VNU Journal of Science, Mathematics - Physics 24 (2008) 67-71

Characterized rings by pseudo - injective modules

Le Van An[1,*], Dinh Duc Tai[2]

[1]*Highschool of Phan Boi Chau, Vinh city, Nghe An, Vietnam*
[2]*Ha Tinh University, Ha Tinh city, Vietnam*

Received 25 July 2007; received in revised form 25 July 2008

Abstract: It is shown that:
(1) Let R be a simple right Noetherian ring, then the following conditions are equivalent:
(i) R is a right SI ring;
(ii) Every cyclic singular right R - module is pseudo - injective.
(2) Let R be a right artinian ring such that every finite generated right R - module is a direct sum of a projective module and a pseudo - injective module. Then:
(i) $R/Soc(R_R)$ is a semisimple artinian ring;
(ii) $J(R) \subset Soc(R_R)$;
(iii) $J^2(R) = 0$.
(3) Let R be a ring with condition (*), then every singular right R - module is isomorphic with a direct sum of pseudo - injective modules.

1. Introduction

Throughout this note, all rings are associative with identity, and all modules are unital right modules. The socle and the Jacobson radical of M are denoted by $Soc(M)$ and $J(M)$. Given two R - modules M and N, N is called M - injective if for every submodule A of M, any homomorphism $\alpha : A \longrightarrow N$ can be extended to a homomorphism $\beta : M \longrightarrow N$. A module$N$ is called injective if it is M - injective for every R - module M. On the other hand, N is called quasi - injective if N is N - injective. For basic properties of injective modules we refer to [1-4].

We say N is M - pseudo - injective (or pseudo - injective relative to M) if for every submodule X of M, any monomorphism $\alpha : A \longrightarrow N$ can be extended to a homomorphism $\beta : M \longrightarrow N$. N is called pseudo - injective if N is N - pseudo - injective. We have the following implications:

Injective $\Rightarrow$ quasi - injective $\Rightarrow$ pseudo - injective.

We refer to [5-8] for background on pseudo - injective modules.

Let M be a module. A module $Z(M)$ is called singular submodule of M if $Z(M) = \{x \in M \mid xI = 0$ for some essential right ideal of $R\}$. If $Z(M) = M$ then M is called singular module, while if $Z(A) = 0$

* Corresponding author. Tel: 84-0383569442
Email: levanan_na@yahoo.com

then A is called nonsingular module. A ring R is called a right (left) SI ring if every singular right R - module is injective. For basic properties of singular (nonsingular) modules and SI rings we refer to [3] and [9].

For a ring R consider the following conditions:

(*) Every cyclic right R - module is a direct sum of a projective module and a pseudo - injective module.

(**) The direct sum of every family of pseudo - injective right R - modules is also pseudo - injective.

(***) Every singular right R - module is a pseudo - injective module.

In [10, Theorem 1], it was shown that a simple ring R is right SI iff every cyclic singular right $R-$module is quasi- continuous. In this paper, we give characteristics of SI rings by class modules pseudo - injective. We prove that a simple ring R is right SI iff every cyclic singular right R - module is pseudo - injective. Note that, every quasi - injective modules is quasi- continuous and pseudo - injective module; but pseudo - injective module is not quasi- continuous and quasi - continuous module is not pseudo - injective. We give also characteristics of artinian rings by class modules pseudo - injective

2. The results

Theorem 2.1. *Let R be a simple right Noetherian ring, then the following conditions are equivalent:*

(i) R is a right SI ring;

(ii) Every cyclic singular right R - module is pseudo injective.

Proof. $(i) \Longrightarrow (ii)$ is clear.

$(ii) \Longrightarrow (i)$. Let R be a simple right Noetherian ring whose cyclic singular right $R-$ modules are pseudo injective. We show that R is a right SI ring. If $Soc(R_R) \neq 0$, then $Soc(R_R) = R$, and hence R is a simple artinian ring. We are done. Next consider the case $Soc(R_R) = 0$. We prove that any artinian right R - module A is semisimple.

Assume that $A \neq 0$, we imply $A \cong F/K$ with F is a free module. We show that K is an essential submodule of F. Assume K is not essential in F, there is a submodule T of F such that $K \cap T = 0$, i.e., $K \oplus T$ is a submodule of F. Note that $A \cong F/K \supset (K \oplus T)/K \cong T$. Since A is a artinian module, thus $Soc(T) \neq 0$. Hence $Soc(F) \neq 0$. But $F = \oplus_{i\in I} R_i$ with $R_i \cong R_R$ for any $i \in I$. We imply $Soc(F) = \oplus_{i\in I} Soc(R_i) = 0$, a contradiction. Therefore K is an essential submodule of F, i.e., A is a singular module. Let X be a cyclic submodule of A. We have $Soc(X) \neq 0$. By R is right noetherian, and X is a finitely generated module, we imply $Soc(X) = X_1 \oplus ... \oplus X_k$ where each X_t is simple module. By [11, Lemma 3.1], we can show that $X \oplus X_1$ is cyclic. By induction, we have $X \oplus Soc(X)$ is cyclic. Hence $X \oplus Soc(X)$ is pseudo injective. By [7, Theorem 2.2] or [5, Corollary 5], thus $Soc(X)$ is X - injective. Therefore $Soc(X)$ is a direct summand of X. Since X is a artinian module, we imply $X = Soc(X) \subseteq Soc(A)$. This shows that A is semisimple.

Now, we prove that every singular cyclic module over R is semisimple, or equivalently, for each essential right ideal C of R, R/C is semisimple. By the above claim, it suffices to that show R/C is artinian. Hence R is a SI ring.

Assume on the contrary that there is an essential right ideal A of R such that R/A is not artinian. Since R is right noetherian, there exists an essential right ideal L of R which is maximal with respect to the condition that $M = R/L$ is not artinian. We show that M is uniform and $Soc(M) = 0$. Assume M is not uniform, there are two submodules P_1, P_2 of R_R such that $L \subset P_1, P_2 \subset R_R$, $L \neq P_1$, $L \neq P_2$ and $P_1 \cap P_2 = L$. Let f be a homomorphism $R \longrightarrow R/P_1 \oplus R/P_2$, $f(r) = (r + P_1, r + P_2)$

then, $Kerf = P_1 \cap P_2 = L$. We have $M = R/L = R/Kerf \cong Imf \subseteq R/P_1 \oplus R/P_2$. Note that, R/P_1 and R/P_2 are artinian, thus R/L is also artinian, a contradiction. Hence M is uniform. By M is not a simple module, we imply $Soc(M) = 0$. Moreover, for any nonzero submodule N of M, we have $N = N_1/L$ with $L \subset N_1$ and $L \neq N_1$. By $M/N = R/L/N_1/L \cong R/N_1$, thus M/N is a artinian module. Therefore M/N is a semisimple module. Let U and V be submodules of M with $0 \neq U \subset V \subset M$ and $U \neq V \neq M$. Then V/U is a submodule of semisimple artinian module M/U. Hence V/U is semisimple artinian module. We imply $V/U = S_1 \oplus ... \oplus S_t$, where each S_j is simple module. Consider the module $Q = M \oplus V$. Since M is cyclic and $Q/(0 \oplus U) \cong M \oplus (V/U) = M \oplus (S_1 \oplus ... \oplus S_t)$. By [11, Lemma 3.1], we can show that $M \oplus S_1$ is cyclic. By induction, we have $M \oplus (S_1 \oplus ... \oplus S_t)$ is cyclic, i.e., $Q/(0 \oplus U)$ is cyclic. By [10], we can choose $x \in Q$ such that $[xR + (0 \oplus U)]/(0 \oplus U) = Q/(0 \oplus U)$ and xR contains $M \oplus 0$. Note that xR is not uniform. By modularity $xR = xR \cap Q = xR \cap (M \oplus V) = M \oplus W$ where $(0, W) = xR \cap (0, V) \neq (0, 0)$. By M is a singular module, thus Q is also a singular module. Hence xR is a singular cyclic module. Since xR is a pseudo - injective module, i.e., $M \oplus W$ is pseudo - injective. By [7, Theorem 2.2], W is M- injective. Therefore W is a direct summand of M, a contradiction. Hence R/C is artinian for every essential right ideal C of R. Thus R/C is a semisimple module, i.e., R is a right SI ring.

Theorem 2.2. *Let R be a right artinian ring such that every finite generated right R - module is a direct sum of a projective module and a pseudo - injective module. Then:*

(i) $R/Soc(R_R)$ is a semisimple artinian ring;

(ii) $J(R) \subset Soc(R_R)$;

(iii) $J^2(R) = 0$.

Proof. (i) Set $A = R/Soc(R_R)$. By R is a right artinian ring, thus A is a singular right R - module. Set $M = A \oplus Soc(A)$, then M is a singular module. Since M is a finite generated right R - module, we imply $M = X \oplus Y$ where X is a pseudo - injective module and Y is a projective module. Note that Y is a singular module, we have $Y \cong F/K$ with K is an essential submodule of F. By Y is a projective module, we imply K is a direct summand of F. Hence $K = F$, i.e., $Y = 0$. Therefore $M = A \oplus Soc(A)$ is a pseudo - injective module. By [7, Theorem 2.2], $Soc(A)$ is $M-$ injective. By A is artinian, thus A is semisimple artinian ring, proving (i).

(ii) and (iii). By properties (i).

Theorem 2.3. *Let R be a ring with condition (*), then every singular right R - module is isomorphic with a direct sum of pseudo - injective modules.*

Proof. Let R be a ring with condition (*). Let X be a cyclic singular right R - module, then by condition (*) we have $X = P \oplus A$ with P is a projective and A is a pseudo - injective. Since P is singular module, thus $P \cong U/V$ with V is an essential submodule of U. By P is a projective module, we imply V is a direct summand of U. Hence $U = V$, i.e., $P = 0$. Therefore X is a pseudo - injective module.

Finally, if M is a singular right R- module, then $M \cong F/K$ with F is a free module and K is an essential submodule of F. Note that $F \cong \oplus_{i \in I} R_i$ with $R_i \cong R_R$, we have $M \cong F/K = (\oplus_{i \in I} R_i)/K \cong \oplus_{i \in I}(R_i + K)/K$. By $(R_i + K)/K$ is a cyclic singular module, thus it is a pseudo - injective module. Therefore, M is isomorphic with a direct sum of pseudo - injective modules.

Corollary 2.4. *Let R be a ring satisfies conditions (*) and (**), then R satisfies condition (***).*

Proof. By condition (**) and Theorem 2.3.

3. Examples

1) ([8, page 364]) Let $F = \mathbf{Z}_2$ where $\mathbf{Z}$ is the ring of integer numbers and $A = F[X]$. Then $A/(x)$ is a $(A/(x) - A/(x^2))$-bimodule in the natural way, and

$$R = \{\begin{pmatrix} u & 0 \\ v & w \end{pmatrix} \mid u, v \in A/(x), w \in A/(x^2)\}$$

is a ring with usual binary operations. Let M be the right ideal

$$M = \{\begin{pmatrix} 0 & 0 \\ v & w \end{pmatrix} \mid v \in A/(x), w \in A/(x^2)\}.$$

Then M_R is pseudo - injective but not quasi - injective.

2) Consider the following ring

$$R = \begin{pmatrix} \mathbf{C} & \mathbf{C} \\ 0 & \mathbf{R} \end{pmatrix}$$

where $\mathbf{R}$ and $\mathbf{C}$ are the fields of real and complex numbers, respectively. By [12, page 152], every right R-modules is a direct sum of a projective module and a quasi - injective module. Hence R satifies conditions (*) and (***).

3) Consider the following ring

$$R = \begin{pmatrix} \mathbf{R} & \mathbf{C} \\ 0 & \mathbf{C} \end{pmatrix}$$

where $\mathbf{R}$ and $\mathbf{C}$ are the fields of real and complex numbers, respectively. By [12, page 151], every right R-modules is a direct sum of a projective module and a quasi - injective module. Hence R satifies conditions (*) and (***). By [12, page 151], R is right SI ring.

4) Consider the following ring

$$R = \begin{pmatrix} \mathbf{R} & \mathbf{R} \\ 0 & \mathbf{R} \end{pmatrix}$$

where $\mathbf{R}$ is the field of real numbers. By [13, Remark 3.4 (page 464)], R is a CS-semisimple ring. Hence R is a right and left Artinian ring (see [11, Theorem 1.1]). Let $M_i, \forall i \in I$ be pseudo - injective right R-modules, then M_i is CS-module where each $i \in I$. Hence M_i is quasi - injective (see [5, Theorem 5]). By [4], $M = \oplus_{i \in I} M_i$ is a quasi - injective module. Hence M is a pseudo - injective module, i.e., R satifies conditions (**).

Acknowledgments. The author is grateful to Prof. Dinh Van Huynh (Department of Mathematics Ohio University) and Assoc. Prof. Ngo Si Tung (Department of Mathematics Vinh University) for many helpful comments and suggestions. The author also wishes to thank an anonymous referee for his or her suggestions which lead to substantial improvements of this paper.

References

[1] F.W. Anderson, K.R Furler, *Ring and Categories of Modules*, Springer - Verlag, NewYork - Heidelberg - Berlin, 1974.

[2] N.V. Dung, D.V. Huynh, P.F. Smith, R. Wisbauer, *Extending Modules*, Pitman, London, 1994.

[3] K.R. Goodearl, R.B. Warfield, An Introduction to Noncommutative Noetherian Rings, *London Math. Soc. Student Text*, Vol. 16, Cambridge Univ. Press 1989.

[4] S.H. Mohamed, B.J. Müller, Continuous and Discrete Modules, *London Math. Soc. Lecture Note Ser.*, Vol. 147, Cambridge University Press 1990.

[5] A. Al - Ahmadi, N. Er, S.K. Jain, Modules which are invariant under monomorphisms of their injective hulls, *J. Australian Math. Soc.*, 79 (2005) 349.
[6] H.Q. Dinh, On pseudo - injective modules, *AMS Meeting No.990, Binghamton (New York)*, October 11 - 12 (2003).
[7] H.Q. Dinh, A note on pseudo - injective modules, *Comm. Algebra* 33 (2005) 361.
[8] S.K. Jain, S. Singh, Quasi - injective and pseudo - injective modules, *Canad. Math. Bull.*, 18(3) (1975) 359.
[9] K.R. Goodearl, Singular Torsion and the Splitting Properties, Mem. Amer. Math. Soc., 124 (1972).
[10] D.V. Huynh, S.K. Jain, S.R. L ópez-Permouth, When cyclic singular modules over a simple ring are injective, *J. Algebra* 263 (2003) 188.
[11] D.V. Huynh, S.K. Jain, S.R. L ópez-Permouth, When is simple ring Noetherian? *J. Algebra* 184 (1996) 786.
[12] D.V. Huynh, S.T. Rizvi, On some classes of artinian rings, *J. Algebra* 223(2000) 133.
[13] D.V. Huynh, Some remarks on CS modules and SI rings, *Bull. Austral. Math. Soc.* 65 (2002) 461.