial p mapping X_1 and X_2 into two polynomially convex subsets Y_1 and Y_2 of the complex plane such that 0 is a boundary point of both Y_1 and Y_2 and with $Y_1 \cap Y_2 = \{0\}$. If $p^{-1}(0) \cap (X_1 \cup X_2)$ is polynomially convex, then $X_1 \cup X_2$ is polynomially convex.* Several instances of such a situation, motivated by questions of local approximation, were studied by O'Farell, De Paepe and Nguyen Quang Dieu (see [8-10],...).

Let f be a continuous function on D. We denote that $[z^2, f^2; D]$ is the function algebra which consisting of uniform limit on D of all polynomials in z^2 and f^2. Using polynomial convexity theory, it can be shown that $[z^2, f^2; D] = C(D)$ for some choices a C^1 function f, which behaves like $\overline{z}$ near the origin (see [9-11] ...). By the known result about approximation of O'Farrell, Preskenis and Walsh [12] :*if X is polynomially convex subset of the real manifold M, K is a compact subset of X such that $X \setminus K$ is totally real. Then, if f is continuous function on X and f can be uniform approximated by polynomials on K then f can be uniform approximated by polynomials on X*, and the techniques developed in [13], we give a class function f which behaves like $\overline{z}^n$ such that $[z^2, f^2; D] = C(D)$.

2. The main results

We always take the graphs X_1 and X_2 of the form $(*)$. For each $r > 0$ we put

$$X_i^r = X_i \cap \{(z, w) : |z| \leq r\}, \quad i = 1, 2.$$

Now we come to the main results of this paper.

Theorem 2.1. *Let m, n be positive integers with $m > n$. Let φ be a C^1 function which is defined near 0 of the form*

$$\varphi(z) = \begin{cases} \sum_{k=-\infty}^{+\infty} a_k \overline{z}^k z^{m-k} + f(z) & z \neq 0 \\ 0 & z = 0, \end{cases}$$

where $f(z)$ is a C^1 function and $f(z) = o(|z|^m)$. Suppose that there exists $l \leq \frac{m}{2}$ such that

$$|a_l| > \sum_{k \neq l} |a_k| \tag{1}$$

and $\frac{m-2l}{n}$ is integer. Then $X_1 \cup X_2$ is locally polynomially convex at 0.

Proof. Consider the polynomial $p(z, w) = \overline{\alpha} z^{m-2l+n} + \alpha w^{\frac{m-2l}{n}+1}$ with α choose later. Thus $p(X_1) = \overline{\alpha} z^{m-2l+n} + \alpha \overline{z}^{m-2l+n}$ belongs to real axis and

$$p(X_2) = \overline{\alpha} z^{m-2l+n} + \alpha(\overline{z}^n + \sum_{k=-\infty}^{+\infty} a_k \overline{z}^k z^{m-k} + f(z))^{\frac{m-2l}{n}+1} =$$

$$= \overline{\alpha} z^{m-2l+n} + \alpha \overline{z}^{m-2l+n} + \alpha(\frac{m-2l}{n} + 1) \overline{z}^{m-2l} \sum_{k=-\infty}^{+\infty} a_k \overline{z}^k z^{m-k} + o(|z|^m).$$

From $p(X_1) = \overline{\alpha} z^{m-2l+n} + \alpha \overline{z}^{m-2l+n} \in \mathbf{R}$, we obtain

$$\operatorname{Im} p(X_2) = \operatorname{Im}(\alpha(\frac{m-2l}{n} + 1) \overline{z}^{m-2l} \sum_{k=-\infty}^{+\infty} a_k \overline{z}^k z^{m-k} + o(|z|^m)).$$

Choose $\alpha = i\frac{\overline{a_l}}{|a_l|}$. It follows that

$$\mathrm{Im}p(X_2) \geq |z|^{2m-2l}(\frac{m-2l}{n}+1)(|a_l| - \sum_{k\neq l}|a_k|) > 0 \qquad (2)$$

for any $z \neq 0$ in a small neighborhood of 0, by (1). It implies that $p(X_2) \cap \mathbf{R} = \{0\}$. On the other hand, from the inquality (2) we see that

$$p^{-1}(0) \cap X_2^r = \{0\}.$$

It is elmentary to check that

$$p^{-1}(0) \cap X_1^r = \{(\rho\exp(i\theta), \rho^n\exp(-ni\theta)) : 0 \leq \rho \leq r\},$$

with a constant θ. Obviously,

$$p^{-1}(0) \cap X_1^r$$

is polynomially convex for r small enough. Thus $p^{-1}(0)\cap(X_1^r \cup X_2^r)$ is polynomially convex for r small enough. By Kallin's lemma (mentioned in introduction) we conclude that $X_1^r \cup X_2^r$ is polynomially convex for r small enough. The proof is completed.

Remark. 1) In the Theorem 1 we can replace X_1 by

$$X_1' = \{(z, \overline{z}^n - \varphi(z)) : z \in D\}.$$

Then, as p in Theorem 1 we obtain the estimate

$$\mathrm{Im}p(X_1') < 0,$$

for any $z \neq 0$ in small neighborhood of 0. On the other hand $p^{-1}(0) \cap (X'^r_1 \cup X_2^r) = \{0\}$ for r small enough. By Kallin's lemma we may conclude that $X_1' \cup X_2$ is locally polynomially convex.

2) This result includes the more restricted case $n = 1$ that is studied by Nguyen Quang Dieu (see [2]).

The following Proposition shows that if we replace $l > \frac{m}{2}$ we may get nontrivial hull of $X_1^r \cup X_2^r$.

Proposition 2.2. *Let n, p be positive integers and*

$$X_1 = \{(z, \overline{z}^n) : z \in D\}; X_2 = \{(z, \overline{z}^n + z^p\overline{z}^{n+p}) : z \in D\}.$$

Then $X_1 \cup X_2$ is not locally polynomially convex at 0.

Proof. For each $t > 0$, let $W_t = \{(z, w) : z^n w = t\}$. Consider the sets

$$P_t := W_t \cap X_1 = \{(z, \overline{z}^n) : |z| = t^{\frac{1}{2n}}\};$$

$$Q_t := W_t \cap X_2 = \{(z, \overline{z}^n + z^p\overline{z}^{n+p}) : |z| = s\},$$

where s is unique positive solution of the equation $s^{2n} + s^{2p+2n} = t$. By the maximum modulus principle we see that the hull of $X_1^r \cup X_2^r$ will contain an open subset of W_t bounded by two closed curves P_t and Q_t for any $t > 0$ small enough and hence $X_1 \cup X_2$ is not locally polynomially convex at 0 .

Theorem 2.3. *Let m be a positive even integer and let n be a odd integer such that $m > n$. Let g be a C^1 function which is defined near 0 of the form*

$$g(z) = \begin{cases} \overline{z}^n + \sum_{k=-\infty}^{+\infty} a_k \overline{z}^k z^{m-k} + f(z)) & z \neq 0 \\ 0 & z = 0, \end{cases}$$

where f is a C^1 function and $f(z) = o(|z|^m)$. Suppose that there exists l such that $\frac{m-2l}{n}$ is positive integer and

$$|a_l| > \sum_{k \neq l} |a_k|. \tag{3}$$

Then the functions z^2 and $g^2(z)$ separate points near 0. Morever, $[z^2, g^2; D] = C(D)$ for D small enough.

We need the next lemma (see [7,8]) for the proof of Theorem 2.1.

Lemma 2.4. *Let X be a compact subset of $\mathbf{C}^2$, and let $\pi : \mathbf{C}^2 \to \mathbf{C}^2$ be defined by $\pi(z, w) = (z^m, w^n)$. Let $\pi^{-1}(X) = X_{11} \cup ... \cup X_{kl} \cup ... \cup X_{mn}$ with X_{mn} compact, and $X_{kl} = \{(\rho^k z, \tau^l w) : (z, w) \in X_{mn}\}$ for $1 \leq k \leq m$, $1 \leq l \leq n$, where $\rho = \exp\left(\frac{2\pi i}{m}\right)$ and $\tau = \exp\left(\frac{2\pi i}{n}\right)$. If $P(\pi^{-1}(X)) = C(\pi^{-1}(X))$, then $P(X) = C(X)$.*

Proof of Theorem 2.3. First we show that the functions z^2 and $g^2(z)$ separate points near 0. Clearly points a and b with $a \neq -b$ are separated by z^2. Now assume that $g^2(z)$ takes the same value at a and $-a$ for some $a \neq 0$. Set

$$h(z) = \begin{cases} \sum_{k=-\infty}^{+\infty} a_k \overline{z}^k z^{m-k} + f(z) & z \neq 0 \\ 0 & z = 0, \end{cases}$$

it follows that $h(a) = -h(-a)$. As m is even, we have

$$\sum_{k=-\infty}^{+\infty} a_k \overline{a}^k a^{m-k} = \frac{-f(a) - f(-a)}{2}.$$

Dividing both sides by $a^{m-l}\overline{a}^l$ we obtain

$$a_l + \sum_{k \neq l} a_k \frac{a^{l-k}}{\overline{a}^{l-k}} = \frac{-f(a) - f(-a)}{2a^{m-l}\overline{a}^l}.$$

By the inequality (3) and the fact that $f(z) = o(|z|^m)$, we arrive at a contradition if we choose the disk D sufficiently small.

Next we consider for a small closed disk D the set $\tilde{X}$ which is the inverse of the compact $X = \{(z^2, g^2(z) : z \in D\}$ under the map $(z, w) \mapsto (z^2, w^2)$. We have $\tilde{X} = X_1 \cup X_2 \cup X_3 \cup X_4$ where

$$X_1 = \{(z, \overline{z}^n + h(z)) : z \in D\};$$

$$X_2 = \{(-z, -\overline{z}^n - h(z)) : z \in D\} = \{(z, \overline{z}^n - h(-z)) : z \in D\};$$

$$X_3 = \{(-z, \overline{z}^n + h(z))) : z \in D\};$$

$$X_4 = \{(z, -\overline{z}^n - h(z)) : z \in D\} = \{(-z, \overline{z}^n - h(-z)) : z \in D\};$$

By Remark 1), $X_1 \cup X_2$ is polynomially convex for D small enough . We have $X_3 \cup X_4$ is the image of $X_1 \cup X_2$ under the biholomorphic map $(z, w) \mapsto (-z, w)$. So $X_3 \cup X_4$ is also polynomially convex with D sufficiently small.

Now we consider the polynomial $q(z, w) = z^n w$. Then q maps $X_1 \cup X_2$ to an angular sector situated near the positive real axis, while p maps $X_3 \cup X_4$ to such sector situated near the negative real axis. The sectors only meet at the origin. Applying Kallin's lemma we get $\tilde{X} = X_1 \cup X_2 \cup X_3 \cup X_4$ is polynomially convex with D small enough. Furthermore, notice that $\tilde{X} \setminus \{0\}$ is totally real (locally contained in a totally real manifold), by an approximation theorem of O'Farrell, Preskenis and Walsh (mentioned in introduction), we get that every continuous function on $\tilde{X}$ can be uniformly approximated by polynomials. By the Lemma 2.4, we see that the same is true for X, which is equivalent to the fact that our algebra equals $C(D)$.

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